Pomeranchuk singularity in a Reggeon field theory with quartic couplings*

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We employ the methods of the renormalization group to investigate the structure of Pomeron Green's functions near $l=1$, $t=0$ in a theory where only four-Pomeron couplings are present. When certain conditions on the renormalized couplings are met, the $l\to 1$, $t\to 0$ behavior of the theory is governed by free effective couplings (the theory is "infrared-free"). In this situation the renormalized Pomeron trajectory is of the form $\alpha(t)=1+At+Bt/(\ln Ct)^5$ for $t>0$, where A, B, and C are constants; the total cross section behaves as $\sigma_T(s)\sim\gamma_1+\gamma_2/(\ln\ln s)^5$, with γ_1 and γ_2 positive.

I. INTRODUCTION

The Reggeon calculus or Reggeon field theories whose study was initiated by Gribov[1] several years ago provides a constructive procedure for investigating the detailed interaction among Regge poles and cuts. It also yields an automatic and natural way to satisfy the discontinuity relations across Reggeon cuts.

In this paper we continue our discussion of the structure of the Pomeranchuk singularity which arises in interacting Reggeon field theories. As in our earlier work[2] we employ the renormalization group to provide a nonperturbative tool for the analysis of the renormalized Reggeon Green's functions in the neighborhood of $l=1$ and $t=0$. Our previous work pointed out the large ambiguity in choosing the appropriate Reggeon field theory within which one ought to cast the Pomeron problem. We encourage the reader to review the detailed motivation for Reggeon field theories as given in Ref. 2 and only remember here that one must choose both a noninteracting Reggeon, to begin with, and then a precise form of the interaction. In Ref. 2 we studied the physically very interesting case of a linear trajectory $\alpha(t)=1+\alpha_0' t$ whose interactions were given by a triple-Pomeron coupling only. Because of the wide range of possibilities in formulating the field theories (a situation hardly special to Reggeons), we feel it is important to study a variety of other theories even when their clear connection to physical processes may be vague.

We shall present here our analysis of the Reggeon theory in which the noninteracting Reggeon has the energy- $(E=1-l)$ momentum $(t=|\vec{k}|^2)$ relation

$$E=\alpha_0'\vec{k}^2, \tag{1}$$

appropriate to a linear trajectory, and where the interaction is taken to be of the $\lambda\psi^4$ variety. It is easy to see from the outset that such a theory will never possess a triple-Pomeron coupling. That coupling is of direct physical importance, as, for example, in inclusive processes. However, it turns out that there are a variety of amusing aspects to the quartic-coupling problem which are not only interesting in themselves but also play a role in the study of the structure of secondary trajectories when Pomeron interactions are accounted for.[3] The additional feature to note is the presence of more than one coupling constant (arising here because of the absence of crossing symmetry in nonrelativistic theories with $E\propto\vec{k}^2$) which makes the infrared behavior of the proper Reggeon vertex functions ($E\to 0$, $|\vec{k}|\to 0$) depend in detail on the direction of approach to the limit and on the precise values of the renormalized couplings.

We find that when certain conditions on couplings are met, the infrared nature of the Green's functions is governed by the effective coupling constants evaluated at zero. That is, the field theory is infrared-free. Under these conditions very mild modifications of the trajectories and Green's functions are present. For example, the bare linear trajectory

$$\alpha(t)=1+\alpha_0' t \tag{2}$$

is modified for t small and positive to

$$\alpha(t)=1+At+Bt/(\ln Ct)^5 . \tag{3}$$

A constant total cross section which would come from (1) is then reached as

$$\sigma_T(s)\underset{s\to\infty}{\sim}\gamma_1+\gamma_2/(\ln\ln s)^5, \tag{4}$$

where $\gamma_1,\ \gamma_2>0$.

II. REGGEON FIELD THEORY WITH ψ^4 COUPLINGS

We begin by recalling that the Reggeon calculus is a technique for building partial-wave amplitudes out of the propagation and interaction of quasiparticles (Reggeons) in two space dimensions and one time dimension. The theory we consider here starts with a linear trajectory

$$\alpha(t)=\alpha_0+\alpha_0' t\ . \tag{5}$$

Following Ref. 2 we choose to cast this in the form

$$\begin{aligned} E(\vec{q}^{\,2}) &= 1-\alpha(\vec{q}^{\,2}) \\ &= (1-\alpha_0)+\alpha_0'\vec{q}^{\,2}, \end{aligned} \tag{6}$$

where, clearly,

$$t=-\vec{q}^{\,2}\ . \tag{7}$$

This is the bare energy-momentum relation of our quasiparticle. It is described by the free action

$$\begin{aligned} A_0 &= \int d^D x\, dt [\tfrac{1}{2} i\psi^\dagger(\vec{x},t)\overleftrightarrow{\partial}_t \psi(x,t) \\ &\qquad -\alpha_0' \nabla\psi^\dagger\cdot\nabla\psi-(1-\alpha_0)\psi^\dagger\psi] \\ &= \int d^D x\, dt\, L_0(\vec{x},t), \end{aligned} \tag{8}$$

where the Reggeon field operator $\psi(\vec{x},t)$ has been written in D space dimensions conjugate to $\vec{q}$ and one time dimension conjugate to E. Physics takes place at $D=2$. The interaction we choose is described by the addition to the free Lagrangian

$$L_1(\vec{x},t)=-\frac{\lambda_{10}}{(2!)^2}(\psi^\dagger)^2(\psi)^2-\frac{\lambda_0}{3!}[\psi^\dagger\psi^3+(\psi^\dagger)^3\psi]. \tag{9}$$

The first term represents the two-to-two process in Fig. 1, while the second term gives the one-Pomeron-in–three-Pomeron-out (and vice versa) amplitude in Fig. 2. The absence of crossing symmetry means that the couplings λ_{10} and λ_0 for these transitions need not be equal.

As with the ψ^3 theory, it is useful to perform some ordinary dimensional analysis on the quantities in $L=L_0+L_1$. Noting separately dimensions (denoted by square brackets) of time as $(\text{energy})^{-1}$ and those of space as $(\text{momentum})^{-1}$ we write

$$[t]=E^{-1}, \tag{10}$$

$$[x]=k^{-1}. \tag{11}$$

Requiring the action to be dimensionless yields

$$[\psi]=k^{D/2}, \tag{12}$$

$$[\lambda]=[\lambda_1]=Ek^{-D}, \tag{13}$$

and, of course,

$$[\alpha']=Ek^{-2}. \tag{14}$$

It is clear that the couplings λ_1 and λ are not dimensionless. We will shortly define dome dimensionless coupling constants to replace the λ's.

Our procedure will be to examine the renormalized proper vertex functions $\Gamma_R^{(n,m)}$ for n incoming and m outgoing Pomerons. These vertex functions are defined as the Fourier transforms of the renormalized Green's functions

$$G_R^{(n,m)}(\vec{x}_1,t_1,\ldots,\vec{x}_{n+m},t_{n+m})=Z^{-(n+m)/2}\langle 0|T(\psi^\dagger(\vec{x}_{n+m},t_{n+m})\cdots\psi^\dagger(\vec{x}_{n+1},t_{n+1})\psi(\vec{x}_n,t_n)\cdots\psi(\vec{x}_1,t_1))|0\rangle \tag{15}$$

with the external legs amputated by multiplication by

$$\prod_{i=1}^{n+m}[G_R^{(1,1)}(E_i,\vec{k}_i)^{-1}]\ . \tag{16}$$

The renormalization constant Z relates the renormalized field operator to ψ by

$$\psi_R=Z^{-1/2}\psi\ . \tag{17}$$

The unrenormalized theory is defined in terms of λ_{10}, λ_0, α_0', α_0, and a possible cutoff will call Λ. We choose $\alpha_0=1$, which means we are dealing with "massless" quasiparticles; there is no energy gap at $\vec{k}^2=0$. We will parameterize the renormalized theory by a set of numbers λ_1, λ, α', and α. These numbers are determined by normalization conditions on a selected set of vertex functions. We choose $\alpha=1$, which means that in the renormalized theory the singularities of the propagator, $G_R^{(1,1)}$, will occur at $E=0$, $\vec{k}^2=0$. This is guar-

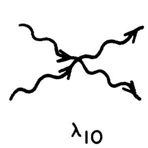

FIG. 1. The transition two Pomerons → two Pomerons has a bare coupling constant λ_{10}.

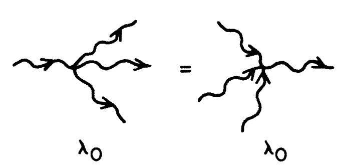

FIG. 2. The transitions three Pomerons → one Pomeron or one Pomeron → three Pomerons. Both have a bare coupling λ_0.

anteed by requiring

$$\Gamma_R^{(1,1)}(E,\vec{k}^2,\alpha',\lambda,\lambda_1,E_N,k_N^2)\big|_{E=0;\vec{k}^2=0}=0\,. \quad (18)$$

Since there is no "mass" scale in the renormalized theory because $\alpha=1$, we have to provide one by choosing a normalization point somewhere in the $(E,\vec{k}^2)$ plane. We will choose the point $E=-E_N<0$ and $\vec{k}^2=k_N^2>0$. This choice keeps us away from the various branch points which arise in perturbation theory in λ_{10} and λ_0. This is pictured in Fig. 3. Normalizing away from $E=0$, $\vec{k}^2=0$ keeps us from any infrared problems. All the vertex functions $\Gamma_R^{(n,m)}$ will depend on E_N and k_N^2 [as indicated in Eq. (18) for $\Gamma^{(n,m)}$], as will the renormalized parameters λ_1, λ, and α'.

The remainder of our normalization conditions are given by

$$\frac{\partial}{\partial E}\, i\Gamma_R^{(1,1)}(E,\vec{k}^2,\alpha',\lambda,\lambda_1,E_N,k_N^2)\bigg|_{E=-E_N;\vec{k}^2=k_N^2}=1\,, \quad (19)$$

$$\frac{\partial}{\partial\vec{k}^2}\, i\Gamma_R^{(1,1)}(E,\vec{k}^2,\alpha',\lambda,\lambda_1,E_N,k_N^2)\bigg|_{E=-E_N;\vec{k}^2=k_N^2}=-\alpha'(E_N,k_N^2), \quad (20)$$

which are not specific to the quartic interaction but are requirements coming from our choice of a linear $E,\vec{k}^2$ relation. Further, we set λ_1 by

$$\Gamma_R^{(2,2)}(E_i,\vec{k}_i,\alpha',\lambda,\lambda_1,E_N,k_N^2)\big|_{E_i=-E_N;\vec{k}_i\cdot\vec{k}_j=(k_N^2/3)(4\delta_{ij}-\eta_i\eta_j)}=\frac{-i\lambda_1(E_N,k_N^2)}{(2\pi)^{D+1}}\,, \quad (21)$$

where $E_1,\vec{k}_1,E_2,\vec{k}_2$ are incoming while $E_3,\vec{k}_3$ and $E_4,\vec{k}_4$ are outgoing, (see Fig. 4), and $\eta_i=+1$ for an incoming Reggeon and $=-1$ for an outgoing Reggeon. Finally, with $E_1,\vec{k}_1$ incoming and $E_2,\vec{k}_2,\ldots,E_4,\vec{k}_4$ outgoing, as in Fig. 5, we set

$$\Gamma_R^{(1,3)}(E_i,\vec{k}_i,\alpha',\lambda,\lambda_1,E_N,k_N^2)\big|_{E_1=-E_N;E_2=E_3=E_4=-E_N;\vec{k}_i\cdot\vec{k}_j=(k_N^2/3)(4\delta_{ij}-\eta_i\eta_j)}=\frac{-i\lambda(E_N,k_N^2)}{(2\pi)^{D+1}}\,. \quad (22)$$

Now for convenience we shall eliminate the dimensional couplings λ and λ_1 in favor of the dimensionless combinations

$$y=\frac{\lambda}{(\alpha')^{D/2}}\,E_N^{D/2-1} \quad (23)$$

and

$$y_1=\frac{\lambda_1}{(\alpha')^{D/2}}\,E_N^{D/2-1}\,. \quad (24)$$

The space dimension $D=2$ seems slated for a special role here. Recall that in the theory with a triple-Pomeron coupling, $D=4$ played this special role. We found there a particular simplicity at $D=4$ and were able to make a perturbation expansion in $4-D=\epsilon$ for all vertex functions. Here the physical number of dimensions is singled out. We can view this special role in a somewhat more general light by noting that if we identify dimensions in time and space (so α' is dimensionless), then at $D=4$ the triple-Pomeron coupling is dimensionless while at $D=2$ the quartic-coupling constants are dimensionless. Further, each of the theories is scale-invariant in its special dimensions.

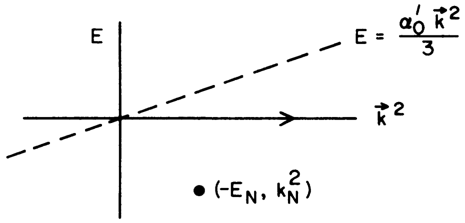

FIG. 3. The $E,\vec{k}^2$ showing the lowest-order perturbation theory cut along $E=\alpha_0'\vec{k}^2/3$ and the normalization point $(-E_N,k_N^2)$ used to define the renormalized theory.

III. RENORMALIZATION-GROUP EQUATIONS FOR THE VERTEX FUNCTIONS

The unrenormalized vertex functions $\Gamma_U^{(n,m)}(E_i,\vec{k}_i,\alpha_0',\lambda_0,\lambda_{10},\Lambda)$ are related to the $\Gamma_R^{(n,m)}$ by

$$\Gamma_R^{(n,m)}(E_i,\vec{k}_i,\alpha',\lambda,\lambda_1,E_N,k_N^2)=Z^{(n+m)/2}\Gamma_U^{(n,m)}(E_i,\vec{k}_i,\alpha_0',\lambda_0,\lambda_{10},\Lambda)\,. \quad (25)$$

The simple observation that the $\Gamma_U^{(n,m)}$ cannot know about the normalization points E_N, k_N^2 for the renormalized Γ_R leads to two independent conditions on Γ_R:

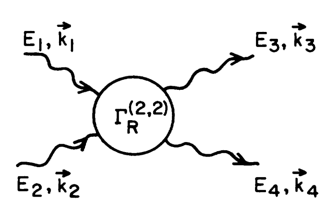

FIG. 4. The definition of momenta in the vertex function $\Gamma_R^{(2,2)}$.

$$\left[k_N^2 \frac{\partial}{\partial k_N^2} + \beta_k(y, y_1) \frac{\partial}{\partial y} + \beta_{1k}(y, y_1) \frac{\partial}{\partial y_1} + \zeta_k(\alpha', y, y_1) \frac{\partial}{\partial \alpha'} - \frac{n+m}{2} \gamma_k(y, y_1)\right] \times \Gamma_R^{(n,m)}(E_i, \vec{k}_i, \alpha', y, y_1, E_N, k_N^2) = 0\,, \quad (26)$$

where

$$\beta_k(y, y_1) = k_N^2 \frac{\partial y}{\partial k_N^2}\bigg|_{\alpha_0',\lambda_0,\lambda_{10},\Lambda\ \text{fixed}}, \quad (27)$$

$$\beta_{1k}(y, y_1) = k_N^2 \frac{\partial y_1}{\partial k_N^2}\bigg|_{\alpha_0',\lambda_0,\lambda_{10},\Lambda\ \text{fixed}}, \quad (28)$$

$$\zeta_k(\alpha', y, y_1) = k_N^2 \frac{\partial \alpha'}{\partial k_N^2}\bigg|_{\alpha_0',\lambda_0,\lambda_{10},\Lambda\ \text{fixed}}, \quad (29)$$

and

$$\gamma_k(y, y_1) = k_N^2 \frac{\partial}{\partial k_N^2} \ln Z\bigg|_{\alpha_0',\lambda_0,\lambda_{10},\Lambda\ \text{fixed}}, \quad (30)$$

These functions could also depend on $\alpha' k_N^2/E_N$, but we choose not to display it. Also one requires

$$\left[E_N \frac{\partial}{\partial E_N} + \beta_E(y, y_1) \frac{\partial}{\partial y} + \beta_{1E}(y, y_1) \frac{\partial}{\partial y_1} + \zeta_E(\alpha', y, y_1) \frac{\partial}{\partial \alpha'} - \frac{n+m}{2} \gamma_E(y, y_1)\right] \times \Gamma_R^{(n,m)}(E_i, \vec{k}_i, \alpha', y, y_1, E_N, k_N^2) = 0\,, \quad (31)$$

with β_E, β_{1E}, ζ_E, and γ_E defined just as in Eqs. (27)–(30) with $E_N\partial/\partial E_N$ replacing $k_N^2\partial/\partial k_N^2$.

These are the important equations in our work. We will turn them into equations governing the $E_i, \vec{k}_i$ variation of the $\Gamma_R^{(n,m)}(E_i, \vec{k}_i)$ by using ordinary dimensional analysis. The dimensions of $\Gamma_R^{(n,m)}$ are

$$[\Gamma_R^{(n,m)}] = E(k^2)^{D(2-n-m)/4}, \quad (32)$$

which allows us to write

$$\Gamma_R^{(n,m)}(E_i, \vec{k}_i, \alpha', y, y_1, E_N, k_N^2) = E_N\left[\frac{E_N}{\alpha'}\right]^{D(2-n-m)/4} \psi_{n,m}\left(\frac{E_i}{E_N}, \frac{\vec{k}_i}{k_N}, \frac{\alpha' k_N^2}{E_N}, y, y_1\right) \quad (33)$$

and leads directly to the result

$$\Gamma_R^{(n,m)}(\xi^\nu E_i, \xi^{\sigma/2}\vec{k}_i, \alpha', y, y_1, E_N, k_N^2) = \xi^{\nu+\sigma(D/4)(2-n-m)}\Gamma_R^{(n,m)}\left(E_i, \vec{k}_i, \xi^{\sigma-\nu}\alpha', y, y_1, \frac{E_N}{\xi^\nu}, \frac{k_N^2}{\xi^\sigma}\right). \quad (34)$$

Using this together with Eqs. (26) and (30) yields our key result:

$$\left\{\xi \frac{\partial}{\partial \xi} - (\sigma\beta_k + \nu\beta_E)\frac{\partial}{\partial y} - (\sigma\beta_{1k} + \nu\beta_{1E}) \frac{\partial}{\partial y_1} + [(\nu-\sigma)\alpha' - (\sigma\zeta_k + \nu\zeta_E)] \frac{\partial}{\partial \alpha'} + \frac{n+m}{2}(\sigma\gamma_k + \nu\gamma_E) - \nu - \sigma\frac{D}{4}(2-n-m)\right\} \times \Gamma_R^{(n,m)}(\xi^\nu E_i, \xi^{\sigma/2}\vec{k}_i, \alpha', y, y_1, E_N, k_N^2) = 0\,. \quad (35)$$

The solution to this equation is fairly standard.[4] Define $t = \ln\xi$; then

$$\Gamma_R^{(n,m)}(\xi^\nu E_i, \xi^{\sigma/2}\vec{k}_i, \alpha', y, y_1, E_N, k_N^2) = \Gamma_R^{(n,m)}(E_i, \vec{k}_i, \tilde{\alpha}'(-t), \tilde{y}(-t), \tilde{y}_1(-t), E_N, k_N^2) \times \exp\left(\int_{-t}^0 dt'\{\nu + \sigma\tfrac{1}{4}D(2-n-m) - \tfrac{1}{2}(n+m)[\sigma\gamma_k(\tilde{y}(t'), \tilde{y}_1(t')) + \nu\gamma_E(\tilde{y}(t'), \tilde{y}_1(t'))]\}\right), \quad (36)$$

where the effective slope and coupling parameters satisfy the auxiliary equations

$$\frac{1}{\tilde{\alpha}'(t)} \frac{\tilde{d}\alpha'(t)}{dt} = (\nu-\sigma) - \left(\sigma\frac{\zeta_k}{\alpha'} + \nu\frac{\zeta_E}{\alpha'}\right), \quad (37)$$

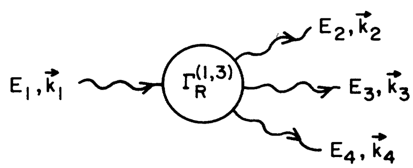

FIG. 5. The definition of momenta in the vertex function $\Gamma_R^{(1,3)}$.

$$\frac{d\tilde{y}(t)}{dt} = -(\sigma\beta_k + \nu\beta_E)\,, \quad (38)$$

and

$$\frac{d\tilde{y}_1(t)}{dt} = -(\sigma\beta_{1k} + \nu\beta_{1E})\,, \quad (39)$$

which are to be solved with the boundary conditions $\tilde{y}(0) = y$, $\tilde{y}_1(0) = y_1$, and $\tilde{\alpha}'(0) = \alpha'$. By dimensional arguments, the β's and the ζ/α''s can depend only on y and y_1.

If we knew the renormalization-group functions β, ζ, and γ and could integrate (37), (38), and

(39), then we could solve for the detailed variation of the $\Gamma_R^{(n,m)}$ in the E_i and $\vec{k}_i$. These functions can only be known in perturbation theory unless one is capable of solving the full-blown field theory directly. We will proceed as usual, then, and evaluate each of these functions in lowest-order perturbation theory. With them known to this order we will solve the auxiliary equations and then determine the behavior of the Γ_R.

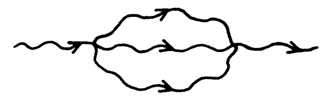

FIG. 6. The lowest-order correction to $\Gamma^{(1,1)}$.

To determine Z and α' we need $\Gamma^{(1,1)}$. The lowest-order perturbation correction to the free propagator is shown in Fig. 6. We find for this

$$i\Gamma_U^{(1,1)}(E,\vec{k}^2,\alpha_0',\lambda_0,\lambda_{10})=E-\alpha_0'\vec{k}^2+\frac{\lambda_0^2}{(\alpha_0')^D}\left(\frac{\alpha_0'\vec{k}^2}{3}-E\right)^{D-1}\frac{\Gamma(1-D)}{(\sqrt{8}\,\pi)^D(3!)^{D/2+1}}, \tag{40}$$

which gives

$$\frac{1}{Z}=1+\frac{\lambda_0^2}{(\alpha_0')^D}\left(\frac{\alpha_0'k_N^2}{3}+E_N\right)^{D-2}\times\frac{\Gamma(2-D)}{(3!)^{D/2+1}(\sqrt{8}\,\pi)^D}. \tag{41}$$

From this we evaluate γ_k and γ_E to lowest order:

$$\gamma_k(y,y_1)=y^2\,\frac{\alpha'k_N^2}{3E_N}\left(1+\frac{\alpha'k_N^2}{3E_N}\right)^{D-3}\times\frac{\Gamma(3-D)}{(\sqrt{8}\,\pi)^D(3!)^{D/2+1}}, \tag{42}$$

and

$$\gamma_E(y,y_1)=y^2\left(1+\frac{\alpha'k_N^2}{3E_N}\right)^{D-3}\frac{\Gamma(3-D)}{(\sqrt{8}\,\pi)^D(3!)^{D/2+1}}. \tag{43}$$

In arriving at these expressions we have not introduced an explicit cutoff Λ to regularize the perturbation theory. Instead we have simply used the dimension D to provide a way to stay away from infinities. The renormalization-group functions are quite regular at $D=2$.

Equation (40) also yields

$$\frac{\alpha'}{\alpha_0'}=1-\frac{2}{3}\,\frac{\lambda_0^2}{(\alpha_0')^D}\left(\frac{\alpha_0'k_N^2}{3}+E_N\right)^{D-2}\times\frac{\Gamma(2-D)}{(\sqrt{8}\,\pi)^D(3!)^{D/2+1}}, \tag{44}$$

and

$$\zeta_k=\frac{2}{9}\,\frac{\alpha'k_N^2}{E_N}\,\alpha'y^2\,\frac{\Gamma(3-D)}{(\sqrt{8}\,\pi)^D(3!)^{D/2+1}}\times\left(1+\frac{\alpha'k_N^2}{3E_N}\right)^{D-3}. \tag{45}$$

and

$$\zeta_E=\tfrac{2}{3}\,\alpha'y^2\,\frac{\Gamma(3-D)}{(3!)^{D/2+1}(\sqrt{8}\,\pi)^D}\left(1+\frac{\alpha'k_N^2}{3E_N}\right)^{D-3}. \tag{46}$$

To find the functions β we need to compute the graphs shown in Fig. 7 for $\Gamma^{(1,3)}$ and those in Fig. 8 for $\Gamma^{(2,2)}$. This gives

$$\Gamma_U^{(1,3)}\big|_{\text{normalization point}}=\frac{-i}{(2\pi)^{D+1}}\left[\lambda_0-\frac{9\lambda_{10}\lambda_0}{4(\alpha_0')^{D/2}}\,\frac{\pi^{D/2}\Gamma(1-D/2)}{3^{D/2}(2\pi)^D}\,(E_N+\alpha_0'k_N^2)^{D/2-1}\right] \tag{47}$$

and

$$\Gamma_U^{(2,2)}\big|_{\text{normalization point}}=\frac{-i}{(2\pi)^{D+1}}\left\{\lambda_{10}-\frac{\lambda_{10}^2}{4(\alpha_0')^{D/2}}\,\frac{\pi^{D/2}\Gamma(1-D/2)}{(2\pi)^D}\left(E_N+\frac{\alpha_0'k_N^2}{3}\right)^{D/2-1}-\frac{\lambda_0^2}{(\alpha_0')^{D/2}}\left[\frac{\alpha_0'k_N^2}{3}\right]^{D/2-1}\frac{\pi^{D/2}\Gamma(1-D/2)}{(2\pi)^D}\right\}. \tag{48}$$

From these we find

$$\beta_k=\frac{9y_1y}{4}\,\frac{\pi^{D/2}\Gamma(2-D/2)}{(2\pi)^D3^{D/2}}\left(1+\frac{\alpha'k_N^2}{E_N}\right)^{D/2-2}\frac{\alpha'k_N^2}{E_N}+\left(2-\frac{D}{3}\right)y^3\,\frac{\Gamma(3-D)}{(\sqrt{8}\,\pi)^D(3!)^{D/2+1}}\left(1+\frac{\alpha'k_N^2}{3E_N}\right)^{D-3}\frac{\alpha'k_N^2}{3E_N}, \tag{49}$$

$$\beta_E = \left(\frac{D}{2}-1\right)y + \tfrac{9}{4}\, y_1 y \frac{\pi^{D/2}\Gamma(2-D/2)}{(2\pi)^D 3^{D/2}}\left(1+\frac{\alpha' k_N^2}{E_N}\right)^{D/2-2} + \left(2-\frac{D}{3}\right)y^3 \frac{\Gamma(3-D)}{(\sqrt{8}\,\pi)^D (3\,!)^{D/2+1}}\left(1+\frac{\alpha' k_N^2}{3E_N}\right)^{D-3}, \tag{50}$$

$$\beta_{1k} = \frac{y_1^2}{4}\,\frac{\pi^{D/2}\Gamma(2-D/2)}{(2\pi)^D}\left(1+\frac{\alpha' k_N^2}{3E_N}\right)^{D/2-2}\frac{\alpha' k_N^2}{3E_N}$$
$$+\, y^2\left(\frac{\alpha' k_N^2}{3E_N}\right)^{D/2-1}\frac{\pi^{D/2}\Gamma(2-D/2)}{(2\pi)^D} + (2-D/3)y^2 y_1 \frac{\Gamma(3-D)}{(\sqrt{8}\,\pi)^D (3\,!)^{D/2+1}}\left(1+\frac{\alpha' k_N^2}{3E_N}\right)^{D-3}\frac{\alpha' k_N^2}{3E_N}, \tag{51}$$

and

$$\beta_{1E} = \left(\frac{D}{2}-1\right)y_1 + \frac{y_1^2}{4}\,\frac{\pi^{D/2}\Gamma(2-D/2)}{(2\pi)^D}\left(1+\frac{\alpha' k_N^2}{3E_N}\right)^{D/2-2} + \left(2-\frac{D}{3}\right)y^2 y_1 \frac{\Gamma(3-D)}{(\sqrt{8}\,\pi)^D (3\,!)^{D/2+1}}\left(1+\frac{\alpha' k_N^2}{3E_N}\right)^{D-3}. \tag{52}$$

Note that for $D\neq 2$ β_E and β_{1E} have a linear zero at $y=0$ or $y_1=0$. This disappears at $D=2$, leaving a higher-order zero.

The slope of the functions β near $y=0$ or $y_1=0$ is positive for $D=2$. This means that zero coupling may be a stable point of our auxiliary equations in the infrared region.[4] More precisely, as we study the $\xi\to 0$ ($t\to-\infty$) limit of (36) the effective couplings $\tilde{y}(-t)$ and $\tilde{y}_1(-t)$ will approach zero. In such a case we may ignore the cubic terms in our β's since they will be negligible in the ξ region of interest. With this in mind we write the characteristic or auxiliary equations for $\tilde{y}$ and $\tilde{y}_1$ needed in Eq. (36):

$$\frac{d\tilde{y}(t)}{dt} = -\frac{3\tilde{y}(t)\tilde{y}_1(t)}{16\pi}\left(1+\frac{\alpha' k_N^2}{E_N}\right)^{-1} \times\left(\sigma\frac{\alpha' k_N^2}{E_N}+\nu\right). \tag{53}$$

and

$$\frac{d\tilde{y}_1(t)}{dt} = -\frac{\tilde{y}_1^2}{16\pi}\left(1+\frac{\alpha' k_N^2}{3E_N}\right)^{-1}\left(\sigma\frac{\alpha' k_N^2}{3E_N}+\nu\right) - \frac{\sigma\tilde{y}^2}{4\pi}, \tag{54}$$

where we have specialized to $D=2$.

Rather than solve these equations in general, we may now note that the presence of k_N^2 has served only to ensure that the last term in (54) was not missed. That is, had we chosen to normalize at $E=-E_N$, $k_N^2=0$ as we did in the triple-coupling problem[2] we would have encountered an infrared divergence in the second graph of Fig. 8 at $D=2$. However, at this stage we may with impunity set $k_N^2=0$, which results in

$$\frac{d\tilde{y}(t)}{dt} = -\frac{3\nu}{16\pi}\tilde{y}\tilde{y}_1, \tag{55}$$

$$\frac{d\tilde{y}_1(t)}{dt} = -\frac{\nu\tilde{y}_1^2}{16\pi} - \frac{\sigma\tilde{y}^2}{4\pi}. \tag{56}$$

To solve these let $\rho=\tilde{y}_1/\tilde{y}$; then

$$\frac{d\rho}{dt} = \frac{\nu\tilde{y}}{8\pi}\left[\rho^2-\frac{2\sigma}{\nu}\right], \tag{57}$$

and

$$\frac{d\tilde{y}}{dt} = -\frac{3\nu}{16\pi}\tilde{y}^2\rho, \tag{58}$$

or

$$\frac{d\rho^2}{d\tilde{y}} = -\frac{4}{3}\,\frac{[\rho^2-2\sigma/\nu]}{\tilde{y}}. \tag{59}$$

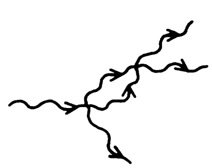

FIG. 7. The lowest-order correction to $\Gamma^{(1,3)}$.

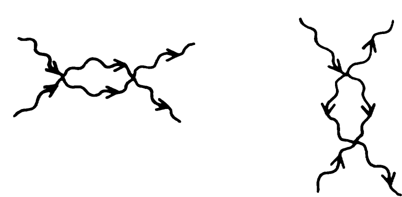

FIG. 8. The lowest-order correction to $\Gamma^{(2,2)}$.

We will seek solutions with a positive effective coupling constant $\bar{y}(t)$. The relation between y and ρ we see to be

$$\bar{y}(t)=y\left[\frac{\rho(0)^2-2\sigma/\nu}{\rho(t)^2-2\sigma/\nu}\right]^{3/4} . \tag{60}$$

If

$$\rho(0)^2=\frac{y_1^{\,2}}{y_1}=\frac{\lambda_1^{\,2}}{\lambda_2}\leq\frac{2\sigma}{\nu}, \tag{61}$$

then

$$\frac{d\rho}{dt}=-\frac{\nu y}{8\pi}\left[\frac{2\sigma}{\nu}-\rho(0)^2\right]^{3/4}\left[\frac{2\sigma}{\nu}-\rho(t)\right]^{1/4} \tag{62}$$

is always negative and $\rho(t)\to 2\sigma/\nu$ as $t\to\infty$. Then $\bar{y}(t)\to\infty$, as we see from (60). This is a case where dropping the higher-order terms in the auxiliary equations for $\bar{y}$ and $\bar{y}_1$ is hardly justified. Furthermore, any confidence one might have in a perturbation expansion in λ or y must be minimal.

If $\rho(0)>(2\sigma/\nu)^{1/2}$, then we see from

$$\frac{d\rho}{dt}=\frac{\nu y}{8\pi}\left[\rho(0)^2-\frac{2\sigma}{\nu}\right]^{3/4}\left[\rho(t)^2-\frac{2\sigma}{\nu}\right]^{1/4} \tag{63}$$

that $\rho(t)$ is ever increasing, and for large t, which is the regime of interest,

$$\rho(t)\underset{t\to\infty}{\sim}\left(\frac{y\nu t}{16\pi}\right)^2\left[\rho(0)^2-\frac{2\sigma}{\nu}\right]^{3/2}. \tag{64}$$

Clearly in this case $y(t)$ goes to zero for large t as t^{-3}. This means that the neglect of higher-order terms in β is allowed and, more interestingly, *the behavior of* Γ_R *for small energies and momenta is governed by zero coupling. This theory is infrared-free.* It is amusing to note that the condition of infrared freedom on the behavior of $\Gamma_R(\xi^\nu E_t,\, \xi^{\sigma/2}\vec{k}_t,\, \alpha',\, y,\, y_1,\, E_N)$,

$$(y_1^{\,2}/y^2)(\nu/2\sigma)>1, \tag{65}$$

depends on the path (that is, σ and ν) one takes in reaching the infrared point as well as the values of the renormalized couplings y_1 and y.

The behavior of $\bar{y}(t)$ and $\bar{y}_1(t)$ for large t is

$$\bar{y}(t)\underset{t\to\infty}{\sim}\frac{1}{y^2t^3}\left(\frac{16\pi}{\nu}\right)^3\frac{1}{[\rho(0)^2-2\sigma/\nu]^{3/2}}, \tag{66}$$

and

$$\bar{y}_1(t)\underset{t\to\infty}{\sim}\frac{16\pi}{\nu t} . \tag{67}$$

From the equation for $\bar{\alpha}'(t)$ and the perturbation expression for ζ we have

$$\bar{\alpha}'(t)\underset{t\to\infty}{\sim}\alpha' C_\alpha e^{(\nu-\sigma)t}(1+C/t^5+\cdots), \tag{68}$$

where

$$C=\frac{y}{135\pi}\left(\frac{16\pi}{y\nu}\right)^5\left[\rho(0)^2-\frac{2\sigma}{\nu}\right]^{-3}, \tag{69}$$

and

$$C_\alpha=\exp\left[-\frac{y}{54\pi}\left(\rho(0)^2-\frac{2\sigma}{\nu}\right)^{3/4}\times\int_{\rho(0)}^{\infty}\frac{dx}{(x^2-2\sigma/\nu)^{7/4}}\right]. \tag{70}$$

We need one more ingredient for the study of $\Gamma_R(\xi^\nu E_t,\, \xi^{\sigma/2}\vec{k}_t,\, \ldots)$. That is the integral

$$\exp\left[-\int_{-t}^{0}dt'\,\frac{n+m}{2}\,(\sigma\gamma_k+\nu\gamma_E)\right]. \tag{71}$$

Using our expressions for γ_k (zero when $k_N^{\,2}=0$) and γ_E we find that (71) behaves as

$$(C_\alpha)^{-3(n+m)/4}\left[1-\frac{3(n+m)}{4}\,\frac{C}{(-t)^5}+\cdots\right] \tag{72}$$

for large t.

IV. PROPERTIES OF THE RENORMALIZED TRAJECTORY

We have now discovered that by a propitious choice of renormalized coupling constants, we can determine the small-E_t, small-$\vec{k}_t$ behavior of the renormalized vertex functions in a perturbative manner since the effective couplings entering the right-hand side of our expression (36) for Γ_R go to zero as $\xi\to 0$. A function of particular interest is the inverse propagator, for its zeros determine the Regge trajectories.

We are instructed to take the renormalized propagator $\Gamma_R^{(1,1)}$ determined to some order in perturbation theory (lowest order will do in our case) and place it into Eq. (36). $\Gamma_R^{(1,1)}$ to $O(y^2)$ is

$$i\Gamma_R^{(1,1)}(E,\vec{k}^2,\alpha',y,y_1,E_N)=E-\alpha'\vec{k}^2+\frac{y^2}{288\pi^2}\times\left\{\left(\frac{\alpha'\vec{k}^2}{3}-E\right)\left[\ln\left(\frac{\alpha'\vec{k}^2}{3E_N}-\frac{E}{E_N}\right)-1\right]\right\} \tag{73}$$

in $D=2$ dimensions.

We now set $\nu=\sigma=1$, for we expect the zero of the inverse propagator to remain approximately linear in the interacting theory. This means that we can confirm the existence of a Pomeron pole in this theory only if $\lambda_1/\lambda=\rho(0)\geq\sqrt{2}$. The quartic-coupling theory is otherwise not infrared-free. The requirement that $\lambda_1/\lambda\geq\sqrt{2}$ is not as stringent as it first appears, since, when λ_1 and λ are small, the vacuum is unstable unless $\lambda_1/\lambda\geq 4/3$.[5]

The renormalized $\Gamma_R^{(1,1)}$ with $\nu=\sigma=1$ is

$$i\Gamma_R^{(1,1)}(\xi E, \xi \vec{k}^2, \alpha', y, y_1, E_N) = \xi (C_\alpha)^{-3/2}\left[1 - \frac{3}{2}\,\frac{C}{(-\ln\xi)^5}\right]$$
$$\times\left\{E - \alpha' C_\alpha \vec{k}^2\left(1 + \frac{C}{(-\ln\xi)^5}\right) + \frac{15}{2}\,\frac{C}{(-\ln\xi)^6}\left[\left(\frac{\alpha' C_\alpha \vec{k}^2}{3} - E\right)\left(\ln\left(\frac{\alpha' C_\alpha \vec{k}^2}{3E_N} - \frac{E}{E_N}\right) - 1\right)\right]\right\} \tag{74}$$

$$= (C_\alpha)^{-3/2}\,\xi\left\{E - C_\alpha \alpha' \vec{k}^2 + \frac{3}{2}\,\frac{C}{(-\ln\xi)^5}\left(\frac{\alpha' C_\alpha \vec{k}^2}{3} - E\right) + \frac{15}{2}\,\frac{C}{(-\ln\xi)^6}\left(\frac{\alpha' C_\alpha \vec{k}^2}{3} - E\right)\left[\ln\left(\frac{\alpha' C_\alpha \vec{k}^2}{3E_N} - \frac{E}{E_N}\right) - 1\right] + \cdots\right\}. \tag{75}$$

The left-hand side is a function of ξE and ξk^2 only. To cast the right-hand side in this form we write

$$\frac{1}{(-\ln\xi)^5} + \frac{5}{(-\ln\xi)^6}\left[\ln\left(\frac{C_\alpha \alpha' \vec{k}^2}{3E_N} - \frac{E}{E_N}\right) - 1\right] = \frac{1}{[-\ln(C_\alpha \alpha' \xi \vec{k}^2/3eE_N - \xi E/eE_N)]^5} + O\left(\left(\frac{1}{-\ln\xi}\right)^7\right). \tag{76}$$

This enables us to conclude that

$$i\Gamma_R^{(1,1)}(E, \vec{k}^2, \alpha', y, y_1, E_N) = (C_\alpha)^{-3/2}\left[E - C_\alpha \alpha' \vec{k}^2 + \tfrac{3}{2}\,\frac{C(\frac{1}{3} C_\alpha \alpha' \vec{k}^2 - E)}{[-\ln(C_\alpha \alpha' \vec{k}^2/3eE_N - E/eE_N)]^5} + \cdots\right] \tag{77}$$

as $E, \vec{k}^2$ go to zero. This function has a zero very close to

$$E = \alpha' C_\alpha \vec{k}^2 + \frac{C C_\alpha \alpha' \vec{k}^2}{[-\ln(-2C_\alpha \alpha' \vec{k}^2/3eE_N)]^5} \tag{78}$$

which yields a Regge trajectory for $y \geq 0$,

$$\alpha(t) = 1 + C_\alpha \alpha' t + \frac{C_\alpha C \alpha' t}{[-\ln(2C_\alpha \alpha' t/3eE_N)]^5}, \tag{79}$$

which is a very moderate modification of the original linear trajectory. For $t \leq 0$ the trajectory becomes complex and there are two trajectories at complex conjugate positions.

V. CONCLUSIONS

We have investigated the behavior of the renormalized Green's functions for Pomerons when the interaction among the Pomerons is only of the ψ^4 type given explicitly in Eq. (9). Under the condition that the ratio of two renormalized couplings be larger than a given value, we found that the effective couplings that determine the infrared behavior of the field theory approach zero. That is, the infrared behavior of a Pomeron field theory with quartic couplings can be determined to high accuracy by the use of perturbation theory around the coupling constants equal to zero.

Within this framework we examined in detail the inverse propagator and found that the interactions give rise to a very mild modification of the noninteracting linear trajectory. If one couples in particles by simply tacking them on the ends of Reggeons (as in Fig. 9), then our $\Gamma_R^{(1,1)}$ of Eq. (77) gives rise to

$$\sigma_T(s) \sim \gamma_1 [1 + \tfrac{3}{2} C/(\ln\ln s)^5 + \cdots], \tag{80}$$

where γ_1 is positive and factorizes. Since the present theory is infrared free, corrections to this result from multiple Pomerons being emitted from the particles are essentially like that of the noninteracting theory; that is, each correction is smaller by powers of $\ln s$ than the term exhibited.

The ψ^4 theory is of a medium amount of interest in its own right but fails to provide a model of significant physical consequence because, by its very definition, it lacks a triple-Pomeron coupling. (The problem of combining both a ψ^3 and ψ^4 is intriguing and is relatively easy to formulate. Its solution has somewhat eluded us because it seems natural to investigate the ψ^3 theory in an expansion about $D = 4$ space dimensions while we have just demonstrated in this paper how the ψ^4 theories are simple in the physical dimensions $D = 2$.) Nevertheless, the quartic-coupling problem has proved instructive in itself by demonstrating

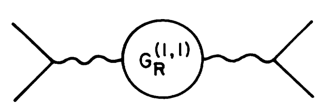

FIG. 9. The coupling of particles (heavy lines) to Pomerons (wavy lines) which interact in all possible ways. This set of graphs give $\sigma_T(s) = \gamma_1 + \gamma_2/(\ln\ln s)^5$. Multiple Pomeron emission by particles is smaller asymptotically by powers of $\ln s$.

how one may substantially alter the infrared behavior of the Green's functions or choosing an inappropriate path by which to approach $E_i \to 0$, $\vec{k}_i \to 0$ or having an unfortunate ratio of renormalized coupling constants. Such trickery did not appear in the ψ^3 theory which possesses a single coupling and provides a richness of solutions not explorable there.

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Feynman rules for gauge theories at finite temperature*

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Feynman's functional formulation of statistical mechanics is used to study general-relativistic quantum field theories at finite temperature. The techniques are then applied to gauge theories. The partition function $\mathrm{Tr}\, e^{-\beta H}$ is discovered to be a gauge-dependent quantity which is meaningless in most gauges. Instead, we define a physically meaningful partition function which is gauge-invariant and only equal to $\mathrm{Tr}\, e^{-\beta H}$ in certain "physical" gauges. Feynman rules for this partition function and for finite-temperature Green's functions are derived for a general gauge.

I. INTRODUCTION

Recently, several authors[1-3] have considered what happens when a system of elementary particles described by a quantum field theory is heated. They have found that symmetries which are spontaneously broken at zero temperature (such as those of the weak interactions) may be restored at sufficiently high temperatures, and have calculated[2,3] the critical temperature at which such a restoration takes place. To do this kind of calculation, one needs to know the Feynman rules for a field theory at finite temperature. For a nongauge theory, these rules can be derived using well-known methods.[4] However, for a gauge theory, a more powerful technique is needed to cope with several new problems that arise. Chief among these is the troublesome fact that the partition function $\mathrm{Tr}\, e^{-\beta H}$ is a gauge-dependent quantity, as we show by an explicit example. The gauge dependence is caused by the appearance, in some gauges, of specious degrees of freedom in H which do not correspond to physical particles. The trace over all states of H is not physically meaningful in these gauges—the specious particles cannot come to equilibrium with a physical heat bath.

It would seem, then, that gauge invariance is completely lost at finite temperature. This is not the case. The functional methods set forth in this paper allow one to calculate the physically meaningful partition function (i.e., $\mathrm{Tr}\, e^{-\beta H}$ in a gauge without specious degrees of freedom—such as the unitarity gauge) using Feynman rules defined in any of the usual gauges (for example, the R_ξ gauges in a spontaneously broken non-Abelian theory). Thus gauge invariance of physical quantities is not lost at finite temperature; we must merely remember that $\mathrm{Tr}\, e^{-\beta H}$ is not in general