neglecting all internal reflections due to discontinuity in the rth derivative of the refraction function W. Hence, correct to the rth order, the wave function can be approximated sufficiently by the positive-frequency solution $\exp(-i\int K_r dt)/K_r^{1/2}$. Physically, this means that the background varies slowly and smoothly enough in the rth derivative that no particle creation occurs to that order (absence of wave reflection). The particle operators are well defined to this limit. This is the basic idea behind the adiabatic quantization method in Ref. 21.

[24]E. Kasner, Am. J. Math. 43, 217 (1921).
[25]C. W. Misner, Phys. Rev. Lett. 22, 1071 (1969).
[26]V. A. Belinskii, E. M. Lifshitz, and I. M. Khalatnikov, Usp. Fiz. Nauk. 102, 463 (1970) [Sov. Phys.—Usp. 13, 745 (1971)].
[27]This method is used to describe general type-IX universes in R. A. Matzner, L. C. Shepley, and J. B. Warran, Ann. Phys. (N.Y.) 57, 401 (1970); M. P. Ryan, *ibid.* 65, 506 (1971); 68, 541 (1971).
[28]J. A. Wheeler, in *Battelle Rencontres, 1967,* edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968). C. W. Misner, in *Magic Without Magic: John Archibald Wheeler, a Collection of Essays in Honor of His 60th Birthday,* edited by John R. Klauder (Freeman, San Francisco, 1972).

PHYSICAL REVIEW D VOLUME 9, NUMBER 12 15 JUNE 1974

Corresponding-states approach to nuclear and neutron-star matter

R. G. Palmer* and P. W. Anderson†
Cavendish Laboratory, Cambridge, England
(Received 11 February 1974)

The properties of nuclear matter and dense neutron-star matter are studied by an approach which largely avoids the microscopic assumptions of nuclear-matter theory. The method is empirical, employing an extended form of the law of corresponding states to deduce the properties of nuclear systems from those of laboratory substances such as helium. It is possible to predict the solidification pressure and density, the compressibility, and the critical temperature of nuclear and neutron-star matter. As previously reported, a comparatively low solidification pressure is found for neutron-star matter, implying a solid core for most neutron stars.

I. INTRODUCTION

There are several reasons for attempting an empirical approach to nuclear matter. The simplicity and relative transparency of the method to be described are worthwhile in themselves, and may allow application to phenomena barely accessible to conventional microscopic techniques. Detailed models of the nucleon-nucleon interaction may be avoided as may the assumptions and results of standard (Brueckner) nuclear matter theory. Indeed, the present approach may be regarded as an independent—albeit weak—test of such theories. By working from the experimental properties of real materials, we let nature compute the bulk of the many-body effects, and only have to concern ourselves with differences between substances. We pay for these advantages of an empirical approach with a large uncertainty in our results, and are only able to make order-of-magnitude estimates. The uncertainties come partly from the approximations used, but also significantly from lack of sufficient source data; there are few real quantum systems.

Our procedure is based on an extension of the quantum law of corresponding states first proposed by de Boer[1] and successfully used by him[2] to predict the properties of He^3 before any was available for study. de Boer's model is not directly applicable to nuclear matter because of the considerable difference between nuclear forces and the Van der Waals forces typical of laboratory quantum systems such as helium. However, we are able to generalize the corresponding-states law to apply to a larger class of interactions by means of an equivalent density transformation. We first apply the extended model to symmetric nuclear matter ($\frac{1}{2}$ neutrons, $\frac{1}{2}$ protons), and then to neutron-star matter, the latter causing more difficulty as less experimental information is available.

Our chief goal is to predict the crystallization pressure and density of neutron-star matter, which bear considerably on the structure and dynamics of neutron stars. A preliminary account of this work[3] has already appeared, based on a less detailed model than described here. Several other authors[4-10] have also examined the crystallization question, using more microscopic techniques. Clark and Chao[5] have followed an approach close to our own—based on corresponding states—but

need to assume particular nuclear potentials and a variational nuclear-matter calculation, both of which we specifically avoid.

In Sec. II we describe de Boer's law of corresponding states. In Secs. III and IV our extensions, the *core shift,* and the *energy shift,* are developed and exemplified by applications to the Bose hard-sphere fluid and to nitrogen. Symmetric nuclear matter is discussed in Sec. V, followed by neutron-star matter in Sec. VI. The accuracy and implications of our results are considered in Sec. VII.

II. CORRESPONDING STATES

The *classical* law of corresponding states is readily understood on dimensional grounds. Consider a group of substances described by additive two-body potentials of the form $v(r) = \epsilon f(r/\sigma)$, where f is a function common to all the substances, which only differ in the scale parameters ϵ and σ. We may express the pressure P, volume per particle V, temperature T, and free-energy per particle F in terms of dimensionless *reduced* quantities:

$$P^* = P\sigma^3/\epsilon \ ,$$

$$V^* = V/\sigma^3 \ ,$$

$$T^* = kT/\epsilon \ ,$$

$$F^* = F/\epsilon \ .$$

Dimensional analysis then implies that

$$F^* = F^*(V^*, T^*)$$

and

$$P^* = P^*(V^*, T^*) \ ,$$

where $F^*(V^*, T^*)$ and $P^*(V^*, T^*)$ are universal functions of their arguments for all substances concerned. The molecular mass m does not enter the equation of state.

On admitting quantum mechanics we allow the construction of a further dimensionless parameter

$$\lambda^2 = \frac{h^2}{m\epsilon\sigma^2} \ . \tag{1}$$

which measures the importance of quantum effects. The equation of state must be modified to

$$F^* = F^*(V^*, T^*, \lambda, \text{ statistics})$$

or (2)

$$P^* = P^*(V^*, T^*, \lambda, \text{ statistics}) \ ,$$

allowing at the same time for differences between Bose and Fermi systems. The result (2) may also be derived from the Schrödinger equation; the energy eigenvalues scale with ϵ for fixed V^*. Except at isolated points, corresponding to phase transitions, we expect F^* and P^* to be smooth functions of λ, and this forms the basis of our predictions.

In the study of nuclear and neutron-star matter we are chiefly interested in the ground-state properties at zero temperature (neutron stars are cold on a nuclear energy scale). Let us define F_0^* and V_0^* to be the equilibrium free energy and volume per particle at zero temperature and pressure. Then

$$F_0^* = F_0^* (\lambda, \text{ statistics})$$

and

$$V_0^* = V_0^* (\lambda, \text{ statistics}) \ ,$$

where F_0^* and V_0^* may be multivalued functions in the presence of several alternative phases. Figures 1 and 2 show experimental values of F_0^* and V_0^* for some substances described approximately by a Lennard-Jones 6-12 potential:

$$v(r) = 4\epsilon[(\sigma/r)^{12} - (\sigma/r)^6] \ .$$

The parameters ϵ and σ are based on second-virial-coefficient determinations,[11] in lieu of any reliable high-density data.[12] F_0^* and V_0^* are the values appropriate to the fluid phase (at $T = P = 0$) for He^3 and He^4, and to the solid phase for the remaining substances. We desire fluid data throughout, but such are not available (being defined by a metastable state for all but helium). The solid-fluid transition at some λ between H_2

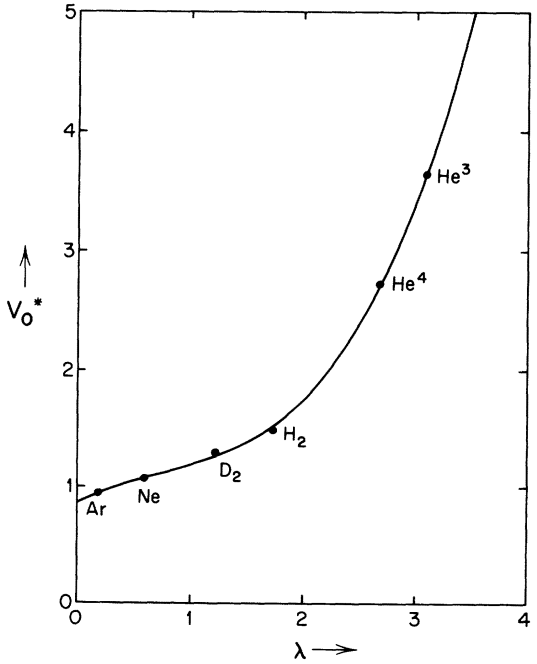

FIG. 1. The reduced equilibrium volume per particle of several substances as a function of the quantum parameter λ.

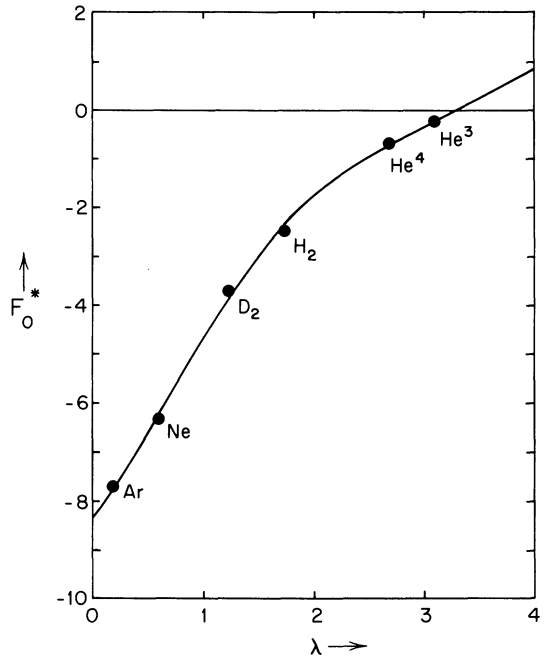

FIG. 2. The reduced equilibrium energy per particle of several substances as a function of the quantum parameter λ.

and He^4 implies a discontinuity in $V_0^*(\lambda)$ of only 5–10% and a small discontinuity in $\partial F^*/\partial\lambda$. F_0^* itself should be continuous. Reasonably smooth curves can thus be drawn through all the points, the solid-phase data at least suggesting the trends of the fluid data. We subsequently only need values of λ beyond the He^4 point, for which we use simple polynomial fits to the $F_0^*(\lambda)$ and $V_0^*(\lambda)$ data. We have used a simple straight line through the He^4 and He^3 points for $F_0^*(\lambda)$. We make no allowance for statistics dependence since no appreciable effect is seen in the data. However, the statistics effect should increase with λ and this, and the lack of further data, causes the uncertainty in the *extrapolation curves* $F_0^*(\lambda)$ and $V_0^*(\lambda)$ to grow rapidly with increasing λ.

It is easy to see how predictions might be made with the corresponding-states law. Given an ϵ and a σ we may predict F_0 and V_0 and any other properties that vary smoothly with λ, provided we have sufficiently accurate extrapolation curves for each property. Alternatively, given two properties such as F_0 and V_0 we may solve numerically the implicit equations

$$F_0/\epsilon = F_0^*(\lambda)\ , \tag{3a}$$

$$V_0/\sigma^3 = V^*(\lambda)\ , \tag{3b}$$

$$\lambda^2 = h^2/m\epsilon\sigma^2 \tag{3c}$$

for ϵ and σ, and use these parameters to predict further properties of the substance. The accuracy of such procedures depends on how well the unknown substance is characterized by the common potential shape, which is always a 6-12 potential in the present work.

The nucleon-nucleon interaction is not well described by a single central potential of the 6-12 shape, but we ignore this for a moment and consider finding ϵ and σ. Directly guessing ϵ and σ from the various potentials is hardly feasible, but suggests a large λ of perhaps 5 to 20. The second procedure described above—solving Eqs. (3)—may be attempted since we know that $F_0 \approx -16$ MeV/nucleon and $V_0 \approx 5.9$ fm³/nucleon for symmetric nuclear matter, from the radii and binding energies of finite nuclei. We obtain $\epsilon = 120$ MeV, $\sigma = 1.15$ fm, $\lambda = 3.2$; a 6-12 material with these parameters (and a molecular weight of 1) would have the same equilibrium energy and density as nuclear matter. These results imply an *effective* 6-12 potential that appears rather unrealistic, being deeper and narrower than most nuclear potential components. No single central component of the Reid soft-core potential[13] is as deep as 120 MeV, and an average potential would be expected to have a depth of about 50 MeV or less. We therefore have little confidence in using the above values of ϵ and σ to make further predictions, and turn instead to an extension of the law of corresponding states.

III. THE CORE SHIFT

One of the main failures of the above approach is the use of an effective 6-12 potential to simulate much broader nuclear potentials. In Fig. 3 the Reid soft-core $^1D_2$ potential is shown with a 6-12 potential of the same depth (ϵ) and zero-crossing point (σ). The $^1D_2$ component is fairly typical, and probably of considerable importance in dense neutron matter,[5] but is not at all well represented by a 6-12 potential. There are no ordinary substances with as wide a potential bowl as nucleons, and so it is not helpful to consider other shapes of reference potential. Nor do ordinary substances show sufficient variability in their potential width for direct extrapolation on a third width parameter to be feasible.

We find that a core-shifted 6-12 potential (*C*-6-12)

$$v_{6-12}(r) = 4\epsilon\left[\left(\frac{\sigma}{r+\Delta\sigma}\right)^{12} - \left(\frac{\sigma}{r+\Delta\sigma}\right)^{6}\right] \tag{4}$$

is well suited to simulating a nuclear potential, at least in the region of the attractive bowl. Figure 3 shows such a potential fitted to the $^1D_2$ nuclear component. The problem thus becomes how to

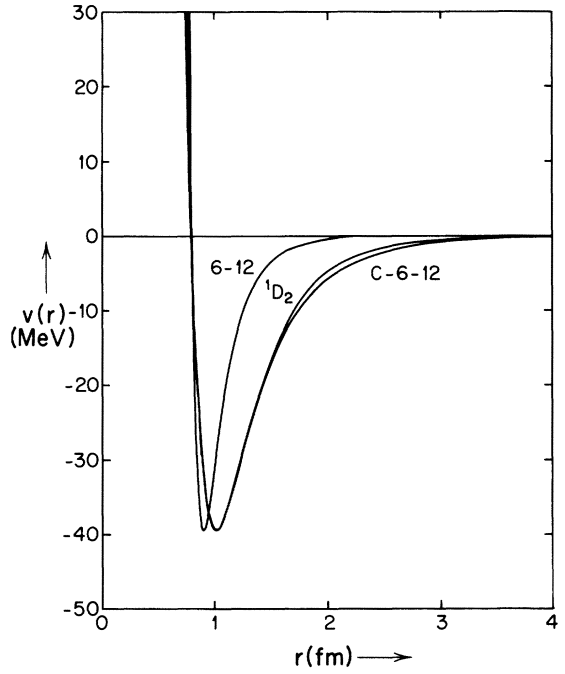

FIG. 3. (a) Reid soft-core $^1D_2$ potential; (b) 6-12 potential with same depth and crossing point as (a); (c) C-6-12 potential fitted to (a).

relate the properties of a system characterized by a potential $v(r+\Delta\sigma)$ to those of one characterized by $v(r)$. We make the following ansatz: *The free energy of a system described by a pair potential $v(r+\Delta\sigma)$ with mean interparticle distance d is similar to that of a system described by $v(r)$ at $d'=d+\Delta\sigma$.* We will make this idea more precise shortly, and will improve it in the next section.

The approximation may best be illustrated by a simple fcc cell model. Figure 4 shows the spherically averaged total potential seen by a particle due to its 12 nearest neighbors, which are presumed fixed. The two cases shown correspond to a pair potential $v(r+\Delta\sigma)$ with nearest-neighbor distance d, and a potential $v(r)$ with nearest-neighbor distance $d+\Delta\sigma$, where $v(r)$ is a 6-12 potential, and $d=0.9\sigma$, $\Delta\sigma=0.5\sigma$.[14] In this simple case the approximation is clearly excellent. In general it is evident that the two systems will be closely similar if

(i) the systems are dense enough for little translational motion to be permitted, the particles being largely confined by their neighbors (the kinetic energy is then directly related to the localization and will be nearly identical in the two cases) and

(ii) the most important contribution to the potential energy is that of the nearest neighbors; the error in our equivalence becomes larger for particle pairs further from the mean distance d (we will allow for this effect in Sec. IV).

In a dense fluid we take $d=\beta V^{1/3}$ for the mean interparticle distance d, where β is a parameter of order unity. We ignore any density dependence of β. To illustrate our core-shift equivalence and to determine β we apply our model to the Bose hard-sphere fluid, for which Monte Carlo data are available.[15,16] Dimensional analysis predicts that the ground-state energy of a Bose hard-sphere fluid may be written as

$$F=\frac{\hbar^2}{m\sigma^2}f(V/\sigma^3)\,,$$

where σ is the hard-sphere diameter, m is the particle mass, V is the volume per particle, and f is an unknown function of $v=V/\sigma^3$. Our core-shift approximation may be used to relate F at two different densities and diameters, and leads to the prediction

$$f(v)=A/(v^{1/3}-1/\beta)^2\,, \tag{5}$$

where A is a constant. In Fig. 5 $f^{-1/2}$ is plotted against $v^{1/3}$ for the Monte Carlo data, and we see good agreement with Eq. (5) except at low density, where we expect our approximation to break down due to the onset of free-particle-like translational motion. The intercept at $f^{-1/2}=0$ gives us $\beta=1.00$

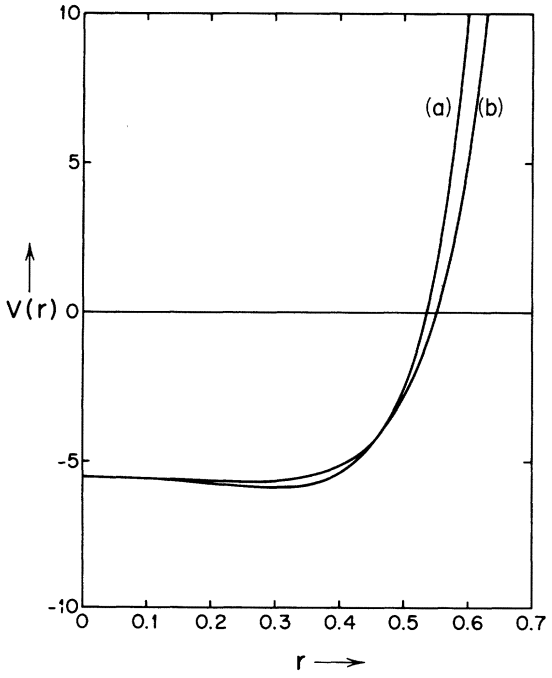

FIG. 4. The total potential, averaged over all directions, seen by a particle due to its nearest neighbors in an fcc lattice. r is the radial distance from the lattice site, in units of σ, and $V(r)$ is the total potential in units of ϵ. (a) is for a 6-12 pair potential at nearest-neighbor distance 1.4σ, and (b) is for a C-6-12 potential with $\Delta=0.5\sigma$ at nearest-neighbor distance 0.9σ.

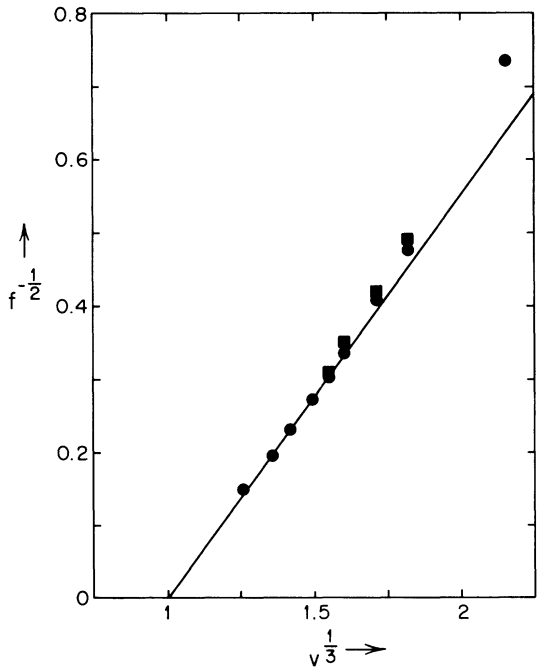

FIG. 5. The equation of state of a Bose hard-sphere fluid at absolute zero. f is the energy in units of $h^2/m\sigma^2$ and v is the volume per particle in units of σ^3. ● are data of Ref. 15. ■ are data of Ref. 16. The straight line is fitted to higher-density points.

± 0.02. We note that this is less than the value near the close-packed volume v_{cp}, where we must have $1/\beta = v_{cp}^{1/3} = 2^{-1/6}$; at such high densities the local packing geometry becomes important.

The core-shift approximation makes possible the prediction of the properties of a system described by a given C-6-12 potential. Conversely, we may determine the parameters ϵ, σ, and $\Delta(=\Delta\sigma/\sigma!)$ of such a potential from knowledge of *three* properties of the system. In the case of nuclear matter we only know *two* macroscopic properties (F_0 and V_0) with sufficient accuracy to make this feasible. The compressibility is not at all well known (see Sec. V) and the surface tension—available from the liquid drop model—is not a good candidate for corresponding states behavior, being very sensitive to potential details and directionality. We therefore fix Δ initially to give C-6-12 potentials with a *shape* similar to typical nuclear potential. This is the main point at which microscopic knowledge of nucleons enters our scheme. Almost all the Reid soft-core potentials possessing a minimum have a shape corresponding to $\Delta = 0.50 \pm 0.05$, as do average potentials which might be appropriate in nuclear matter (various mixtures of attractive and repulsive parts have been tried). The example of Fig. 3 corresponds to $\Delta = 0.52$.

Our model now allows a computation of ϵ and σ for nuclear matter from F_0 and V_0, but we defer this to include first an improvement which forms the subject of the next section.

IV. THE ENERGY SHIFT

One of the largest sources of error in the model described above is the inaccuracy of the core-shift approximation due to the potential energy of further neighbors. The nearest neighbors are treated correctly, but more distant neighbors can give rise to a distinct energy difference between the "equivalent" systems. Figure 6 shows the result of including (essentially) all neighbors in the fcc cell model of Sec. III and Fig. 4. To a good approximation the effect may be considered as a simple *potential* energy shift between the two systems; we ignore any change of shape of the total potential between the systems, which would alter the kinetic energy. Our central thesis then becomes

$$F(V) = \epsilon F^*(V^*) + E(V) , \tag{6}$$

where $F^*(V^*)$ is the reduced equation of state of a 6-12 system, $E(V)$ is the energy shift, and V^* is related to V by

$$d = \beta V^{1/3} ,$$
$$d' = d + \Delta\sigma ,$$
$$V^* = V'/\sigma^3 = (d'/\beta\sigma)^3 \tag{7}$$

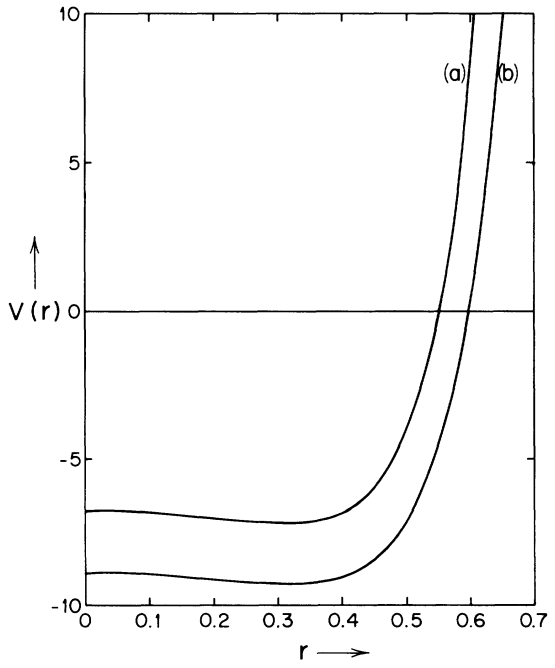

FIG. 6. Total potentials in an fcc lattice, as in Fig. 4, but including all neighbors up to the 10th shell. Including further shells makes virtually no difference.

or

$$V^{*1/3} = V^{1/3}/\sigma + \Delta/\beta \ .$$

Here, and henceforth, we use starred quantities (*) exclusively for properties of 6-12 systems.

The energy shift is given by

$$E(V) = \tfrac{1}{2}[E_1(V) - \epsilon E_2^*(V^*)] \ ,$$

where $E_1(V)$ is the potential energy due to all but nearest neighbors in the nuclear system and $E_2^*(V^*)$ is the corresponding reduced energy in a 6-12 system. The $\frac{1}{2}$ avoids double counting. For fluid phases we compute E_1 and E_2^* from integrals of the form

$$\frac{1}{V}\int_R^\infty v(r)\, d^3r \ ,$$

which is equivalent to setting the radial distribution function $g(r)$ to unity beyond nearest neighbors; this is not a bad approximation at the densities concerned. The radius R defines the regime of further neighbors. For consistency we must have

$$\frac{1}{V}\int_0^R d^3r = z + 1 \ ,$$

where z is the mean number of nearest neighbors, in order that

$$\frac{1}{V}\int_0^\infty [g(r) - 1]\, d^3r = -1$$

as required at zero temperature. This leads to

$$R = \left(\frac{3(z+1)V}{4\pi}\right)^{1/3} = \alpha V^{1/3} \ ,$$

giving $\alpha = 1.34$ to 1.42 for $z = 9\frac{1}{2}$ to 11 as found in helium.[17,18]

It is not necessary to use the tail of the effective C-6-12 potential in calculating $E_1(V)$; instead we can use a more realistic nuclear potential tail. Our effective potential will then be discontinuous, having a different form for the core and tail regions, but this is of no account since the two forms enter the model in quite different ways. For the nuclear potential tail we use the one-pion-exchange potential (OPEP)[19]:

$$V_{\rm OPEP}(r) \simeq 3.5\ \vec{\tau}_1 \cdot \vec{\tau}_2 \times [\vec{\sigma}_1 \cdot \vec{\sigma}_2 + S_{12}(1 + 3/x + 3/x^2)]\, e^{-x}/x \ ,$$

where $x = 0.7r$ for r in fm and v in MeV. For $r \gtrsim 3$ fm all nuclear potentials reduce to the OPEP form; the radius R in our model is always greater than or equal to $\alpha V_0^{1/3} \simeq 2\frac{1}{2}$ fm, so the OPEP should be a reasonable approximation. We must, however, average in some way over spin and isospin, taking account of the strong correlations present in the nucleon fluid. The allowed s-wave pair states (the dominant long-range components) have $(\vec{\tau}_1 \cdot \vec{\tau}_2)\ (\vec{\sigma}_1 \cdot \vec{\sigma}_2) = -3$ and we take this for our average, ignoring the tensor term S_{12}. In fact the tensor term will add further attraction, but the admixture of higher momentum states will counter this. We thus use

$$v_{\rm tail}(r) = -10.5\, e^{-0.7r}/0.7r$$

as our average, commenting that our results are anyway insensitive to this assumption since the $E_2^*(V^*)$ term is dominant in the energy shift.

The inclusion of the energy shift complicates the computation of ϵ and σ from F_0 and V_0. Without the energy shift F_0 and V_0 corresponded to a minimum in the free energy $F^*(V^*)$ of the equivalent 6-12 system, but now we have for the point F_0, V_0:

$$\frac{\partial F}{\partial V} = 0 = \epsilon \frac{\partial F^*}{\partial V^*}\frac{\partial V^*}{\partial V} + \frac{\partial E}{\partial V} \ .$$

We cannot therefore immediately relate F_0, V_0 to the extrapolation curves $F_0^*(\lambda), V_0^*(\lambda)$, which correspond to $\partial F^*/\partial V^* = 0$. We need to know more of the 6-12 equation of state $F^*(V^*)$ than the location of the zero-pressure equilibrium point. This knowledge can only be based on the experimental data for He^3 and He^4, which we must attempt to extrapolate with respect to λ. We define

$$X = [\,V^* - V_0^*(\lambda)]/V_0^*(\lambda)$$

and

$$Y = [F^* - F_0^*(\lambda)]^{-1} \ ,$$

and estimate $Y(X, \lambda)$ by linear extrapolation of Y with respect to X from the values of λ (He^4 and He^3) at which $Y(X, \lambda)$ is known.[20] This is a fairly arbitrary procedure, but does not lead to great errors if λ is not too far from the helium values. In any case the energy shift is quite a small correction to our model, and so V^* is close to V_0^*.

Our model is now complete. To find ϵ and σ from experimental values of F_0 and V_0 (and chosen Δ) we adjust ϵ and σ until Eq. (6) predicts an $F(V)$ which has a minimum at F_0, V_0, using the prescriptions given above for $F^*(V^*)$ and $E(V)$.

We have applied a similar model to the prediction of F_0 and V_0 for solid nitrogen at absolute zero, starting from a C-6-12 potential. Such a potential (with negative Δ) has been called the *Kihara potential*.[21] We find ϵ, σ, and Δ by fitting the theoretical and experimental second virial coefficients, obtaining $\epsilon = 130\ ^\circ$K, $\sigma = 2.99$ Å, $\Delta = -0.19$. Our model has to be modified somewhat: For a solid phase we use simple fcc lattice sums for E_1 and E_2^*; for small λ our $Y(X, \lambda)$ extrapolation is unsuitable and we use instead a quadratic compressibility term, obtaining the compressibility from corresponding-states extrap-

olation[22]; and we use the high density value $\beta = 2^{1/6}$. We then obtain $F_0 = -1730$ cal/mole, $V_0 = 26.3$ cc/mole, which are considerably closer to the experimental values ($F_0 = -1660 \pm 60$ cal/mole, $V_0 = -26.2 \pm 1.5$ cc/mole) than are those ($F_0 = -1460$ cal/mole, $V_0 = 29.6$ cc/mole) predicted by a direct corresponding-states application with a pure 6-12 potential ($\epsilon = 96$ °K, $\sigma = 3.72$ Å). The method can be applied similarly to other systems not well fitted by a simple 6-12 system, although there are few cases in which Δ is much different from zero.

V. SYMMETRIC NUCLEAR MATTER

The approach of the previous sections allows an immediate computation of ϵ and σ for symmetric nuclear matter ($\frac{1}{2}$ neutrons, $\frac{1}{2}$ protons), using $F_0 = -16$ MeV, $V_0 = 5.9$ fm^3. With $\Delta = 0.50 \pm 0.05$, $\beta = 1.00 \pm 0.02$, $\alpha = 1.38 \pm 0.04$, we obtain $\epsilon = 56 \pm 10$ MeV, $\sigma - \Delta\sigma = 0.88 \pm 0.05$ fm, $\lambda = 3.07 \pm 0.05$. We have included in the bounds a generous allowance for the uncertainty in the extrapolation curves $F_0^*(\lambda)$ and $V_0^*(\lambda)$, obtained by artificially perturbing these curves. Errors in the core shift and energy shift do not appreciably increase the total uncertainty; in fact most of the uncertainty comes from that in Δ. If nuclear matter *were* fully characterized by a single effective potential of the C-6-12 shape, then the results above for ϵ, σ, and λ would follow. Our results seem considerably more plausible than the large ϵ and σ found in Sec. II from the unmodified corresponding-states law (we should note that it is $\sigma - \Delta\sigma$, not σ, that represents the zero-crossing point of the potential). Indeed the depth and core size of the effective potential are about where we would intuitively put them, and we have chosen the shape (Δ) independently.

We now consider what predictions may be made with our results. In principle we can compute the free energy $F(V)$ of nuclear matter for any V, but in practice this is likely to be quite inaccurate far from V_0, relying as it does on the crude $Y(X, \lambda)$ extrapolation. We content ourselves with estimating the compressibility parameter

$$K = r_0^2 \left. \frac{d^2F}{dr_0^2} \right|_{V_0}, \tag{8}$$

where $r_0 \propto V^{1/3}$, which gives the curvature of $F(V)$ at equilibrium point. Our best estimate is $K = 320$ MeV, but the uncertainty is large; being pessimistic about the various inaccuracies and approximations we find $200 < K < 600$ MeV. The result is quite reasonable, agreeing better with recent empirical estimates (around 300 MeV) from finite nuclei[23] than with nuclear matter theory (typically 150–200 MeV) or Migdal's[24] Fermi liquid approach ($K \sim 700$ MeV).

Our original goal was to predict the solidification pressure P_s and density ρ_s of nuclear matter and, particularly, neutron-star matter. These are difficult to treat microscopically because they depend on small energy differences between solid and fluid phases, which must usually be treated by different methods. We may construct extrapolation curves $P_s^*(\lambda)$ and $\rho_s^*(\lambda)$ for the reduced solidification pressure and density of 6-12 systems, using not only the He^4 and He^3 points but also the negative values of P_s^* obtained by extrapolating the melting lines of H_2, D_2, and Ne back to absolute zero. However it is still not straightforward to obtain P_s for nuclear matter. It is not even a simple matter of differentiating Eq. (6) with respect to V, allowing for a factor $\sigma^3 \partial V^*/\partial V \neq 1$, because P_s depends on both solid and liquid properties. We must consider the conditions for equilibrium between these phases, assuming that our model applies separately to each. We first make the following approximations:

$$\text{(i)} \quad \sigma^3 \frac{V_{\text{liq}}^*}{V_{\text{liq}}} = \sigma^3 \frac{V_{\text{sol}}^*}{V_{\text{sol}}} = Q^3 ,$$

$$\text{(ii)} \quad \left(\frac{\partial E}{\partial V}\right)_{\text{liq}} = \left(\frac{\partial E}{\partial V}\right)_{\text{sol}} = \frac{E_{\text{liq}} - E_{\text{sol}}}{V_{\text{liq}} - V_{\text{sol}}} ,$$

where V_{liq} and V_{sol} represent the volumes per particle of the liquid and solid phases when in equilibrium, and the remaining quantities are evaluated at these points [V^* being related to V by Eq. (7)]. We define Q by (i). The first approximation is supported by the data for helium,[20] the two quantities differing by less than 5% for $\Delta = 0.5$. The second approximation will also be good if the liquid and solid densities are not greatly different. In any case $\partial E/\partial V$ only contributes a small fraction (~10%) of the total pressure.

Approximations (i) and (ii) allows us to write the conditions for equilibrium (equal pressure and Gibbs energies) as

$$\begin{aligned} \frac{-\sigma^3}{\epsilon Q^3}\left(P_s + \frac{\partial E}{\partial V}\right) &= \frac{1}{Q}\left(\frac{\partial F^*}{\partial V^*}\right)_{\text{liq}} \\ &= \frac{1}{Q}\left(\frac{\partial F^*}{\partial V^*}\right)_{\text{sol}} \\ &= \frac{F_{\text{liq}}^* - F_{\text{sol}}^*}{V_{\text{liq}}^* - V_{\text{sol}}^*} , \end{aligned} \tag{9}$$

which may be compared to the equations for the 6-12 solidification pressure

$$-P_s^* = \left(\frac{\partial F^*}{\partial V^*}\right)_{\text{liq}} = \left(\frac{\partial F^*}{\partial V^*}\right)_{\text{sol}} = \frac{F_{\text{liq}}^* - F_{\text{sol}}^*}{V_{\text{liq}}^* - V_{\text{sol}}^*} \quad (10)$$

The $1/Q$ factors in the second and third terms of Eq. (9) imply that the correct construction in the equivalent 6-12 system for the nuclear solidification point is *not* a double tangent [as for Eq. (10)], but a chord which meets the $F_{\text{liq}}^*(V^*)$ and $F_{\text{sol}}^*(V^*)$ curves at points where the slope is Q times the chord's slope (see Fig. 7). If the solid and liquid curves are of similar shape in this region this chord will be nearly parallel to the double tangent, and anyhow the slope of the chord will be largely dictated by the relative displacement of the solid and liquid curves. We therefore take

$$\frac{\sigma^3}{\epsilon Q^3}\left(P_s + \frac{\partial E}{\partial V}\right) = P_s^*(\lambda) \ . \quad (11)$$

The error in this approximation may be estimated by solving Eq. (9) exactly with helium data (and $\Delta = 0.5$). We find less than 1% difference between the left-hand sides of Eqs. (9) and (10).

To evaluate P_s from Eq. (11) we need to know the solidification density $\rho_s = 1/V_s$ (in the fluid phase, say) since both Q and $\partial E/\partial V$ are density dependent. We find this self-consistently from our equation of state for the fluid phase [Eq. (6)] by adding to Eq. (11) the condition

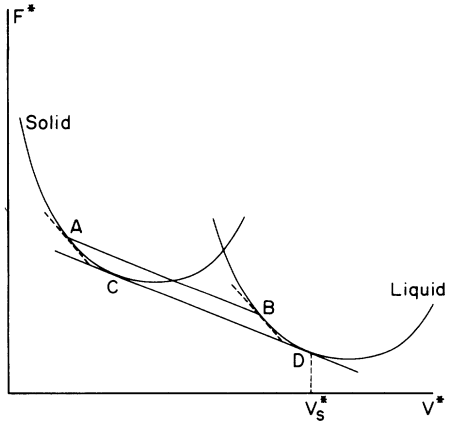

FIG. 7. Construction, in the equivalent 6-12 system, for the liquid-solid equilibrium in a nuclear system. The chord AB illustrates the required construction, joining points where the gradient (broken lines) is Q times the gradient of AB. If the shapes of the solid and liquid curves are fairly similar, AB will be nearly parallel to the double tangent CD.

$$P_s = -\frac{\partial F}{\partial V}(V_s) \ .$$

An alternative procedure is to compute V_s from an extrapolation curve for $V_s^*(\lambda)$, making some allowance for the volume reduction due to the factors $1/Q$ in Eq. (9). This gives results for P_s agreeing with our chosen method to within 15%.

For symmetric nuclear matter we obtain $P_s = 16 \pm 8$ MeV/fm^3 and $\rho_s = 0.32 \pm 0.02$ nucleons/fm,[25] the uncertainties allowing for everything but the error in describing nuclear matter by a single effective potential (see Sec. VII). The result for P_s is not the same as that previously reported,[3] for the earlier model did not include the energy shift, and made no allowance for the effect of the core shift on P_s. Our value for ρ_s represents about twice infinite nuclear matter density ($1/V_0$). Small nuclei have densities higher than $1/V_0$ due to the surface tension (but reduced by Coulomb effects), and it might be wondered whether the surface pressure could be enough to cause crystallization. We estimate the internal pressure due to surface tension to be $\gamma/(2\pi r^3) \sim 2.2A^{-1/3}$ MeV/fm^3 from the surface energy γ of about $19A^{2/3}$ MeV. Thus we do *not* predict solid nuclei, but the margin is not very great. Pauling[26] once proposed such an idea, and Cook[27] has recently put forward an fcc lattice structure for nuclei, but these models must be considered unlikely.

Another property of nuclear matter that we can easily compute is the critical temperature. In general the critical properties belong to too low a density regime for our core-shift approximation to be valid, but the critical *temperature* should not be greatly affected by the core shift, and will be approximately the same as that of the equivalent 6-12 system. This is intuitively clear from a kinetic theory approach, and is supported by detailed calculations employing the core shift and a Van der Waals equation of state[28]; these calculations are crude, but indicate a 10% correction to T_c and much larger corrections to P_c and V_c. We thus take $kT_c \simeq \epsilon T_c^*(\lambda)$ for nuclear matter using an extrapolation curve for $T_c^*(\lambda)$.[1] This gives $kT_c = 18 \pm 3$ MeV, which is in good agreement with a recent Thomas-Fermi calculation.[29]

The existence of a finite critical temperature shows that ordinary nuclear matter must be regarded as a liquid phase below its critical point.[30] This phase is distinct from a neutron-proton gas, which casts doubt on formulations of nuclear matter theory which do not distinguish the two.[31]

VI. NEUTRON-STAR MATTER

We now consider the application of our model to nuclear matter containing only 5% protons, which

we call *neutron-star matter*. The matter comprising the central region of large neutron stars is of similar composition (together with electrons and muons), the exact fraction of protons depending on the equation of state adopted. The difficulty with this case is lack of knowledge of any experimental properties from which to deduce ϵ and σ. In the case of symmetric nuclear matter we used F_0 and V_0, but for neutron star matter these are unknown and may not even exist; most nuclear matter theories predict a monotonically decreasing $F(V)$. In our earlier report[3] we attempted to calculate F_0 and V_0 for neutron star matter from the empirical mass formula, but we now regard this as inadequate, even with the addition of a quadratic compressibility term. Instead we believe that it is better to extrapolate ϵ and σ themselves.

Let us suppose that we can adequately describe nuclear matter by two effective pair potentials, v_0 and v_1, for total isospin $T=0$ and 1, respectively. By performing our analysis at two different compositions within the range for which F_0 and V_0 are known, we can find effective C-6-12 potentials for two different linear combinations of v_0 and v_1, and hence obtain an effective potential for any other composition.

The effective potential for a fraction x of protons is given by

$$v(r,\delta)=\frac{1-\delta^2}{1+\mu}\,v_0(r)+\frac{\mu+\delta^2}{1+\mu}\,v_1(r),$$

where $\delta=1-2x$ and μ is a parameter which depends on the degree of isospin correlation.[32] If we obtain a C-6-12 potential for two values of δ then the effective potential at a third δ may be computed by linear extrapolation with respect to δ^2. The potential so obtained will not, in general, be exactly of the C-6-12 shape, but the difference is not large, and it is easy to refit a C-6-12 potential to the result. In practice this amounts very nearly to extrapolating ϵ and σ linearly with δ^2, and so this is the procedure we adopt, incurring negligible error. For accurate extrapolation we require our two reference δ^2's to be as widely separated as possible, and we therefore use the values corresponding to $x=\frac{1}{2}$ (symmetric nuclear matter) and $x=0.35$. For the latter point F_0 and V_0 may be extracted from the masses and radii of heavy nuclei in conjunction with a compressible empirical mass formula. This has not been carried out on a purely empirical basis (most mass laws assume constant density), but Myers and Swiatecki[33,23] have developed a semiempirical model involving a few theoretical parameters. We adopt their results, which imply $F_0=13.0$ MeV and $V_0=6.50$ fm^3 for $x=0.35$. The method of Sec. IV then leads to $\epsilon\sim 53$ MeV and $\sigma\sim 1.8$ fm (Ref. 34) for $x=0.35$. Linear extrapolation with respect to δ^2 from these and our $x=0\,5$ results gives us $\epsilon=28\pm 8$ MeV, $\sigma-\Delta\sigma=1.03\pm 0.09$, $\lambda=3.7\pm 0.2$ for neutron-star matter with $x=0.05$ (we assume $\Delta=0.5$ still). The extrapolation is over a considerable range of δ^2 and therefore carries with it a considerable uncertainty, which we have attempted to include in the above bounds. The procedure is very sensitive to variation of F_0 and V_0 between $x=0.5$ and $x=0.35$, and better data are desirable. We do believe, however, that the quadratic dependence of ϵ and σ on δ^2 is well founded over the whole range of δ^2, which is not the case in the alternative procedure of directly extrapolating F_0 and V_0 with respect to δ^2.

We may use Eq. (6) to obtain an equation of state of neutron-star matter. We find that there is still a shallow minimum at $F_0=12\pm 8$ MeV, $V_0=18^{+12}_{-6}$ fm^3/nucleon, with a compressibility parameter $K=50$ to 250 MeV. Nuclear matter theory predicts no minimum here, as previously noted, but some other approaches[33,35] also show a shallow minimum.

Applying the method of Sec. V we find $P_s=6\pm 3$ MeV/fm^3, $\rho_s=0.18\pm 0.05$ nucleons/fm^3. Prediction of these quantities—which are of direct relevance to neutron-star structure—was the original aim of our empirical approach.

The critical temperature is harder to predict for neutron-star matter since the $T_c^*(\lambda)$ curve approaches zero in the vicinity of $\lambda=3.7$. The effect of the core shift and of otherwise minor inaccuracies may thus be crucial. It is safe to state $kT_c<6$ MeV for neutron-star matter, acknowledging that a finite T_c might not exist at all.

VII. DISCUSSION

We have presented a model of nuclear matter that allows order-of-magnitude predictions with very few microscopic assumptions. It has been necessary to make several approximations in reaching the results, but we believe all these to be plausible, and have estimated the errors involved.

A number of criticisms have been leveled against our corresponding-states approach to nuclear matter. It has frequently been stated[6,7,10] that nuclear potentials are not of the 6-12 shape, and indeed not even of the Yukawa shape in the repulsive region. However, our C-6-12 potential with OPEP tail can approximate nuclear potentials with good accuracy except perhaps at very short range where the repulsion becomes very strong (> 100 MeV). At the densities considered this part of the core is little sampled by the nucleon wave

functions and contributes little to the total energy. At higher densities the core shape would indeed become very important.

A second criticism[5,7,9] of our method questions the reliability of extrapolation with respect to λ far from laboratory values. Through the use of the coreshift we do not have to compare nuclear matter directly with a 6-12 system with the same quantum parameter ($\lambda' = [h^2/m\epsilon(\sigma-\Delta\sigma)^2]^{1/2} \sim 6$ for nuclear matter), but with an equivalent 6-12 system at a more reasonable λ of ~3–4, not very far from helium. The effect of statistics dependence on the reduced thermodynamic properties, as in Eq. (2), is an important problem at large λ, made worse by lack of sufficient data to estimate the effect. Nosanow and Parish[8] find considerable statistics dependence in their Monte Carlo estimates of the solidification density, but Schiff's[9] calculated correction for the Bose-Fermi difference is quite small even at $\lambda \sim 10$. We have made reasonable allowances for the consequent uncertainties in the extrapolation curves, and note also that the experimental point most heavily weighted in our method is that of He^3, which at least belongs to the Fermi case. A *strong* statistics dependence, such as observed by Nosanow and Parish,[8] might nevertheless appreciably increase our estimated solidification density and pressure.

A further criticism[5,7,9] of our approach concerns the assumption that nuclear matter may be adequately described by one or two central effective potentials, and we believe this to be the major remaining source of inaccuracy. The effective potentials must represent an average of the various components really present, belonging to different spin, isospin, and angular momentum states. In a given phase at fixed density this is a reasonable assumption, but it is necessary to consider a range of densities, and solid-liquid equilibria. The quantititative study of these effects is not possible within the present framework, but we make some qualitative remarks.

The worst flaw is probably the computation of ϵ and σ from properties of one density (V_0^{-1}) for use at another (ρ_s). The optimum effective potential will not be the same at these two densities, becoming less attractive at higher density due to the increasing importance of high angular momentum states. Clark and Chao[5] are able to allow for this effect at the expense of further microscopic assumptions, which provide *theoretical* data on nuclear matter at all densities. We prefer to be more empirical, and estimate the likely magnitude of the effect by halving ϵ before computing P_s and ρ_s. This approximately halves P_s, but hardly affects ρ_s, and this is probably an overestimate of the effect as σ will increase at high densities, partially compensating ϵ.

The spin and isospin dependence of the real potential may have some significance for the solid-liquid equilibrium, favoring ordered (solid) states and thus lowering P_s and ρ_s. The solid may take further advantage of this and form a molecular solid, perhaps of alpha particles. The L dependence is not likely to have much effect on the equilibrium itself, for the density discontinuity is small. Velocity dependence will favor fluid states, raising P_s and ρ_s. Overall, however, we believe that our estimates for P_s and ρ_s are if anything too high, being most affected by the spin-isospin dependence. Our results are nevertheless lower than the most recent microscopic calculations[6] based on quantum crystal theory, but agree quite well with the variational estimates of Nosanow and Parish.[8] Different methods are still a long way from agreement.

There are, of course, other phenomena which may occur in dense neutron-star matter. Hyperons will undoubtedly appear at sufficiently high density, but will not at first alter our conclusions greatly; we may find a hyperonic crystal. Pion condensation also seems likely, and will probably serve to enhance neutron crystallization.[36]

The consequences of a low solidification density, attainable in neutron stars, are now well known and we merely list some possible implications. The transition to a solid phase softens the equation of state of neutron-star matter at high densitits, thus reducing the maximum neutron-star mass. Knowledge of the maximum mass is of considerable importance in the identification of massive objects such as the collapsed partners in x-ray binaries. The onset of crystallization reduces the superfluid fraction of a neutron star, which may explain the extent to which the Vela pulsar is able to regain its original period after speed-up.[37] The presence of a solid core, oblate due to rotation, has been invoked to explain several phenomena, including the large size and high frequency of quakes in Vela, via a corequake mechanism, and the thirty-five day cycle of *Her-X-1*, by means of a neutron-star wobble activating an accretion gate.[38] Such a wobble has also been proposed by Dyson[39] as a source of gravitational radiation.

ACKNOWLEDGMENTS

Most of this work was done at the Cavendish Laboratory, with the support of the Science Research Council (RGP) and the AFOSR (PWA) under

contract 73-2449. The work was completed at Princeton University, supported in part by AFOSR (RGP) under contract F44620-71-C-0108. We thank A. B. Pippard for originally calling de Boer's theory to our attention.

---

*Present address: Dept. of Physics, Princeton University, Princeton, New Jersey 08540.
†Also at Bell Laboratories, Murray Hill, New Jersey 07974.

[1]J. de Boer, Physica (The Hague) 14, 139 (1948).
[2]J. de Boer and R. J. Lunbeck, Physica (The Hague) 14, 510 (1948).
[3]P. W. Anderson and R. G. Palmer, Nat. Phys. Sci. 231, 145 (1971).
[4]R. L. Coldwell, thesis, Univ. of Washington, 1969 (unpublished); R. L. Coldwell, Phys. Rev. D 5, 1273 (1972).
[5]J. W. Clark and N.-C. Chao, Nat. Phys. Sci. 236, 37 (1972); J. W. Clark, in symposium on present status and novel developments in the many-body problem, Rome, September, 1972 (unpublished).
[6]V. Canuto and S. M. Chitre, Phys. Rev. Lett. 30, 999 (1973); V. Canuto and S. M. Chitre, Phys. Rev. D 9, 1587 (1974).
[7]V. R. Pandharipande, in proceedings of the Sixth Texas Symposium on Relativistic Astrophysics, New York, 1972 (unpublished); V. R. Pandharipande, Nucl. Phys. A217, 1 (1973).
[8]L. Nosanow and L. Parish, in proceedings of the Sixth Texas Symposium on Relativistic Astrophysics, New York, 1972 (unpublished); L. Nosanow and L. Parish, in proceedings of the 1974 annual APS/AAPT meeting, Chicago, February, 1974 (unpublished).
[9]D. Schiff, Nat. Phys. Sci. 243, 130 (1973).
[10]S. Cochran and G. V. Chester, Report No. 2127, the Materials Science Center, Cornell University (unpublished).
[11]J. O. Hirschfelder, C. F. Curtis, and R. Bird, *Molecular Theory of Gases and Liquids* (Wiley, New York, 1954).
[12]In Ref. 3 we used the "zero-point crystal data" of I. H. Hillier, M. S. Islam, and J. Walkley, J. Chem. Phys. 43, 3705 (1965). These were based on a quantum cell model which predicts $F_0^*(\lambda)$ and $V_0^*(\lambda)$. The parameters ϵ and σ were chosen to give exact agreement with experiment for each substance. While such a single-particle model is probably adequate for small λ, it is certainly quite inadequate in the helium region.
[13]R. V. Reid, Ann. Phys. (N.Y.) 50, 411 (1968).
[14]These values are typical of our results for nuclear matter.
[15]J.-P. Hansen, D. Levesque, and D. Schiff, Phys. Rev. A 3, 776 (1971).
[16]M. H. Kalos, D. Levesque, and L. Verlet, Report No. COO-3077-35, Courant Inst., New York Univ. (unpublished).
[17]D. G. Henshaw, Phys. Rev. 119, 9 (1960).
[18]W. L. Gordon, C. H. Shaw, and J. G. Daunt, J. Phys. Chem. Solids 5, 117 (1958).
[19]G. Breit, Rev. Mod. Phys. 34, 766 (1962).
[20]J. Wilks, *Properties of Liquid and Solid Helium* (Clarendon, Oxford, 1967).
[21]T. Kihara, Rev. Mod. Phys. 25, 831 (1953).
[22]N. Bernardes, Phys. Rev. 120, 807 (1960).
[23]W. D. Myers and W. J. Swiatecki, Phys. Lett. 30B, 451 (1969).
[24]A. B. Migdal, *Theory of Finite Fermi Systems and Applications to Atomic Nuclei* (Interscience, New York, 1967).
[25]1 nucleon/fm^3 = 1.67×10^{15} g/cm^3; 1 MeV/fm^3 = 1.6×10^{27} atmospheres.
[26]L. Pauling. Proc. Natl. Acad. Sci. USA 54, 989 (1965).
[27]N. D. Cook, report (unpublished).
[28]R. G. Palmer, thesis, Univ. of Cambridge, 1973 (unpublished).
[29]E. Hilf, private communication.
[30]It is certainly much denser than the critical density on any reasonable assumptions.
[31]G. A. Baker and J. Kahane, J. Math. Phys. 10, 1647 (1969).
[32]We assume that the dependence of μ on δ is small.
[33]W. D. Myers and W. J. Swiatecki, Ann. Phys. (N.Y.) 55, 395 (1969).
[34]The uncertainties here are similar to those for $x=0.5$, and the values at $x=0.5$ and $x=0.35$ are of course correlated, depending on the choice of Δ, etc. within the bounds. Our extrapolated results for $x=0.05$ take this correlation into account.
[35]R. A. Weiss and A. G. W. Cameron, Can. J. Phys. 47, 2171 (1969).
[36]R. G. Palmer, E. Tosatti, and P. W. Anderson, Nat. Phys. Sci. 245, 119 (1973).
[37]D. Pines, J. Shaham, and M. Ruderman, Nat. Phys. Sci. 237, 83 (1972).
[38]F. Lamb, C. J. Pethick, and D. Pines, in proceedings of the Sixth Texas Symposium on Relativistic Astrophysics, New York, 1972 (unpublished).
[39]F. Dyson, in proceedings of the Sixth Texas Symposium on Relativistic Astrophysics, New York, 1972 (unpublished).