Scalar waves in the mixmaster universe. II. Particle creation*†

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Solutions to the complete time-dependent scalar wave equation in the mixmaster universe are presented. Differential equations for the mode amplitudes are given corresponding to field expansions in terms of the asymmetric- and the symmetric-top wave functions and for both minimal and conformal couplings of the scalar field to gravity. The theory of quantized fields in the mixmaster space is summarized. Creation and annihilation operators in the Heisenberg picture develop in time under a general Bogolubov transformation which expresses mathematically the physical processes of particle creation and mode mixing. The problem of particle creation is discussed here in the light of the classical theory of wave propagation. In the adiabatic limit when the expansion of the universe is slow, little production takes place. Higher-order expressions for the production amount are derived by the method of successive WKB approximations. Finally, numerical solutions of the wave equation for the uncoupled modes are presented. The effects of geometric shape, level energy, and particle mass on production are studied for the characteristic "small oscillation" and "bounce" solutions of the mixmaster universe. The anisotropic dynamics of the background gives rise to strong directional effects in particle creation. Production of particles of higher mass or energy is less abundant and less sensitive to the shape or level. The process of mode mixing, which is a distinct feature of the mixmaster universe, will be studied in a later paper.

I. INTRODUCTION

In an earlier paper,[1] hereafter referred to as I, we have presented solutions of the scalar wave equation for a fixed background in the mixmaster universe, assuming the three principal curvatures of the universe to be time-independent. The solutions are found to be equivalent to that of the quantum-mechanical problem of an asymmetric rotator. In this paper, we turn to the solution of the complete time-dependent wave equation. In Sec. II, we consider the wave equation in which both the Laplace-Beltrami operator and the basis function are dependent on time through some external parameter. This corresponds to the field expansion in terms of the asymmetric-top wave functions discussed in I. Interesting physical phenomena such as wave coupling become immediately apparent. Alternatively, if the time-independent symmetric-top wave functions are used as basis functions, the expansion coefficients (mode amplitudes) satisfy a set of simple coupled wave equations. The physical significance of "conformally coupled" wave equation as compared to the "minimally coupled" wave equation is brought into attention. In Sec. III, we recapitulate the main results on the quantization of the scalar field in a closed anisotropic universe as presented in Ref. 2 and introduce the physical notions of particle creation and mode mixing. The rest of this paper deals with particle creation in uncoupled modes. The phenomenon of mode mixing and its implications will be treated in paper III.

The modes of oscillation of scalar waves in a time-dependent background can be described in analogy to a system of parametric oscillators. In Sec. IV, the problem of particle creation is discussed in the light of the classical theory of wave propagation. In a statically bounded expansion of the universe, particle creation can be visualized as wave reflection off a potential barrier arising from the time variation of the background geometry. If the relative change of the natural frequency of a particular mode is small compared to the characteristic expansion rate of the universe, the adiabatic approximation is applicable. In this limit, the variation of the adiabatic invariant remains small. Little particle creation takes place. Violation of the adiabatic condition gives rise to the coupling of waves and the mixing of positive- and negative-frequency components. These processes are understood in the context of quantum field theory as particle creation and mode mixing. In Sec. V, we use the method of successive WKB approximations to derive higher-order expressions for the production amount. Following this approach, one is able to take into consideration more rapid variation of the background and can in principle obtain as close an approximation as desired to the exact solution.

The behavior of particle creation is brought into full display with a complete solution of the wave equation. In Sec. VI, we present the results of numerical integration of the combined wave equations and the field equations on a computer and discuss their implications. The effects of geometric shape, level energy, and particle mass on production are studied for some characteristic solutions of the mixmaster universe. The results obtained in this paper will be employed in the discussion of paper III on mode mixing and in our subsequent investigations on the possible mechanism of anisotropy damping due to these effects.

II. THE TIME-DEPENDENT WAVE EQUATION

A. Expansion in asymmetric-top functions

Our treatment here follows the method originally introduced by Guttinger and later expounded by Tolman.[3] Consider the Laplace-Beltrami operator on the three-dimensional space with a form dependent on some time-varying parameters $\alpha(t)$. In our case, these are the three principal curvatures of the universe $l_i(t)$, or, equivalently, the volume V and the asymmetry parameters α, β [Fig. 2 in I]. (Here, and throughout this paper, I refers to Ref. 1, and all equations, tables, and figures referring to it are written with the prefix I.) A certain configuration of the spatial geometry is described by and is equivalent to a set of values of α. For any configuration one can determine a set of energy eigenfunctions $u(x, \alpha)$ with the help of the equations [Eq. (I11)]

$$^{(3)}\Delta(x, \alpha)u_n(x, \alpha) + k_n{}^2(\alpha)u_n(x, \alpha) = 0 \;, \tag{1}$$

where $k_n(\alpha)$ are the eigenvalues of the system with that value of the parameter α. The solution of the wave equation for the system [Eq. (I10)]

$$\ddot{\Phi}^s(x, t) - {}^{(3)}\Delta(x, \alpha)\Phi^s(x, t) + \mu^2\Phi^s(x, t) = 0 \tag{2}$$

could then be expressed as a superposition of steady-state solutions in the form [Eq. (I12)]

$$\Phi^s(x, t) = \sum_n c_n u_n^s(x, \alpha)e^{-i\omega_n t} \;, \quad n = (J, \chi, M, \gamma) \tag{3}$$

where the amplitudes c_n are a set of constants which will in general be complex numbers, and $\omega_n = (k_n{}^2 + \mu^2)^{1/2}$ is the energy of the nth level.

Consider now that the external parameter α, instead of being held constant, is made to change with time. The wave operator itself would then be a function of time through its dependence on α, and one should have to write the wave equation in the form, appropriate for nonconservative systems [Eq. (I9)],

$$\ddot{\Phi}(x, t) + 2\Gamma(t)\dot{\Phi}(x, t) - {}^{(3)}\Delta(x, \alpha(t))\Phi(x, t) + \mu^2\Phi(x, t) = 0 \qquad \left(\Gamma \equiv \frac{1}{2}\sum_{i=1}^{3}\frac{\dot{l}_i}{l_i}\right). \tag{4}$$

As a form of solution for this equation one takes the superposition expressed by

$$\Phi(x, t) = \sum_n c_n(t)u_n(x, \alpha(t))\exp\left[-i\int^t \omega_n(t')dt'\right] \;, \tag{5}$$

where at each time t the steady-state solution corresponding to the instantaneous value of α holds, and the amplitude changes with time as dictated by the wave equation. Combining the two factors giving an explicit dependence on t,

$$C_n(t) \equiv c_n(t)\exp\left(-i\int^t \omega_n dt'\right) \;, \tag{6}$$

one writes

$$\Phi(x, t) = \sum_n C_n(t)u_n(x, \alpha(t)) \;. \tag{7}$$

The quantities $C_n(t)$ are then the mode amplitudes for the different states that correspond to the instantaneous forms of the eigenfunctions $u_n(x, \alpha)$. To determine how these probability amplitudes depend on time we first substitute the proposed solution (7) in the wave equation (4). Carrying out the differentiation, multiplying through by $u_m^*(x, \alpha)$ and integrating over the configuration space, while making use of the normalization and orthogonality of the eigenfunction $u_n(x, \alpha)$, we obtain

$$\ddot{C}_m(t) + 2\Gamma(\alpha)\dot{C}_m(t) + \omega_m{}^2(\alpha)C_m(t) + \sum_n R_{mn}^{(1)}[2\dot{\alpha}\dot{C}_n + (\ddot{\alpha} + 2\Gamma\dot{\alpha})C_n] + \sum_n R_{mn}^{(2)}[\dot{\alpha}^2 C_n] = 0 \;. \tag{8}$$

The coupling constants

$$R_{mn}^{(1)} \equiv \int u_m^* \frac{\partial u_n}{\partial \alpha}\, dV \;, \quad R_{mn}^{(2)} \equiv \int u_m^* \frac{\partial^2 u_n}{\partial \alpha^2}\, dV \tag{9}$$

(dV is the group invariant measure $dV = \sin\theta\, d\theta d\phi d\psi$) govern the rate of exchange between levels m and n of the system, where m, n denote different sets of quantum numbers (J, M) and symmetry types γ characterizing a given wave function. To see what these quantities are, let us recall the form of the original wave function u_n as given in Eq. (I26):

$$u_{\chi M,\gamma}^J(\theta, \phi, \psi, l_i) = \sum_K a_{K\chi}(l_i)w_{KM,\gamma}^J(\theta, \phi, \psi) \;, \tag{10}$$

where $w_{KM,\gamma}^J(\theta, \phi, \psi)$ is the properly symmetrized symmetric-top wave function (the Wang function) and $a_{K\chi}(l_i)$ are the elements of the expansion coefficient matrix $\mathfrak{A}$ dependent on time explicitly through the l_i. ($\mathfrak{A} = (a_{K\chi})$ is the transformation matrix that diagonalizes the Hamiltonian, H, i.e., $\mathfrak{A}^{-1}H\mathfrak{A} = \Lambda$ [see (I32)], and is orthonormal with respect to column multiplication, i.e., $\sum_K a_{K\chi} a_{K\chi'} = \delta_{\chi\chi'}$. χ is a labeling index number for the eigenvalues of the asymmetric-top wave function within a given J value.) Since the only dependence on l_i in the wave function $u^J_{\chi M,\gamma}$ is through the coefficients $a_{K\chi}$ and since the Wang functions are properly orthonormalized with respect to the indices (J, K, M, γ), the only contribution of R_{mn} comes from levels of different χ but within the same angular momentum numbers (J, M) and belonging to the same symmetry type (γ). Hence

$$R^{(1)}_{\chi\chi'}(\alpha) = \sum_K a^*_{K\chi}(\alpha) a_{K\chi'}{}'(\alpha)$$

and similarly (11)

$$R^{(2)}_{\chi\chi'}(\alpha) = \sum_K a^*_{K\chi}(\alpha) a_{K\chi'}{}''(\alpha) ,$$

where

$$a_{K\chi}{}' \equiv \frac{da_{K\chi}}{d\alpha} , \quad a_{K\chi}{}'' \equiv \frac{d^2 a_{K\chi}}{d\alpha^2} ,$$

and α here stands for some deformation parameter. The R's will be nonzero only for levels belonging to the same energy submatrix; hence the number of terms in the summation is equal to the rank of the submatrix minus one (excluding self-coupling). For example, for $J=1$ (see Table I1) all three levels are independent of each other; so is the case with the levels $J_\chi = 2_1, 2_0, 2_{-1}$. However, coupling exists between the two levels 2_2 and 2_{-2}. For $J=3$, three groups of levels couple pairwise but not among themselves. They are $(3_3, 3_{-1})$, $(3_2, 3_{-2})$, and $(3_1, 3_{-3})$. The 3_0 level is independent. Following this we can also deduce that, for example, when $J=5$, there are three groups of three coupled levels and one coupled pair, all four groups being independent of each other. In general, there exist three n-tuples (of B_a, B_b, B_c types) and one $(n+1)$-tuple (of A type) for even $J=2n$, or one $(n-1)$-tuple for odd $J=2n+1$ $(n=1, 2, \ldots)$. For high J, the multiplets are grouped into 4 classes each containing approximately $\frac{1}{2}J$ coupled levels.

As the mode expansion is made in terms of the asymmetric-top wave functions $u^J_{\chi M,\gamma}(\theta, \phi, \psi, t)$, which is itself time-dependent through the transformation matrix coefficients $a_{K\chi}(l_i(t))$, the coupling of modes as exemplified by the terms involving $R^{(1)}_{mn}$ and $R^{(2)}_{mn}$ in the wave equation (8) arises precisely from the change of such coefficients. As will be shown in the following, it actually proves simpler to describe the system in terms of the symmetrized symmetric-top wave functions (the Wang functions), which are time-independent.

B. Expansion in symmetric-top functions

Substituting the expansion

$$\Phi^J_{M,\gamma}(x, t) = \sum_K \frac{f_K(t)}{[V(t)]^{1/2}} w^J_{KM,\gamma}(x) \tag{12}$$

(where $w^J_{KM,\gamma}$ is the Wang function) into Eq. (I9) we obtain an equation for the coefficients f_K (here we have suppressed the J, M, γ indices):

$$\frac{d^2 f_K}{dt^2} - U(t) f_K + H_{KK'}(t) f_{K'} = 0 , \tag{13}$$

$$U(t) = \Gamma^2(t) + \dot{\Gamma}(t) , \tag{14}$$

where $H_{KK'}$ is one of the four $O^\pm$, $E^\pm$ energy submatrices in the eigenvalue equation $Hw_K = H_{KK'} w_{K'}$ (see Appendix A in I)

$$H = -{}^{(3)}\Delta + \mu^2 . \tag{15}$$

From Eq. (7) and (12), it is easy to relate f_K to C_χ, i.e.,

$$f_K(t) = \sqrt{V} \sum_{\chi=-J}^{J} C_\chi(t) a_{K\chi}(l_i(t)) . \tag{16}$$

In fact, if we put Eq. (16) into (13) we recover Eq. (8), with $R^{(1)}$ and $R^{(2)}$ arising from the time derivatives of $a_{K\chi}$.

Notice that with a change to the time element $d\tau$ defined by

$$dt = V(\tau) d\tau , \tag{17}$$

where $V = l_1 l_2 l_3$ is the volume of the universe, the scalar-wave equation contains no first-order time-derivative terms:

$$\frac{\partial^2 \Phi(x, \tau)}{\partial \tau^2} + V^2(\tau)[-{}^{(3)}\Delta(\tau, x) + \mu^2] \Phi(x, \tau) = 0 . \tag{18}$$

Under the expansion

$$\Phi(x, \tau) = \sum_K h_K(\tau) w_K(x) \tag{19}$$

we obtain an equation similar to (13) for h_K, i.e.,

$$\frac{d^2 h_K(\tau)}{d\tau^2} + \sum_{K'} \mathcal{H}_{KK'}(\tau) h_{K'}(\tau) = 0 , \tag{20}$$

where

$$\mathcal{H}_{KK'} = V^2(\tau) H_{KK'} .$$

C. Conformally coupled wave equation

So far in our discussion we have treated the wave equation (I7),

$$(\Box - \mu^2)\Phi(x, t) = 0 , \tag{21}$$

as the simplest fully covariant generalization of the Klein-Gordon equation describing a spin-zero field "minimally coupled" to a curved background.[4] A modified form of Eq. (21), the so-called "conformally coupled" wave equation,[5]

$$(\Box-\mu^2-\tfrac{1}{6}R)\Phi(x,t)=0\ , \tag{22}$$

was treated earlier in Ref. 2. It contains an additional term involving the four-dimensional curvature R, and is conformally invariant in the massless case $\mu=0$.[6] If an isotropic metric like the Robertson-Walker type is assumed, and the conformally coupled equation is adopted, no massless particles will be produced. As was remarked earlier, this is a consequence of the conformal flatness of the space and the conformal invariance of the equation. Therefore, for significant particle production, it is essential that an anisotropic expansion be postulated.[7,8] This is especially so in the case of the empty space, because the two types of equations (21) and (22) become indistinguishable (as $R=0$) and thus they are both conformally invariant when $\mu=0$.

The question as to whether Eq. (21) or (22) more correctly describes the physical field remains largely open, the answer still awaiting better observations and deeper understanding. For the present purpose, since we are dealing with an empty background, as long as the dynamics of the universe is governed by the Einstein equations ($R_{ij}=0 \Leftrightarrow R=0$), the question does not arise. However, when one investigates the problem of anisotropy damping, the energy-momentum content due to the particles produced will be allowed to react back on the geometry, thus modifying the spacetime significantly. Also, since it is in the case of conformal coupling that isotropy tends to reduce the rate of particle creation, one expects to find the effect of anisotropy damping in such case most evident. For this reason we shall also work with the conformally coupled equations in the following.

In (22), the four-dimensional scalar curvature R of the mixmaster spacetime is given by (cf. Appendix A of Ref. 16)

$$^{(4)}R=\sum_{i=1}^{3}\ddot{l}_i/l_i+\ddot{V}/V-\frac{(l_1^4+l_2^4+l_3^4)-\frac{1}{2}(l_1^2+l_2^2+l_3^2)^2}{V^2}\ . \tag{23}$$

Equation (4) now is replaced by

$$\ddot{\Phi}+2\Gamma\dot{\Phi}-{}^{(3)}\Delta\Phi+(\mu^2+\tfrac{1}{6}R)\Phi=0\ . \tag{24}$$

The forms of Eq. (13) and (20) remain the same, except for the change in H in Eq. (15), which now contains an extra term

$$H=-{}^{(3)}\Delta+\mu^2+\tfrac{1}{6}R\ . \tag{25}$$

In the conformally coupled equation, a more convenient time coordinate as used in Ref. 8 is the η time defined by

$$dt=v(\eta)d\eta,\quad v\equiv V^{1/3}\ . \tag{26}$$

Thus, defining $\Psi=v\,\Phi$ and denoting differentiation with respect to η by a prime, the wave equation becomes

$$\frac{\partial^2\Psi}{\partial\eta^2}+v^2\left(-{}^{(3)}\Delta+\mu^2+\tfrac{1}{6}R-\frac{v''}{v^3}\right)\Psi=0\ . \tag{27}$$

Under the expansion

$$\Psi(x,\eta)=\sum_K\psi_K(\eta)w_K(x)\ , \tag{28}$$

we obtain in place of (20)

$$\frac{d^2\psi_K}{d\eta^2}+\sum_{K'}\left[(\Omega^2)_{KK'}+Q\delta_{KK'}\right]\psi_{K'}=0\ , \tag{29}$$

where

$$\Omega^2=v^2(-{}^{(3)}\Delta+\mu^2) \tag{30}$$

and

$$Q(\eta)=v^2R/6-v''/v$$
$$=\frac{1}{18}\left[\left(\frac{l_1'}{l_1}-\frac{l_2'}{l_2}\right)^2+\left(\frac{l_2'}{l_2}-\frac{l_3'}{l_3}\right)^2+\left(\frac{l_3'}{l_3}-\frac{l_1'}{l_1}\right)^2\right]-\frac{1}{6v^4}\left[l_1^4+l_2^4+l_3^4-\tfrac{1}{2}(l_1^2+l_2^2+l_3^2)^2\right]\ .$$

III. QUANTIZED FIELDS, PARTICLE CREATION, AND MODE MIXING

After having obtained useful forms of the wave equation, and before going on to seek their solutions, we pause here for a brief description of the theory of quantized fields in curved space and introduce the physical notions of cosmological particle creation.[4,8-10] The principal ideas and results pertaining to a closed, homogeneous, and anisotropic spacetime like the mixmaster has been treated in detail in Ref. 2. In order to facilitate better continuity and uniformity of our presentation, we will recapitulate only the main points and write down the useful formulas.

The Lagrangian density of a scalar field minimally coupled to a curved spacetime is given by

$$\mathcal{L}=-\tfrac{1}{2}(-g)^{1/2}(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi+\mu^2\Phi^2)\ . \tag{31}$$

The metric is here treated as an unquantized external field. The scalar field is quantized canonically by imposing the equal-time commutation relations

$$\begin{aligned}&[\Phi(x,t),\Phi(x',t)]=0=[\pi(x,t),\pi(x',t)]\ ,\\&[\Phi(x,t),\pi(x',t)]=i\delta^{(3)}(x-x')\ ,\end{aligned} \tag{32}$$

where

$$\pi = \partial \mathfrak{L}/\partial \dot{\Phi} = (-g)^{1/2}\dot{\Phi} \qquad (33)$$

is the conjugate momentum to Φ.

Since the field $\Phi(x, t)$ is quantized, the wave equation (4) now becomes an operator equation. The solution can be given in at least two ways. In terms of the *time-independent* symmetric-top wave functions w_k

$$\Phi(x, t) = \frac{1}{\sqrt{2}} \sum_k [C_k(t) w_k(x) + C_k^\dagger w_k(x)] \ , \qquad (34)$$

where the w_k, as usual, are the time-independent symmetric-top wave functions and the C_k are operators satisfying the coupled equations [in τ coordinates; cf. Eq. (20)]

$$\frac{d^2 C_k}{d\tau^2} + \sum_{k'} \mathcal{K}_{kk'} C_{k'} = 0 \ . \qquad (35)$$

Alternatively, in terms of the asymmetric-top wave functions u_j regarded as *instantaneously static* (say, at t_0, a fixed background),

$$\Phi(x, t_0) = \sum_j (2E_j)^{-1/2} [A_j(t_0) u_j(x) + A_j^\dagger(t_0) u_j^*(x)] \ . \qquad (36)$$

Here we use the collective indices $k = (J, K, M, \gamma)$ and $j = (J, \chi, M, \gamma)$ to denote the symmetric- and asymmetric-top indices, respectively.

The creation and annihilation operators A_j, $A_j^\dagger$ are defined in a way such that the states with a fixed number of quanta in each mode at a given time are the eigenstates of the Hamiltonian of the field theory at that time. As a consequence of (32), they satisfy the commutation relations

$$[A_j, A_{j'}] = 0 = [A_j^\dagger, A_{j'}^\dagger] \ ,$$
$$[A_j, A_{j'}^\dagger] = \delta_{jj'} \ . \qquad (37)$$

The time development of the operators $A_j(t) A_j^\dagger(t)$ is described by a general Bogolubov transformation:

$$A_j(t) = \sum_{j'} [\mathfrak{U}_{jj'}(t) A_{j'}(t_0) + \mathfrak{V}_{jj'}(t) A_{j'}^\dagger(t_0)] \ , \qquad (38)$$

where $\mathfrak{U}_{jj'}$ and $\mathfrak{V}_{jj'}$ are complex c-number functions of time.

The canonical nature of A_j, $A_j^\dagger$ yield the following familiar conditions for the transformation coefficients:

$$\sum_i (\mathfrak{U}_{ji} \mathfrak{U}_{j'i}^* - \mathfrak{V}_{ji} \mathfrak{V}_{j'i}^*) = \delta_{jj'} \ , \qquad (39a)$$

$$\sum_i (\mathfrak{U}_{ji} \mathfrak{V}_{j'i} - \mathfrak{V}_{ji} \mathfrak{U}_{j'i}) = 0 \ . \qquad (39b)$$

The coefficients of the Bogolubov transformation $\mathfrak{U}_{jj'}$, $\mathfrak{V}_{jj'}$ give a complete description of the system. In Sec. III of Ref. 2, they are calculated in terms of the eigenvalues E_j of the mixmaster Laplacian operator (see I), the expansion coefficients $\mathfrak{A}_{kj}$ [Eq. (10)], and the c-number field quantities h_k and $\dot{h}_k$ [Eq. (19)]. In this way, the dynamics of the field is again reduced to a c-number problem, the essence of it amounting to solutions of the classical field equations [Eq. (20) or (13)].

For a general time-dependent metric like the mixmaster, the transformation matrices are in general nondiagonal. Hence, if one starts out with a pure positive-frequency wave component in a certain mode j_1, then at some later time one will find a certain amount of negative-frequency j_1 component, as well as some positive- and negative-frequency waves of some other mode j_2. These effects correspond to particle creation and mode mixing, respectively [the amounts are measured by $\mathfrak{V}_{j_1 j_1}$ and $\mathfrak{U}_{j_1 j_2}$, $\mathfrak{V}_{j_1 j_2}$, respectively, as one can easily see from Eq. (38)]. The latter is a new phenomenon absent in simpler model universes, such as the Robertson-Walker, Kasner, and Taub universes. Wave processes in the mixmaster universe are rather complicated. In order to understand the problem in some depth, it is easier for the purpose of presentation to deal with the two processes separately. Thus in this paper II we direct our attention to uncoupled modes only and discuss particle creation associated with them. Wave coupling and mode mixing will be discussed in a following paper (III).

In the basis of the symmetric-top wave functions, the uncoupled modes in the mixmaster space contain only the low-lying levels $J = 1, 2$; $K = -1, 0, 1$ (see I and Sec. II). There, like the simple model universes, separation of variables in the scalar wave equation allows for mode decomposition and the field amplitudes develop in time obeying a wave equation of the form (20) with diagonal $\mathcal{K}_{kk'}(t)$

$$\ddot{\phi}_k(t) + W_k^2(t) \phi_k(t) = 0 \ . \qquad (40)$$

(A dot here denotes a time derivative.) Since there is no mode coupling, the process is described by a diagonal Bogolubov transformation [cf. Eq. (38)]:

$$A_k(t) = \alpha_k^*(t) A_k(t_0) + \beta_k(t) A_{-k}^\dagger(t_0) \ . \qquad (41a)$$

The transformation coefficients satisfy the condition [we set $j = j'$ in Eq. (39a)]

$$|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1 \ . \qquad (41b)$$

The quantity $|\beta_k(t)|^2$ gives the average number of particles present in the kth mode at time t if the state is the vacuum at t_0.[4]

IV. PARAMETRIC OSCILLATORS, BARRIER REFLECTION, AND PARTICLE CREATION

The wave equation (40) has been a subject of extensive study. With arbitrary time dependence of the natural frequency $W(t)$, exact solution does not exist except for certain special functional forms. Various methods of approximation prevail, however.[11] It is not our intention here to try to cover different approaches; rather, as one way to understand the problem, we review the analogy between the process of particle creation and the excitation of parametric oscillators[12] and relate them to the scattering of classical waves[13] over a barrier. (This relationship was implicitly assumed in Ref. 8.) The amount of particle creation in the adiabatic limit is shown to be small.

A. Parametric oscillators

Consider a one-dimensional classical oscillator with variable frequency $W(t)$. Assume that $W(t)$ takes the constant values $W_\pm$ as $t\to\pm\infty$. The real solution of the oscillator equation

$$\ddot{\phi}(t)+W^2(t)\phi(t)=0$$

has the asymptotic form as $t\to\pm\infty$

$$\phi_\pm(t)=\tfrac{1}{2}(a_\pm e^{iW_\pm t}+a_\pm^* e^{-iW_\pm t})\ . \tag{42}$$

A complex solution of the wave equation also exists which asymptotically approaches

$$\begin{aligned}\Phi_-(t)&=e^{iW_-t}+Re^{-iW_-t},\quad \text{for } t\to-\infty\\ \Phi_+(t)&=Te^{iW_+t},\quad \text{for } t\to+\infty\end{aligned} \tag{43}$$

where R and T are related to (a_+, a_-) by

$$\begin{aligned}a_-&=1+R^*\ ,\\ a_+&=T\ .\end{aligned} \tag{44}$$

In an alternative view, Eq. (43) describes the problem of reflection over a one-dimensional barrier. If we regard t as a spatial variable and $W(t)$ as the potential function, then the wave equation corresponds to the time-independent Schrödinger equation and ϕ becomes a state function. Thus, in the picture of *barrier reflection*, Eq. (43) describes the situation where a wave of unit amplitude comes in from $t=-\infty$ traveling to the right and is reflected at a potential barrier. This results in a reflected wave of amplitude R to the left and a transmitted wave of amplitude T to the right [Fig. 1(a)].

In the *particle creation* problem, the splitting of waves are arranged differently [Fig. 1(b)]:

$$\begin{aligned}\phi_-&=e^{-W_-t},\quad \text{for } t\to-\infty\\ \phi_+&=\alpha e^{-W_+t}+\beta e^{iW_+t},\quad \text{for } t\to+\infty\ .\end{aligned} \tag{45}$$

The assumption of asymptotically constant oscillator frequency corresponds to the situation of a statically bounded expansion of the universe.[4] That is, the universe evolves in such a way that

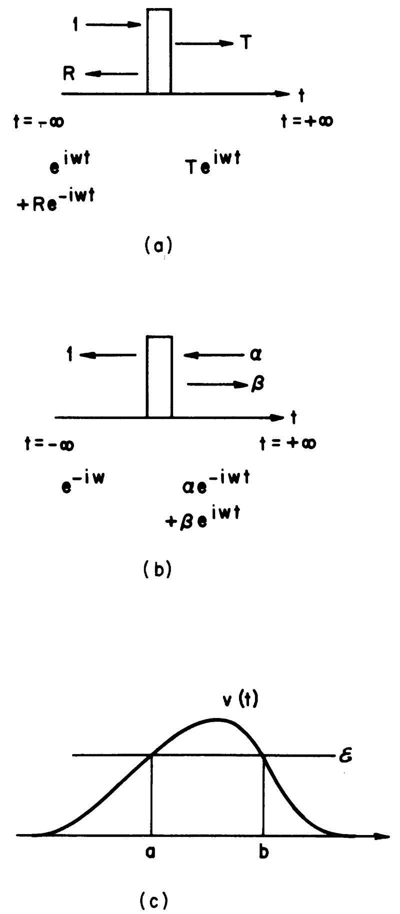

FIG. 1. (a) Wave scattering. R and T are the reflection and transmission coefficients. (b) Particle creation. $|\beta|^2$ measures the amount of production. (c) Barrier reflection. The "effective potential" arises from the time variance of the background.

it is static in the beginning and the end, but can vary smoothly in the time in-between. A static metric is necessary at $t=-\infty$ for an unambiguous definition of the vacuum and at $t=+\infty$ for an unambiguous determination of produced particles.[7]

In (45), β measures the degree of mixing of the positive- and the negative-frequency components. In the context of quantum field theory, $|\beta|^2$ gives the amount of particle production (in a particular mode). The condition for the conservation of current density (the continuity condition) yields the following relation between R and T in the scattering problem:

$$W_-(1-|R|^2)=W_+|T|^2\ . \tag{46}$$

If there is no change in the frequency, then

$$|T|^2+|R|^2=1\ .$$

This corresponds, for the particle creation problem, to the relation

$$|\alpha|^2-|\beta|^2=1\,, \tag{47}$$

which was obtained earlier [Eq. (41b)] from the Wronskian condition.

B. Quasiclassical approximation

In the case in which the natural frequency $W(t)$ of a particular mode of the system changes sufficiently slowly in time relative to the change in the background, e.g., when

$$\left|\frac{1}{W^2}\frac{dW}{dt}\right| \ll 1\,, \tag{48}$$

a good approximation to the solution of the wave equation is given by the (first-order) WKB approximation (also known as the quasiclassical, or adiabatic, approximation). We now use this method to calculate the rate of particle creation under the adiabatic conditions.

Consider a wave of energy $\mathcal{E}$ propagating towards a one-dimensional potential barrier described by the potential function $\mathcal{V}(t)$. If the quasiclassical approximation is assumed to hold in the regions outside and inside the barrier, the solution of the wave equation

$$\frac{d^2}{dt^2}\,\phi(t)+W^2(t)\phi(t)=0 \tag{49}$$

can be written as [see Fig. 1(c)]

$$\begin{aligned}\phi(t)&=\frac{A_1}{[W(t)]^{1/2}}\exp\left(i\int_a^t W\,dt\right)\\&\quad+\frac{A_2}{[W(t)]^{1/2}}\exp\left(-i\int_a^t W\,dt\right)\quad(t<a)\\&=\frac{B_1}{[W(t)]^{1/2}}\exp\left(i\int_b^t W\,dt\right)\\&\quad+\frac{B_2}{[W(t)]^{1/2}}\exp\left(-i\int_b^t W\,dt\right)\quad(t>b)\,,\end{aligned} \tag{50}$$

where $W(t)=[\mathcal{E}-\mathcal{V}(t)]^{1/2}$ and $a<b$ are the classical turning points. The wave amplitude before and after the scattering are related by the matrix relation[14]

$$\begin{pmatrix}A_1\\A_2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2\theta+1/2\theta & i(2\theta-1/2\theta)\\ -i(2\theta-1/2\theta) & 2\theta+1/2\theta\end{pmatrix}\begin{pmatrix}B_1\\B_2\end{pmatrix} \tag{51}$$

or

$$A=SB\,,$$

where the "opacity parameter"

$$\theta=\exp\left[i\int_a^b W(t)\,dt\right] \tag{52}$$

measures the height and thickness of the barrier as a function of energy. Since the scattering matrix S is unitary, B can be obtained from A via the action of $S^\dagger$. In the wave-reflection problem [Fig. 1(a)] the initial and final conditions are

$$\begin{aligned}&A_1=1\,,\quad B_1=T\,,\\&A_2=R\,,\quad B_2=0\,.\end{aligned} \tag{53}$$

From (57), the transmission and reflection coefficients are given by[15]

$$\begin{aligned}&T=2/(2\theta+1/2\theta)\,,\\&R=-\tfrac{1}{2}i(2\theta-1/2\theta)T\,,\end{aligned} \tag{54}$$

whereas in the particle creation problem [Fig. 1(b), the initial and final conditions are

$$\begin{aligned}&A_1=0\,,\quad B_1=\beta\,,\\&A_2=1\,,\quad B_2=\alpha\,.\end{aligned} \tag{55}$$

One obtains easily from (57) the amount of particles produced:

$$|\beta|^2=\tfrac{1}{4}\left(2\theta-\frac{1}{2\theta}\right)^2. \tag{56}$$

For a "high and broad" barrier, $\theta\gg 1$ and $|\beta|^2\simeq\theta^2$.

Thus, in the adiabatic limit, the "opacity parameter" of the barrier gives an approximate measure of the amount of particles created. The "barrier" here arises from the variation of the time-dependent background in a statically bounded universe. One can therefore gain a good deal of knowledge about the qualitative behavior of particle creation by studying the details of the effective potential. In the wave equation (13) with diagonal H, the effective potential $W_k^{\,2}(t)\equiv -U(t)+\omega_k^{\,2}(t)$ consists of two parts: $U(t)=\Gamma^2(t)+\dot{\Gamma}(t)$, which is related to the logarithmic dependence of volume on time, is a purely geometric quantity, while $\omega_k^{\,2}(t)=k^2+\mu^2$ depends on the particular energy level and on the mass of the particle created. It should be noted that here, as different from potential scattering, k^2 is regarded as a fixed parameter and μ^2 as a varying parameter.

The above approach is best illustrated by diagrammatic methods. From a solution of the Einstein's field equations, the time dependence of the volume is used to calculate $U(t)$, and from a solution of the scalar field equation one obtains the eigenvalues k^2 as a function of time through their dependence on l_i. The sum of these two terms yields the effective potential $W^2(t)$. Then, just by examining the profile of the potential barrier, one can infer the qualitative behavior of particle creation. For example, in Fig. 1(c), the area of $\mathcal{V}$ above the line $\mathcal{E}$ (corresponding to μ^2) measures the amount of production. From this, one can draw the conclusion almost immediately that for the same geometry, in the same level, fewer massive particles are produced than light ones [cf. Appendix B of Ref. 16].

For the model universes considered here, particle creation is not so easy to analyze since the quasiclassical approximation is not valid near the cosmological singularity. Hence the asymptotic form for $t\to-\infty$ in Eq. (45) does not hold and an unambiguous definition of the vacuum state near the singularity does not exist. To study particle creation in the mixmaster universe, one must make *ad hoc* assumptions about the initial state. Such will be done in Sec. VI so that numerical solutions can be found.

C. Adiabatic invariants

Adiabatic invariants are physical quantities which change so slowly in response to slight variations of the external conditions that they may be taken as constants of motion.[17] The action $J=E/W$ (the ratio of the energy to frequency) of a harmonic oscillator is such a quantity.[18] If the external conditions can be described by an analytic function of time, the variation of the adiabatic invariant of the classical oscillator can be shown to be exponentially small.[19] Thus, from (42), the asymptotic values of the action for $t\to\pm\infty$ are given by

$$J_\pm=\tfrac{1}{2}W_\pm a_\pm a_\pm^* \,, \tag{57}$$

which can be written with the help of Eqs. (44) and (46) as

$$\begin{aligned} J_-&=\tfrac{1}{2}W_-|1+R|^2 \,,\\ J_+&=\tfrac{1}{2}W_-(1-|R|^2)\,. \end{aligned} \tag{58}$$

The variation is thus

$$\Delta J=J_--J_+=W_-(|R|^2+\mathrm{Re}R)\,. \tag{59}$$

In the adiabatic approximation, the reflection amplitude was given earlier[15] by (54). Substituting it into (59) one obtains

$$\begin{aligned} J&=W_-(e^{-4\sigma}+\sin 2\rho\, e^{-2\sigma})\,,\\ \rho+i\sigma&=\int^{t_0}W(t)dt\,, \end{aligned}$$

where ρ and σ are real numbers. We see that the variation of the adiabatic invariant is an exponentially small quantity. Besides, the variation depends on the initial phase, owing to the presence of the term $\mathrm{Re}R$ in Eq. (59). The result is proved to be valid also for a quantum oscillator. Furthermore, it can be shown that if the kth derivative of the function $W(t)$ has a finite discontinuity on the real axis, the variation of the adiabatic invariant is proportional to the kth power of the small parameter $|(1/W^2)(dW/dt)|$.[20] In the context of quantum field theory, the adiabatic invariant here corresponds to the particle number created in a given mode. The above result states that in the limit of an infinitely slow expansion of the universe, the average number in each mode before and after the expansion differs by an exponentially small quantity (cf. Ref. 4, Sec. D).

V. HIGHER-ORDER WKB APPROXIMATIONS

In the Sec. IV it was shown that there is little particle creation in a slowly changing universe. The adiabatic limit was reached as the first-order WKB approximation. As higher-order approximations are taken into account, the solution depicts an increasingly rapid varying background.[20] In this section we make use of the formalism of successive WKB approximations to calculate particle creation up to the fourth order (in T^{-1}, where T is a long time characteristic of the system). The results obtained here are useful in studying the problem of regularization of the energy-momentum tensor by the adiabatic method.[21]

Higher-order WKB approximations to the wave equation (49) are obtainable by introducing the transformations[22]

$$\begin{aligned} dt_r&=W_{r-1}dt_{r-1}\equiv K_r\,dt\quad (W_0\equiv W,\quad t_0\equiv t)\,,\\ \phi_r&=W_{r-1}{}^{1/2}\phi_{r-1}=K_r{}^{1/2}\phi\,,\\ K_r&\equiv W_0W_1\cdots W_{r-1}\,. \end{aligned} \tag{60}$$

The rth-order WKB equation is then given by $(r\geq 1)$

$$\frac{d^2}{dt_r{}^2}\,\phi_r+W_r{}^2\phi_r=0\,,$$

where

$$\begin{aligned} W_r{}^2&=1+\epsilon_{2r}\,,\\ \epsilon_{2r}&=-\frac{1}{W_{r-1}{}^{1/2}}\,\frac{d^2}{dt_r{}^2}\,(W_{r-1}{}^{1/2})\,. \end{aligned} \tag{61}$$

If $|\epsilon_{2r}|\ll 1$, the solution of the wave equation (49) correct up to the rth order of derivatives of the parameter W^2 with respect to t_r is then [from (60)]

$$\phi=\frac{A\exp(i\int K_r\,dt)+B\exp(-i\int K_r dt)}{K_r{}^{1/2}}\,, \tag{62}$$

where A and B are complex constants.

In the rest of this paper we shall study the conformally coupled wave equations of the form

$$\chi''+(\Omega^2+Q)\chi=0\,, \tag{63}$$

where χ denotes a general amplitude function of a particular mode and a prime denotes differentiation with respect to a general time ζ. Equation (63) stands for either Eqs. (20) and (25) in τ time or Eq. (29) in η time, while χ stands for h or ψ, respectively. The general solution can be written in the WKB form

$$\chi(\zeta)=\frac{\alpha}{(2\Omega)^{1/2}}\,e_- + \frac{\beta}{(2\Omega)^{1/2}}\,e_+\,,$$
$$e_\pm \equiv \exp\left(\pm i\int \Omega\, d\zeta\right)\,. \tag{64}$$

With the introduction of two functions α, β, one has the freedom to impose one additional condition. This is chosen as

$$\chi'(\zeta)=-i\Omega\left(\frac{\alpha}{(2\Omega)^{1/2}}\,e_- - \frac{\beta}{(2\Omega)^{1/2}}\,e_+\right). \tag{65}$$

From the Wronskian condition (continuity equation)

$$\chi'^*\chi-\chi'\chi^*=i \tag{66}$$

it follows that

$$|\alpha|^2-|\beta|^2=i\,. \tag{67}$$

Define $s\equiv|\beta|^2$ as the amount of particle production. We want to derive an expression for s correct to the fourth order. First, invert (64) and (65) to find

$$\beta=(\tfrac{1}{2}\Omega)^{1/2}\left(\chi-\frac{i}{\Omega}\,\chi'\right)e_-\,.$$

Thus, making use of Eq. (66),

$$s\equiv\beta\beta^*=\frac{\Omega}{2}\,|\chi|^2+\frac{|\chi'|^2}{2\Omega}-\frac{1}{2}\,. \tag{68}$$

Then, one seeks to express χ in terms of ϵ_2 and ϵ_4 defined in Eq. (61). To the fourth order, the wave function χ can be approximated by the positive-frequency solution[23] [Eq. (62), $A=0$, $B=1$]

$$\chi=(2K_4)^{-1/2}\exp\left(-i\int K_4 d\zeta\right)\,, \tag{69}$$

where

$$K_4=\Omega\left[\left(1+\frac{Q}{\Omega^2}\right)(1+\underline{\epsilon}_2)(1+\underline{\epsilon}_4)\right]^{1/2}\,.$$

Note that $\underline{\epsilon}_r$ is written here instead of ϵ_r. To the fourth order (in powers of time derivatives, T^{-4}),

$$K_4\simeq\Omega\left(1+\frac{Q}{\Omega^2}+\underline{\epsilon}_2+\underline{\epsilon}_4+\underline{\epsilon}_2\,\frac{Q}{\Omega^2}\right)^{1/2}$$
$$\equiv\Omega(1+\epsilon_2+\epsilon_4)^{1/2}\,, \tag{70}$$

where

$$\epsilon_2\equiv\epsilon_2+Q/\Omega^2\,,$$
$$\epsilon_4\equiv\underline{\epsilon}_4+\underline{\epsilon}_2 Q/\Omega^2\,.$$

Thus

$$|\chi|^2=(2K_4)^{-1}$$
$$\simeq(2\Omega)^{-1}(1-\tfrac{1}{2}\epsilon_{2(2)}-\tfrac{1}{2}\epsilon_{2(4)}-\tfrac{1}{2}\epsilon_{4(4)}+\tfrac{3}{8}\epsilon_{2(2)}{}^2)\,. \tag{71}$$

The number in the parenthesis following ϵ_2, ϵ_4 denotes the order of T^{-1}. Taking the time derivative of (69), one derives

$$|\chi'|^2=(2K_4)^{-1}\left[K_4{}^2+\frac{1}{4}\left(\frac{d}{d\eta}\ln K_4\right)^2\right]\,.$$

The term in the large parenthesis reduces to (in third order) $\sim\Omega'/\Omega+\frac{1}{2}\epsilon_2{}'$. Combining (71), one gets

$$|\chi'|^2\simeq(2\Omega)^{-1}\{\Omega^2+[\tfrac{1}{2}\Omega^2\epsilon_{2(2)}+\tfrac{1}{4}(\Omega'/\Omega)^2]+\tfrac{1}{2}\Omega^2\epsilon_{4(4)}+\tfrac{1}{2}\Omega^2\epsilon_{2(4)}-\tfrac{1}{8}\,\Omega^2\epsilon_{2(2)}{}^2-\tfrac{1}{8}(\Omega'/\Omega)^2\epsilon_{2(2)}+\tfrac{1}{4}(\Omega'/\Omega)\epsilon_{2(2)}{}'\}\,. \tag{72}$$

Now, all that remains is to derive expressions of ϵ_2 in terms of Q, Ω, and their derivatives. (It is readily seen that terms of fourth order in ϵ_4 and ϵ_2 cancel in the expression for s). This involves straightforward differentiation steps starting from the definition of (61). (The details of this calculation can be found in Appendix B of Ref. 21.) To the second order, they are given by

$$\epsilon_{2(2)}=\frac{Q}{\Omega^2}-\frac{\bar\Omega^2}{4}-\frac{\bar\Omega'}{2\Omega}\,, \tag{73a}$$

$$\epsilon_{2(2)}{}'=\frac{Q'}{\Omega^2}-2Q\,\frac{\bar\Omega}{\Omega}-\frac{\bar\Omega''}{2\Omega}\,, \tag{73b}$$

where

$$\bar\Omega\equiv\frac{\Omega'}{\Omega^2}\,.$$

Substituting (73) into (71) and (72) and selecting terms of same order and putting them into (68), we obtain to the second and the fourth order, respectively,

$$s_{(2)}=\tfrac{1}{16}\bar\Omega^2\,, \tag{74a}$$

$$s_{(4)}=\frac{1}{16}\left(-\frac{1}{2\Omega^2}\bar\Omega\bar\Omega''+\frac{1}{4\Omega^2}\bar\Omega'^2+\frac{1}{2\Omega}\bar\Omega'\bar\Omega^2+\frac{3}{16}\bar\Omega^4\right.$$
$$\left.+\frac{Q^2}{\Omega^4}+\frac{Q'}{\Omega^3}\bar\Omega-\frac{Q}{\Omega^3}\bar\Omega'-3\frac{Q}{\Omega^2}\bar\Omega^2\right). \tag{74b}$$

This result agrees with that given in Appendix II of Ref. 8. The method presented here can be extended to calculate higher-order terms, but as an illustration of the method the above calculation should serve well enough. We turn now to a complete solution of the wave equation.

VI. NUMERICAL SOLUTIONS AND DISCUSSION

We begin by reducing the second-order equation (63) for χ to two first-order coupled equations for α and β. This is done in the following way.[8] Differentiating (64) once and subtracting from it Eq. (65), one obtains an equation containing α' and β'. Then differentiating (65) once and substituting in (63), one obtains a second equation containing α' and β'. Solving these two equations for α' and β' in terms of α and β, one gets

$$\begin{aligned} \alpha' &= \frac{1}{2}\left(\frac{\Omega'}{\Omega} - i\frac{Q}{\Omega}\right) e_+^2\beta - i\frac{Q}{2\Omega}\alpha , \\ \beta' &= \frac{1}{2}\left(\frac{\Omega'}{\Omega} + i\frac{Q}{\Omega}\right) e_-^2\alpha + i\frac{Q}{2\Omega}\beta . \end{aligned} \tag{75}$$

To get equations for $|\beta|^2$, the amount of particle production, one then introduces the three real variables

$$\begin{aligned} s &= |\beta|^2 , \\ p &= \alpha\beta^* e_-^2 + \alpha^*\beta e_+^2 , \\ q &= i(\alpha\beta^* e_-^2 - \alpha^*\beta e_+^2) . \end{aligned} \tag{76}$$

After a little manipulation, the following system of three linear equations is obtained:

$$\begin{aligned} \frac{ds}{d\zeta} &= \frac{1}{2}\frac{\Omega'}{\Omega} p + \frac{1}{2}\frac{Q}{\Omega} q , \\ \frac{dp}{d\zeta} &= \frac{\Omega'}{\Omega}(1+2s) - \left(\frac{Q}{\Omega} + 2\Omega\right) q , \\ \frac{dq}{d\zeta} &= \frac{Q}{\Omega}(1+2s) + \left(\frac{Q}{\Omega} + 2\Omega\right) p . \end{aligned} \tag{77}$$

The initial conditions $s = p = q = 0$ for $\zeta = \zeta_0$ are specified under the assumption that no particles have been created before the time $\zeta = \zeta_0$. This would be the case should one make the spacetime flat at times prior to ζ_0. In the Minkowski-space epoch ($\zeta < \zeta_0$) the vacuum state is well defined and no particle creation occurs. This setup of initial conditions circumvents the previously mentioned serious problems that arise when one attempts to define the vacuum state near the cosmological singularity. It is quite artificial, as one commonly visualizes the universe at its earliest state as possessing a strong gravitational field and large curvature, far from being flat. Therefore, one should not expect the method used here as capable of yielding exact numbers of cosmological bearing; our main aim here is to understand the characteristic features of particle creation particular to anisotropic universes. For this purpose the present approach is fully suitable.

The quantities Ω, Q appearing in the above equations are functions of l_i and l_i': Q is a purely geometric quantity, while Ω is related to the mass of the particle (μ^2) and its energy (k^2), which are eigenvalues of the Helmholtz equation. Expressions for some low-lying levels in the mixmaster universe have been given explicitly in I in terms of l_i. The time dependence of l_i is, of course, solvable from Einstein's equations. Therefore, a combined solution of the field equation and the scalar wave equation should yield results for s, the amount of particle production.

A. The Kasner universe

We study first the simple Kasner universe,[24] which is a spatially homogeneous and anisotropic universe of Bianchi type I (flat). The three radii of the universe assume power-law time dependence,

$$l_i \sim t^{p_i} \quad (i = 1, 2, 3) , \tag{78a}$$

where the three indices satisfy the relations

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1 . \tag{78b}$$

Equivalently, in place of the triplet (p_1, p_2, p_3), one can use a single parameter u ($1 \leq u \leq \infty$) to describe a particular history of the universe. (See, e.g., Ref. 26).

The Helmholtz equation in the Kasner universe is separable in the basis functions

$$w_k(x) = e^{ikx} , \tag{79a}$$

and the corresponding eigenvalues are given by

$$k = \left(\frac{k_1^2}{l_1^2} + \frac{k_2^2}{l_2^2} + \frac{k_3^2}{l_3^2}\right)^{1/2} , \tag{79b}$$

where k_i are the components in the ith direction.

In what follows, the wave equation (63) is solved using the τ coordinate because the radii functions then assume a simple form

$$l_i = e^{\Lambda p_i \tau} , \quad V = l_1 l_2 l_3 = e^{\Lambda\tau} , \tag{78c}$$

where Λ is some constant (an expansion parameter). The quantities Ω^2 and Q are given by [Eqs. (20) and (25)]

$$\begin{aligned} \Omega_k^2 &= V^2(\mu^2 + k^2) , \\ Q^{(\tau)}_{\text{Kasner}} &= \tfrac{1}{6} V^2 R \\ &= \tfrac{1}{3}[(\rho_1 + \rho_2 + \rho_3)'' - (\rho_1'\rho_2' + \rho_2'\rho_3' + \rho_1'\rho_3')] , \end{aligned} \tag{80}$$

where $\rho_i \equiv l_i'/l_i$, and a prime now denotes differentiation with respect to τ. It can easily be checked that whenever l_i satisfies the vacuum Einstein equations (78c), $Q = 0$. Thus, in an empty space, the minimally and conformally coupled equations are identical.

The coupled equations (77) are solved on an IBM 360/91 computer using a fourth-order Runge-Kutta integration routine. Two Kasner solutions characterized by the parameter $u=1.25$ (case A) and $u=10$ (case B) have been studied. [The p_i parameters are correspondingly (−0.328, 0.590, 0.738) for case A and (−0.090, 0.099, 0.991) for case B.] The results of numerical integration, with the *ad hoc* assumption of a vacuum state at $\tau_0=-10$, are displayed in Figs. 2 and 3. One sees that for an expanding universe, there is always one radius of curvature decreasing and two increasing. In Fig. 2(a), $l_1 \gg l_2 \approx l_3$ depicts a prolate configuration. For the time interval considered it shows the evolution of a "long thin rod" into a nearly isotropic configuration, whereas in Fig. 3(a) $l_1 \approx l_2 \gg l_3$ depicts an oblate configuration and the evolution of a "large thin disk". In the solution of the wave equation we have chosen a particular low-lying symmetric mode $(k_1, k_2, k_3)=(1,1,1)$ for consideration, and the effects of geometric shape and mass on particle production are studied.

1. *Shape effect*

In case A particle production increases exponentially to a large quantity, while in case B it maintains an oscillatory behavior at a low level (recall that τ is an exponential time). The effect of geometric shape is apparent here. For one thing, the change of l_i is more drastic in case A than B. For another, the rapid contraction along the longer axis (in 1-direction) in case A is more apt to produce low-energy particles (long-wavelength excitation) than the small-scale slow variations in case B. As one can see from Figs. 2(b) and 3(b), before $\tau \sim -3$, when the change is most rapid, the production in one case is about three times the other. After $\tau \gtrsim -2$, both cases tend to a more symmetric configuration where the conformal flatness of space grows in effect and (massless) particle production is reduced.

2. *Mass effect*

Relative to the same background, in the same energy range, particles of higher mass are produced in less quantity. This result is in agreement with the conclusion reached earlier (Sec. IV C) for a nearly adiabatic situation. Compared to the short deBroglie wavelength of high-energy or high-mass particles, the background expands rather slowly. Thus, particle creation is scarce and remains nearly constant. In fact, in both cases as illustrated, the high mass ($\mu^2=100$) production approaches a constant as the volume of the universe becomes very large. As an example of the effect of conformal invariance, one also notices that the decrease in the production of massless particles is faster than that of higher mass.

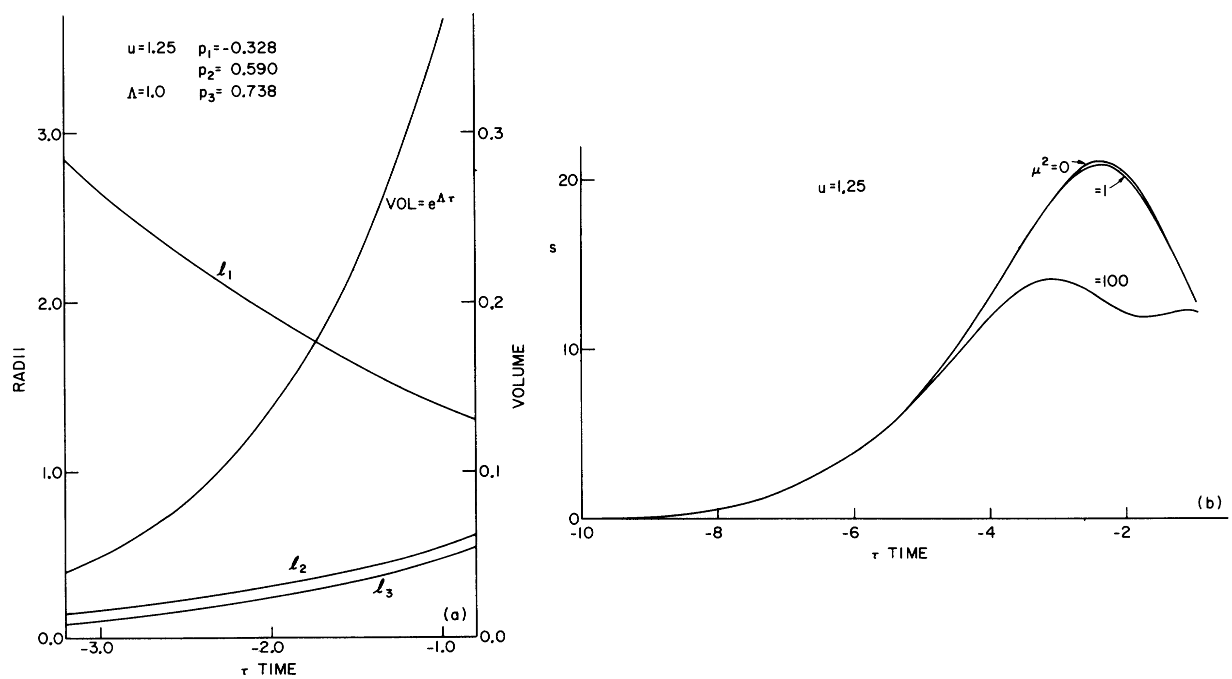

FIG. 2. (a) Kasner solution, $u=1.25$, a prolate configuration. l_i are the radii curvatures. VOL denotes the volume. (b) Particle production s as a function of τ time for particle mass $\mu^2=0,1,100$. Numerical integration starts from $\tau=-10$, before which the space is assumed to be flat and no particle creation takes place.

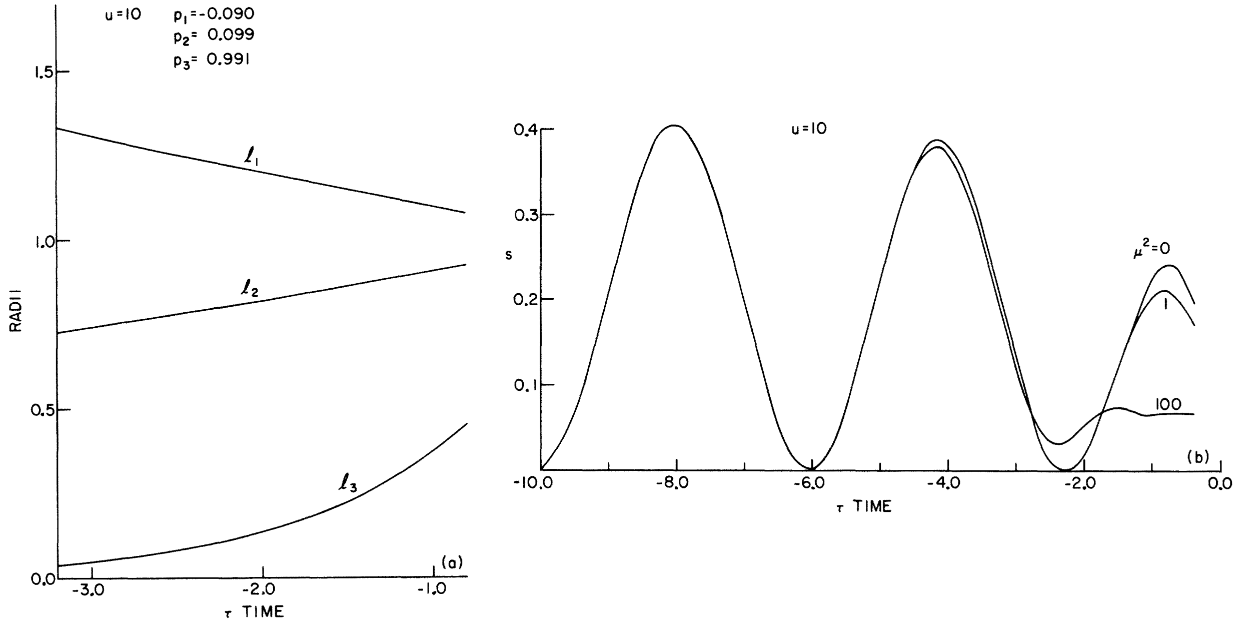

FIG. 3. (a) Kasner solution, $u = 10$, an oblate configuration. (b) Particle production s.

In the above examples, a symmetric mode is chosen for the reason that the influences of geometric shape and particle mass can be brought to clearer display with little interference from the directional level effects. If we take into consideration different (k_1, k_2, k_3), then anisotropic effects will become more apparent. For instance, in case A, owing to strong compression in the 1-direction, particles of higher k_1 components are expected to be enhanced. The level effect will be treated in more detail in the mixmaster case.

B. Mixmaster universe

The simpler Kasner solution serves as a good introduction to the discussion of the mixmaster universe, in which case the spatial geometry has discernable important effects on particle creation.

The solution of the Einstein field equations for the mixmaster universe has been discussed quite extensively.[25,26] We adopt the diagrammatic method introduced by Misner,[25,27] because it describes the dynamics in a more easily understandable way. A simple derivation of the field equations and the geometric quantities are collected in Appendix A of Ref. 16. The three radii of curvature l_1, l_2, l_3 define the configuration of the universe uniquely. An equivalent set consists of the volume V and the shape parameters β_+, β_- as defined in Fig. 2 of I. Hence, factoring out the volume dependence (which is a simple function of time), the dynamics of this anisotropic universe can be represented by a trajectory in the (β_+, β_-) shape-parameter space (the so-called minisuperspace[28]). For the type-I Kasner universe, the point which describes the configuration of the universe (the universe point) executes free motion in the shape parameter space. For the type-IX mixmaster universe, the anisotropy in the spatial curvature induces a "potential wall" of three-fold symmetry [Fig. 4(a)] within which the motion of the universe point is confined. The equipotentials expand outwards as the universe contracts and move inwards as the universe expands. The closer a universe point gets to the origin, the more it approximates an isotropic configuration. Along the $\alpha = 0°$, $120°$, $240°$ directions, the equipotential lines channel out. In this way, once the universe point migrates into any such "corner," it will keep bouncing for a while before it drifts out. In general, when the volume is increasing, the universe expands in two directions and contracts in one (Kasner behavior). As depicted in Fig. 4(b), a "bounce" corresponds physically to the shifting of the contraction direction. A "small oscillation" solution describes the situation in which the two nearly equal longer radii oscillate while the short one varies monotonically.

The initial conditions are chosen so that $s = 0$ at times $t < t_0$. The time t here denotes real cosmic time. Our initial time of integration $t_0 = 0$ is chosen so that the initial volume is equal to unity. (The scales of t and l_i are arbitrary and only relative.) Since t is a simple, one-valued function of η [Eq. (26)], t_0 can be chosen arbitrarily. The present choice of t_0 equal to zero is for simplicity's sake

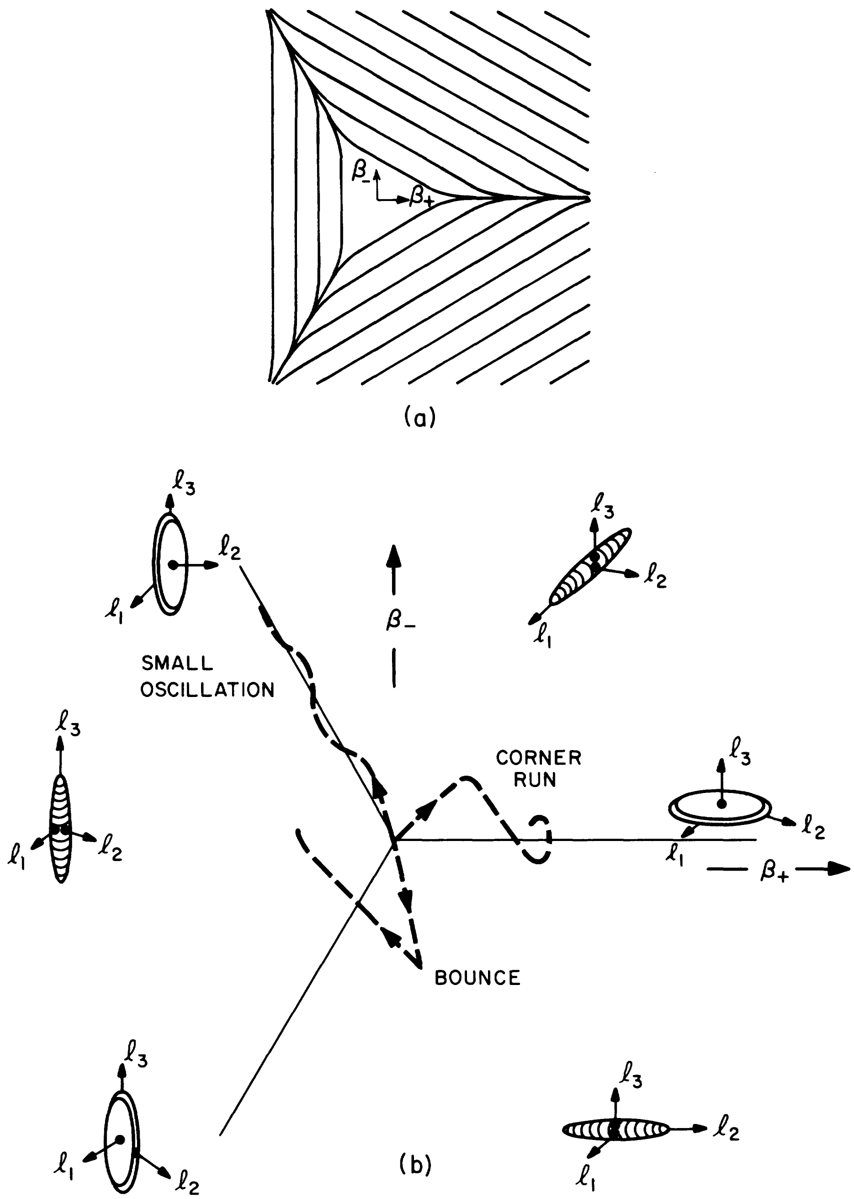

FIG. 4. (a) Anisotropy equipotential in the mixmaster universe (courtesy C. W. Misner). See Ref. 25 for explanations. β_+, β_- are the shape parameters. (b) Characteristic solutions of the mixmaster universe (courtesy D. J. Okerson). The dashed lines are the trajectories of the universe point in the shape-parameter space ("minisuperspace") describing different world histories.

and in no way implies the singularity. At this initial instant, the universe point is assumed to start at the origin of the shape-parameter space, where $(\beta_+, \beta_-)_0 = (0, 0)$ corresponds to an isotropic configuration. Three selected sample mixmaster solutions are shown in Figs. 5(a), 6(a), and 7(a), with initial values $(\beta_+', \beta_-')_0 = (2, 1), (-1, 2), (1, -2)$, respectively. In all three cases, the initial velocities of the universe point $\beta' = (\beta_+'^2 + \beta_-'^2)^{1/2}$ are set equal. The case (2, 1) in Fig. 5(a) corresponds to a "corner run" solution, but after three bounces against the potential wall, the universe point changes direction and drifts out. Notice that the amplitude of the oscillations decrease. This is because the equipotentials contract inwards with an expansion of the universe, thus "pushing in" the bounces with them. The behavior of the three radii are shown in Fig. 5(b). One sees that the 1-2 radii oscillate in larger amplitude while the 3 radius decreases in time. In Fig. 6(a), with an initial direction (−1, 2) set only slightly deviated from the 2 symmetry axis, the universe point

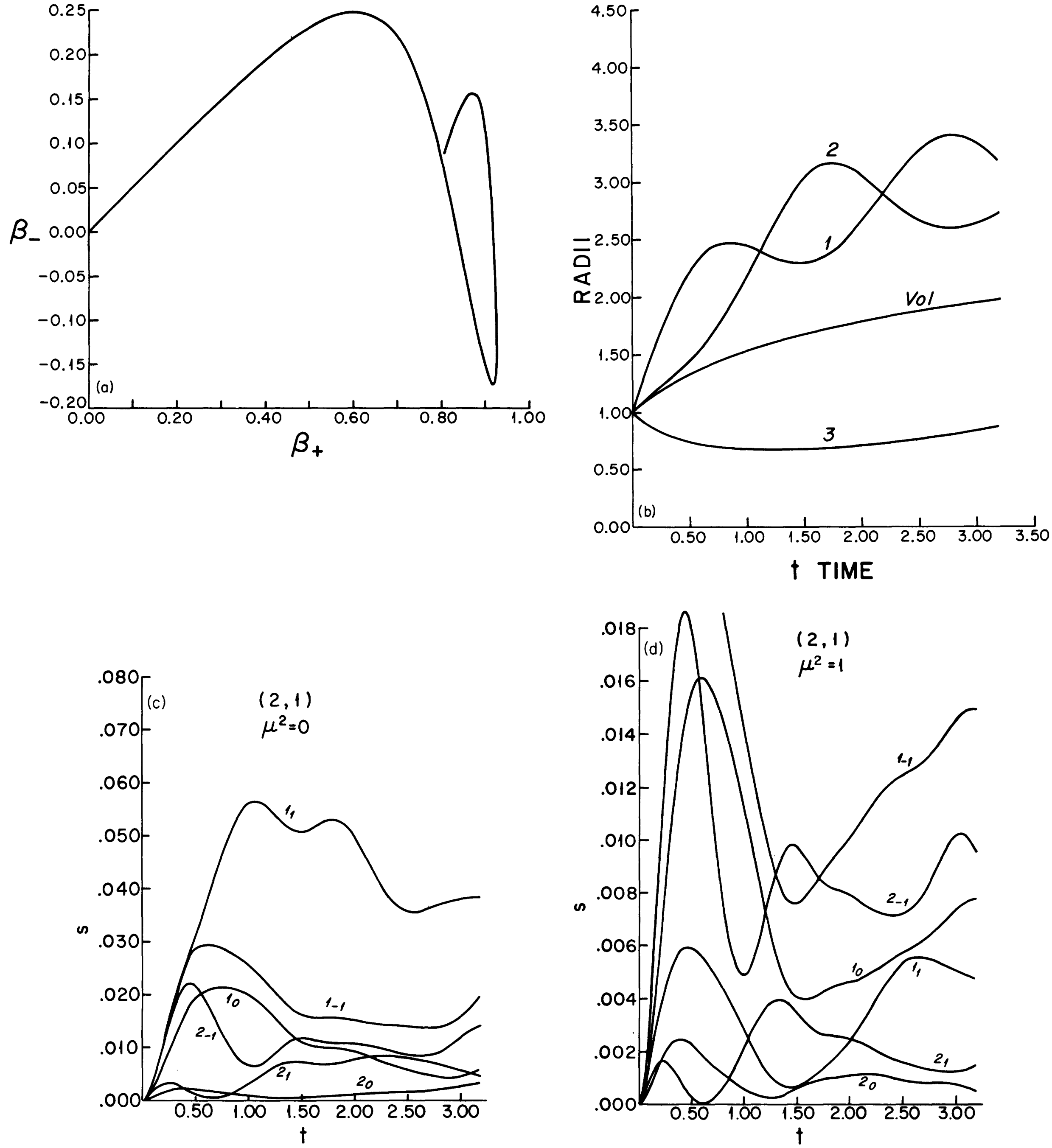

FIG. 5. (a) A "corner run" solution. The world trajectory starts at $(\beta_+, \beta_-)_0 = (0, 0)$, an isotropic configuration, with initial velocity $(\beta_+', \beta_-')_0 = (2, 1)$. A prime denotes $d/d\eta$. (b) Curvature radii and volume as a function of cosmic time t. The initial time of integration is set arbitrarily at $t_0 = 0$. It is not the cosmological singularity. (c) Particle production s for mass $\mu^2 = 0$ in six uncoupled levels labeled by J_K. (d) Particle production s for $\mu^2 = 1$.

drifts deep into the two-channel. Physically, this corresponds to a nearly oblate configuration [Fig. 6(b)] where two expanding major radii (1 – 3) oscillate in turn and the minor radius (2) contracts monotonically. The picture is a small oscillating "pancake." If the initial direction of the previous run is reversed, i.e., $(\beta_+', \beta_-')_0 = (1, -2)$, one sees in Fig. 7(a) that the universe point bounces onto the perpendicular potential wall in the 2-direction and gets reflected. Afterwards, it continues to travel until it executes a second bounce off the wall in the 1-direction. These bounces occur in

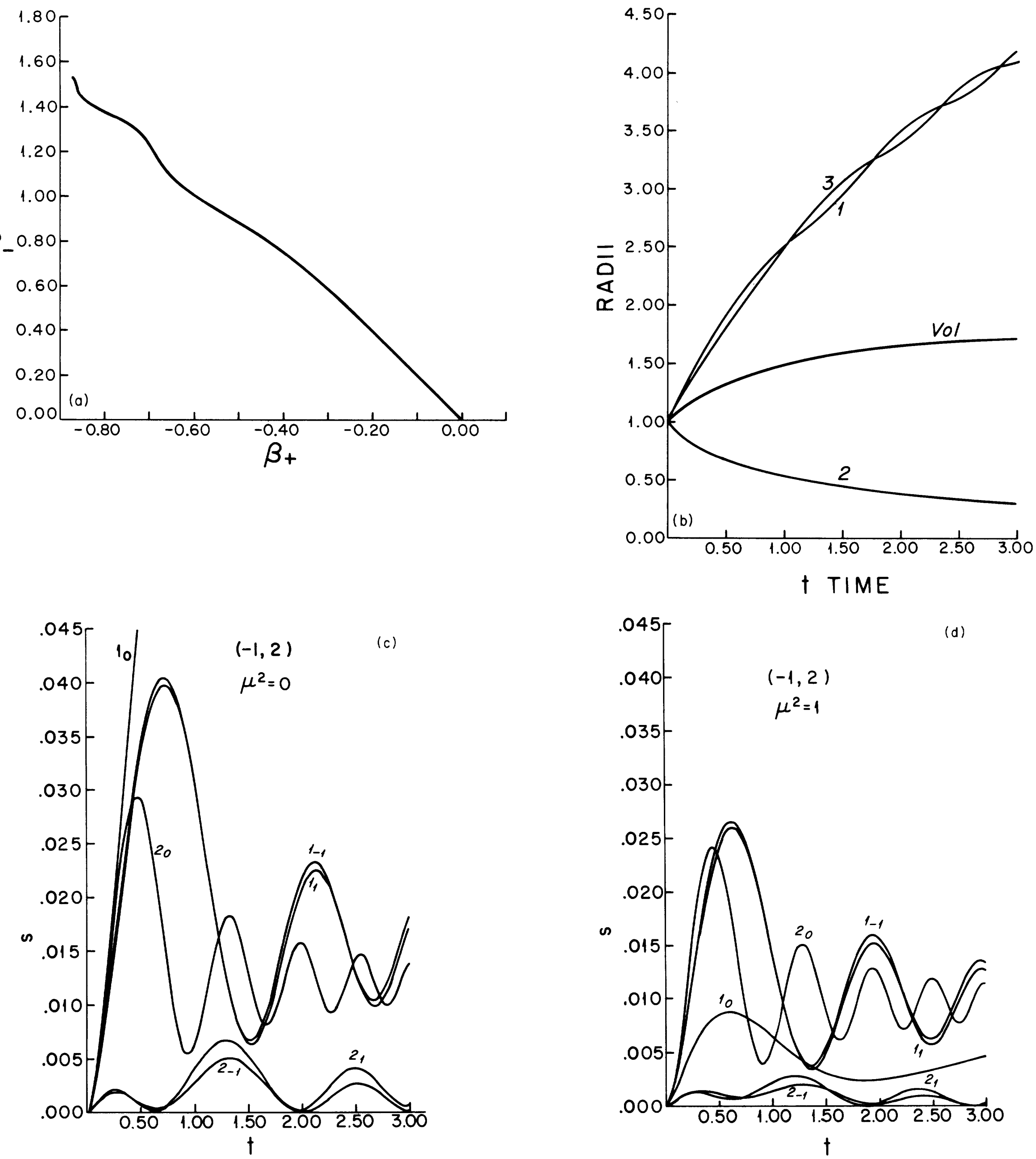

FIG. 6. (a) A "small oscillation" solution $(\beta_+', \beta_-')_0 = (-1, 2)$. (b) Curvature radii and volume functions. (c) Particle production for $\mu^2 = 0$. (d) Particle production for $\mu^2 = 1$.

Fig. 7(b) at $t \sim 0.6$ and 2.8 where l_2 and l_3 are at their respective maxima. The two cases 6(a) and 7(a) are characteristic solutions of the mixmaster universe and are called the "small oscillation" and the "bounce" solutions, respectively.

The wave equation (77) is solved for the six uncoupled modes $J = 1, 2,\ K = -1, 0, 1$. Results are shown in Figs. 5–8. Again, we divide our discussion into three distinctive effects.

1. Shape effect

Compare the massless particle production curves in Figs. 5(c), 6(c), and 7(c). Here the effect

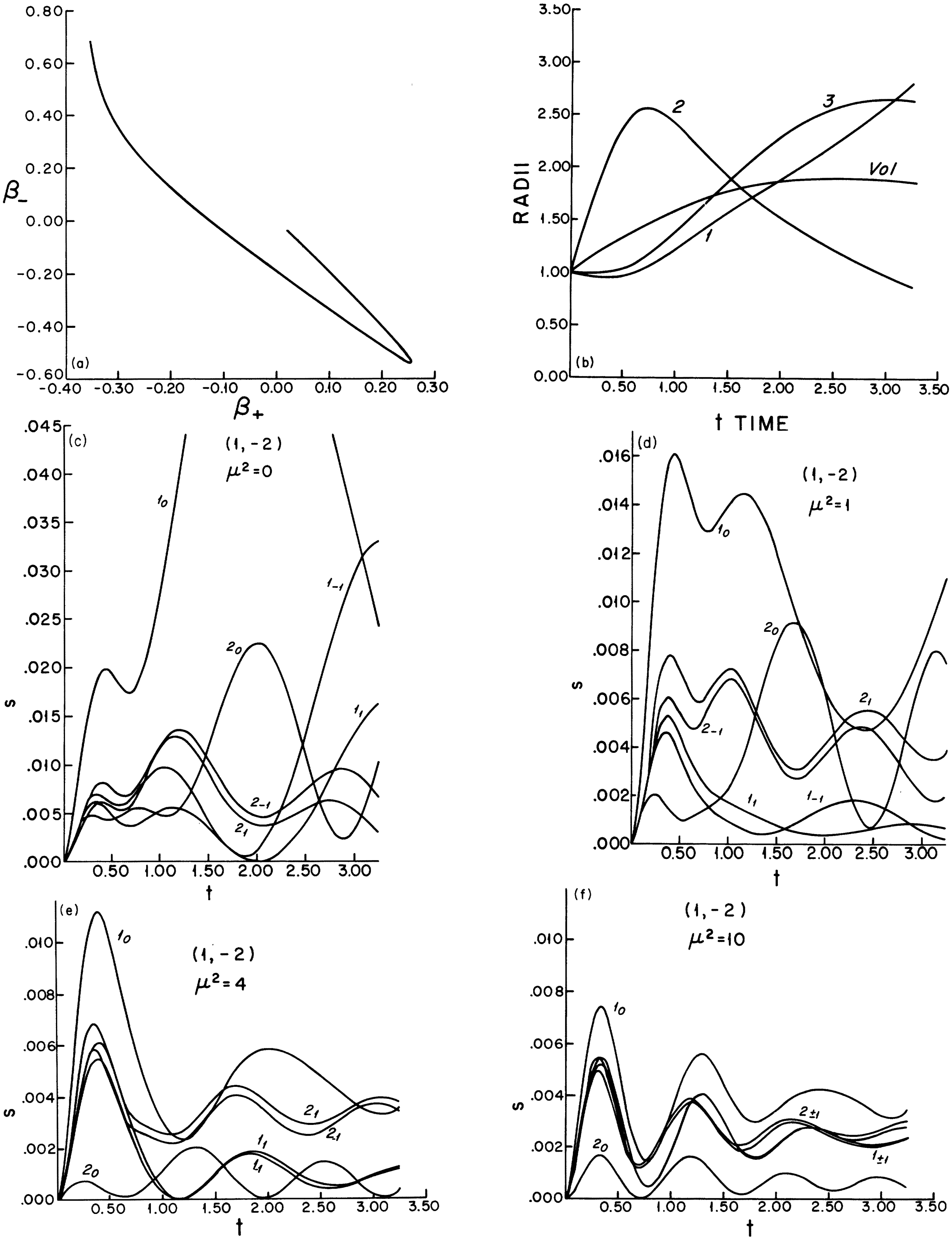

FIG. 7. (a) A "bounce" solution $(\beta_+', \beta_-')_0 = (1, -2)$. (b) Curvature radii and volume functions. (c) Particle production for $\mu^2 = 0$. (d) Particle production for $\mu^2 = 1$. (e) Particle production for $\mu^2 = 4$. (f) Particle production for $\mu^2 = 10$.

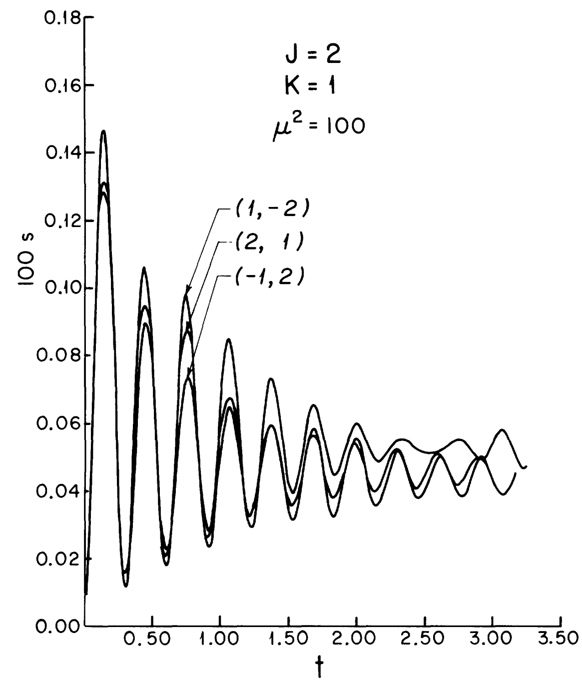

FIG. 8. Particle production in the $J=2$, $K=1$ level for $\mu^2=100$ for the three characteristic solutions $(\beta_+', \beta_-')_0 = (2,1)$, $(-1,2)$, and $(1,-2)$.

of the geometric shape does not appear as drastic as that in the flat Kasner universe. This is because the addition of the curvature tends to soften the extremities of the time dependence. (Thus, near the cosmic singularity when spatial components of the curvature become small compared to their time derivatives, the anisotropic effect magnifies in significance.) However, qualitative features characteristic of the different eras do show up. In the "small oscillation" solution [Fig. 6(c)] the oscillatory feature of the production curve reflects the periodic behavior of the background. There is more production in the first cycle, because the change in the short radius l_2 is more rapid there. Deep into the corner, the contraction turns more gradual and the production lessens. This phenomenon occurs in the bounce solution too. In Fig. 7(c), we see that the second bounce elicits more production. This happens when the universe point begins to migrate into the channel and the 2-radius starts a steep contraction [Fig. 7(b)]. Thus, the "squeezing" of the flat "pancake" enhances particle production with large momentum in the perpendicular direction.

2. *Level effect*

The expressions for the energy levels in the mixmaster universe can be found in Table I of paper I. Because of the anisotropic background, the directional effect of levels shows up manifestly. The remarks in the last subsection (VI A 2) on the Kasner universe also apply here. For example, in the small oscillation solution [Fig. 6(c)], within the triplet $1_{\pm 1,0}$, the one level 1_0 which corresponds to the direction perpendicular to the disk has the highest production; the other two levels $1_{\pm 1}$ which correspond to the two oscillating directions are oscillatory and nearly degenerate. The same happens for the $J=2$ multiplets, although of a smaller amplitude. A similar situation occurs in the corner run solution (2, 1), although the axes of symmetry are now in the 3-direction, thus making the 1_1 level extraordinary compared to 1_0 and 1_{-1} levels.

3. *Mass effect*

These effects are interrelated. In general, one expects to find large production of particles of rest mass μ in those modes of energy k when the time rate of change of the background (ω_g) is comparable with the natural frequency ω_0 of the system, $\omega_0=(k^2+\mu^2)^{1/2}$ (for minimal coupling). If one measures the change of background by $\omega_g \simeq |\dot{l}/l|$, then the optimal condition for particle creation is $\omega_0 \lesssim \omega_g$. Therefore, under the same background, production of particles of higher mass or energy will be relatively fewer. This corresponds in the wave picture to the nearly adiabatic situation where high frequency scalar waves, being considered as "small ripple" perturbations on the background, suffer little parametric amplification from the vibration of the universe. For high enough mass ($\mu^2 \gg k^2$, Q) the effects due to differences in shape and level are much obliterated. Compare Figs. 7(c)–7(f), where production is plotted for various masses $\mu^2=0, 1, 4, 10$, respectively. Observe that the over-all production decreases as particle mass increases. At the same time, the directional effects due to the levels gradually disappear. In Fig. 8, particle production in a particular level 2_1 at $\mu^2=100$ is plotted for all three solutions. It is seen that they nearly coincide. The five other levels under study also show the same behavior. Average amount of production reaches a low level of constant value ($\sim 5\times 10^{-4}$). As one scans through Figs. 7(c)–7(f) and 8, one sees the oscillatory behavior gradually developing. The period is related to the characteristic time $1/\mu$ of the particle. For very heavy mass or high energy, the production curve will approach a horizontal line. The high-energy behavior will be discussed further in paper III, within the framework of asymptotic-level analysis.

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neglecting all internal reflections due to discontinuity in the rth derivative of the refraction function W. Hence, correct to the rth order, the wave function can be approximated sufficiently by the positive-frequency solution $\exp(-i\int K_r dt)/K_r^{1/2}$. Physically, this means that the background varies slowly and smoothly enough in the rth derivative that no particle creation occurs to that order (absence of wave reflection). The particle operators are well defined to this limit. This is the basic idea behind the adiabatic quantization method in Ref. 21.

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Corresponding-states approach to nuclear and neutron-star matter

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The properties of nuclear matter and dense neutron-star matter are studied by an approach which largely avoids the microscopic assumptions of nuclear-matter theory. The method is empirical, employing an extended form of the law of corresponding states to deduce the properties of nuclear systems from those of laboratory substances such as helium. It is possible to predict the solidification pressure and density, the compressibility, and the critical temperature of nuclear and neutron-star matter. As previously reported, a comparatively low solidification pressure is found for neutron-star matter, implying a solid core for most neutron stars.

I. INTRODUCTION

There are several reasons for attempting an empirical approach to nuclear matter. The simplicity and relative transparency of the method to be described are worthwhile in themselves, and may allow application to phenomena barely accessible to conventional microscopic techniques. Detailed models of the nucleon-nucleon interaction may be avoided as may the assumptions and results of standard (Brueckner) nuclear matter theory. Indeed, the present approach may be regarded as an independent—albeit weak—test of such theories. By working from the experimental properties of real materials, we let nature compute the bulk of the many-body effects, and only have to concern ourselves with differences between substances. We pay for these advantages of an empirical approach with a large uncertainty in our results, and are only able to make order-of-magnitude estimates. The uncertainties come partly from the approximations used, but also significantly from lack of sufficient source data; there are few real quantum systems.

Our procedure is based on an extension of the quantum law of corresponding states first proposed by de Boer[1] and successfully used by him[2] to predict the properties of He^3 before any was available for study. de Boer's model is not directly applicable to nuclear matter because of the considerable difference between nuclear forces and the Van der Waals forces typical of laboratory quantum systems such as helium. However, we are able to generalize the corresponding-states law to apply to a larger class of interactions by means of an equivalent density transformation. We first apply the extended model to symmetric nuclear matter ($\frac{1}{2}$ neutrons, $\frac{1}{2}$ protons), and then to neutron-star matter, the latter causing more difficulty as less experimental information is available.

Our chief goal is to predict the crystallization pressure and density of neutron-star matter, which bear considerably on the structure and dynamics of neutron stars. A preliminary account of this work[3] has already appeared, based on a less detailed model than described here. Several other authors[4-10] have also examined the crystallization question, using more microscopic techniques. Clark and Chao[5] have followed an approach close to our own—based on corresponding states—but