lead to a quantum theory, with $\hbar$ as the derived constant.

CONCLUSION

We have shown, via a simplified model, that one should not dismiss *a priori* the Planck length as being irrelevant to observational physics. Breakdown of special relativistic dynamics in this model occurs at $10^{19}-10^{20}$ eV, within the observational regime of high-energy cosmic-ray events. Possible manifestations of the lattice structure could be anisotropies in distribution, cutoffs in energy, and anomalous stability of composite systems.

ACKNOWLEDGMENT

Appreciation is expressed to Battelle Seattle Research Center for hospitality and for preparation of this manuscript.

*Permanent address.

[1]See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 43.
[2]J. Linsley, Phys. Rev. Lett. 10, 146 (1963).
[3]K. Suga, H. Sakuyama, S. Kawaguchi, and T. Hara, Phys. Rev. Lett. 27, 1604 (1971).
[4]W. Heisenberg, Z. Physik 110, 251 (1938).
[5]H. S. Snyder, Phys. Rev. 71, 38 (1947).
[6]E. J. Hellund and K. Tanaka, Phys. Rev. 94, 192 (1954).
[7]A. Schild, Can. J. Math. 1, 29 (1949).
[8]E. L. Hill, Phys. Rev. 100, 1780 (1955).
[9]C. Lanczos, Phys. Rev. 134, B476 (1964); J. Math. Phys. 7, 316 (1966).
[10]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1965), Vol. III, Chaps. 13 and 16.
[11]K. Greisen, Phys. Rev. Lett. 16, 748 (1966).
[12]G. T. Zatsepin and V. A. Kuzmin, Zh. Eksp. Teor. Fiz. Pis'ma Red. 4, 114 (1966) [JETP Lett. 4, 78 (1966)].
[13]F. W. Stecker, Phys. Rev. Lett. 21, 1016 (1968).
[14]M. A. Markov, Zh. Eksp. Teor. Fiz. 51, 878 (1966) [Sov. Phys.—JETP 24, 584 (1967)].

PHYSICAL REVIEW D VOLUME 9, NUMBER 12 15 JUNE 1974

Masses and spins in curved space-time*

S. Malin

Department of Physics and Astronomy, Colgate University, Hamilton, New York 13346

(Received 10 January 1974)

The Poincaré and de Sitter groups are well known to imply a deep connection between the structure of space-time and particle properties (mass and spin) for cosmological models with constant four-dimensional curvature. An analogous relationship is constructed for any isotropic *spatially* homogeneous cosmological model. The spin is proved to be constant; a time variation of the masses of all physical systems may well exist. It is shown that the variation $m \sim t^{-1}$, which arises in the context of the Friedmann model, is not precluded by present observational and experimental evidence.

I. INTRODUCTION

The investigation of the representations of the Poincaré group[1] implies a deep relationship between the geometry of space-time and the fundamental properties of particles. The Poincaré group itself is the group of motion in flat space-time. It is indeed remarkable that the eigenvalues of its Casimir operators can be interpreted in terms of the masses and spins of particles.

Investigations of the representations of the de Sitter group[2,3] have revealed a similar relationship for curved space-time with constant curvature. The de Sitter group is the group of motion of the de Sitter space-time, and the eigenvalues of its Casimir operators can, again, be interpreted in terms of masses and spins of particles (Sec. II).[4]

On the other hand, cosmological models, the curvature of which is not a constant, do not have corresponding groups of motion.[5] Consequently, the group-theoretical approach, which works so well for flat and constant-curvature space-times, cannot be applied, and one is left in the dark as to the relationship between the geometrical structure and the fundamental properties of particles. This problem is all the more intriguing because all the cosmological models with nonvanishing matter density and zero cosmological constant fall under this category.

The present paper presents a derivation of such a relationship for cosmological models which are *isotropic, spatially homogeneous* and otherwise arbitrary. It is shown (Sec. III) that for such models it is always possible to define, at any time t_0, *an associated de Sitter space-time*, having the following properties:

(a) The constant (four-dimensional) curvature of the associated de Sitter space-time is the same as the four-dimensional curvature of the cosmological model at time t_0.

(b) The hypersurface t = const of the associated de Sitter space-time, for an appropriately chosen t, is identical with the hypersurface t_0 = const of the cosmological model.

It is then postulated that *the masses and spins of particles in a cosmological model at time* t_0 *are given by the eigenvalues of the Casimir operators in the associated de Sitter space-time.* The hypothesis is based on the following consideration: In spite of the curvature of space-time, the eigenvalues of the Casimir operators of the Poincaré group do correspond to the masses and spins of particles. This amazing correspondence can be understood if the flat space-time of the Poincaré group is taken as a limiting case of curved space-time, and the Poincaré group representations are obtained from the de Sitter group representations by the process of contraction,[6-8] as the curvature K of the de Sitter space approaches zero. It seems, therefore, that the associated de Sitter space-time, as defined in Sec. II, is a better approximation to the reality of the cosmological models, since the limit $K \to 0$ is not taken.

The consequences of this hypothesis are investigated in Sec. IV and lead to the possibility that *the masses of particles* (and, in fact, all physical systems) *vary in time and are inversely proportional to the four-dimensional curvature of the cosmological model.*

This time variation is model-dependent; in the representative case of a Friedmann universe the variation $m \sim t^{-1}$ is obtained. In Sec. V the observational and experimental consequences of a time variation of this nature are investigated, following a recent review paper by Dyson.[9] It is shown that the wealth of available data from beta-decay experiments, planetary orbit observations, interplanetary ranging experiments, solar and stellar evolution considerations as well as cosmological data does not preclude such a time variation of all masses.

II. THE CASIMIR OPERATORS OF THE DE SITTER AND POINCARÉ GROUPS

The de Sitter group O(4, 1), the group of motion in a de Sitter universe, is also the group of transformations in five-dimensional Euclidean space with coordinates $\xi_1, \ldots, \xi_5$ which leave invariant the hypersurface

$$\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 = b^2 . \tag{2.1}$$

Following Gürsey,[3,10] we denote the infinitesimal operators of the de Sitter group by J_{ab} ($a, b = 1, \ldots, 5$). They obey the following commutation relations[11]:

$$-i[J_{ab}, J_{cd}] = \delta_{ad} J_{bc} - \delta_{ac} J_{bd} + \delta_{bc} J_{ad} - \delta_{bd} J_{ac} . \tag{2.2}$$

The Casimir operators of the group are given by

$$C_1 = J_{ab} J_{ab} , \tag{2.3}$$

$$C_2 = U_a U_a , \tag{2.4}$$

where the quantities U_a are defined by

$$U_a = \epsilon_{abcde} J_{bc} J_{de} \tag{2.5}$$

(ϵ_{abcde} is the totally antisymmetric tensor with five indices).

The Poincaré group is obtained from the de Sitter group by the process of contraction,[6-8] as $b \to \infty$. Let us define

$$\Pi_\mu = \frac{1}{b} J_{5\mu} , \tag{2.6}$$

$$V_\lambda = -\tfrac{1}{2} \epsilon_{5\lambda\kappa\mu\nu} \Pi_\kappa J_{\mu\nu} , \tag{2.7}$$

$$W_a = \frac{1}{8b} U_a , \tag{2.8}$$

$$I_1 = -\frac{1}{2b^2} C_1 , \tag{2.9}$$

$$I_2 = -\frac{1}{64b^2} C_2 = -W_a W_a = -V_\lambda V_\lambda - W_5^2 . \tag{2.10}$$

If we furthermore define

$$P_\mu = \lim_{b \to \infty} \Pi_\mu \tag{2.11}$$

then it follows from Eq. (2.2) that

$$[P_\mu, P_\nu] = 0 , \tag{2.12}$$

$$-i[P_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu} P_\nu - \delta_{\lambda\nu} P_\mu , \tag{2.13}$$

and therefore the infinitesimal operators $P_\mu, J_{\mu\nu}$ are the generators of the Poincaré group. Furthermore, by Eqs. (2.6), (2.9), and (2.11)

$$I_1 = -\Pi_\mu \Pi_\mu - \frac{1}{2b^2} J_{\mu\nu} J_{\mu\nu} , \tag{2.14}$$

$$\lim_{b \to \infty} I_1 = P_\mu P_\mu = m^2 , \tag{2.15}$$

where m is the mass, and by Eqs. (2.7), (2.8), and (2.10)

$$\lim_{b\to\infty} I_2 = m^2 s(s+1), \tag{2.16}$$

where s corresponds to the spin, for a system with nonvanishing rest mass.

Equation (2.14) is particularly interesting. It shows that in de Sitter universe the invariant quantity is not the mass of a particle, but some combination of the mass and the total angular momentum. This feature of the de Sitter space-time is of great theoretical interest, but of no practical significance, because, since b is the radius of curvature of the universe, the second term of (2.14) is numerically negligible compared with the first. [For an electron the ratio of the two terms is

$$\left(\frac{L}{b m_e c}\right)^2 \sim 10^{-77}\left(\frac{L}{\hbar}\right)^2, \tag{2.17}$$

where L is the angular momentum of an electron, m_e is its mass, and $b \sim 10^{10}$ light years.]

III. ISOTROPIC AND SPATIALLY HOMOGENEOUS COSMOLOGICAL MODELS AND THE ASSOCIATED DE SITTER UNIVERSES

As pointed out by Robertson and Noonan,[12] "the observational evidence allows us to assume the existence of a congruence of fundamental world lines which fills the universe." If the universe is also assumed to be isotropic and spatially homogeneous, then it is possible to choose a canonical coordinate system (t, x_1, x_2, x_3) such that the metric tensor in the coordinate system is of the form[13]

$$ds^2 = dt^2 - S^2(t)\,\frac{dx_1^2 + dx_2^2 + dx_3^2}{(1+\frac{1}{4}kr^2)^2}, \tag{3.1}$$

where

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \tag{3.2}$$

and on the fundamental world lines $ds^2 = dt^2$. The coordinate t in the canonical frame of reference is called "cosmic time."

The isotropic, spatially homogeneous cosmological models are classified into the following three well-known types, according to the sign of k:

(i) $k > 0$ *(spherical space).* The hypersurfaces $t =$ const have constant positive curvature.

(ii) $k = 0$ *(Euclidean space).* The hypersurfaces $t =$ const have zero curvature.

(iii) $k < 0$ *(pseudospherical space).* The hypersurfaces $t =$ const have constant negative curvature.

The four-dimensional curvature of these models is, in general, a function of the cosmic time t and is given, in terms of $S(t)$, by

$$K(t) = -\tfrac{1}{12} g^{\alpha\beta} R_{\alpha\beta} = -\frac{1}{2S^2}(S\ddot{S} + \dot{S}^2 + k), \tag{3.3}$$

where $R_{\alpha\beta}$ is the Ricci tensor.

All three types have, as special cases, models for which the four-dimensional curvature K is constant and negative, $K = -1/b^2$:

(i) The Lanczos universe,

$$S(t) = b\cosh\frac{t}{b}, \tag{3.4}$$

describes a universe which contracts to a volume $2\pi^2 b^3 C^3$ and then expands.

(ii) The de Sitter universe

$$S(t) = \frac{1}{c}\,e^{t/b} \tag{3.5}$$

[$S(t)$ is determined only up to a constant; the constant $1/c$ is chosen by convention].

(iii) The function

$$S(t) = b\sinh\frac{t}{b} \tag{3.6}$$

describes a universe which contracts to a singular event at $t = 0$ and then expands.

Because of the uniqueness theorem for space-time with constant four-dimensional curvature, *all the above-mentioned three special cases can be transformed into each other, and are, in fact, for a given value of* $K = -1/b^2$, *the same universe.* We will refer to this universe as the de Sitter universe. The three space-times are different from each other only in that different congruences are used for the fundamental world lines.

Given a cosmological model, which is isotropic and spatially homogeneous, we now proceed to define the associated de Sitter universe at a given cosmic time t_0.

An isotropic, spatially homogeneous cosmological model is given by a function $S(t)$ in Eq. (3.1). Denote its value and the value of its first and second derivatives at a given time t_0 by

$$S_0 \equiv S(t_0),\quad \dot{S}_0 \equiv \frac{d}{dt}S(t)\Big|_{t=t_0},\quad \ddot{S}_0 = \frac{d^2}{ds^2}S(t)\Big|_{t=t_0}. \tag{3.7}$$

The associated de Sitter universe at time t_0 is now defined as the de Sitter universe with the four-dimensional curvature

$$K_0 = -\frac{1}{2S_0^2}(S_0\ddot{S}_0 + \dot{S}_0^2 + k) \equiv -\frac{1}{b_0^2}. \tag{3.8}$$

Because of Eq. (3.3) this de Sitter universe has the same four-dimensional curvature as the cosmological model at cosmic time t_0. Furthermore, because of Eqs. (3.4)–(3.6), the hypersurface t_0 = const of the cosmological model is the same as the hypersurface t_D = const of the associated de Sitter universe, where t_D is given by

$$t_D = \begin{cases} b_0 \cosh^{-1}\left(\dfrac{S_0}{b_0}\right), & \text{if } k>0 \\ \dfrac{1}{2b_0}\ln(c^2 S_0), & \text{if } k=0 \\ b_0 \sinh^{-1}\left(\dfrac{S_0}{b_0}\right), & \text{if } k<0 . \end{cases} \qquad (3.9)$$

In the general case of cosmological models, the four-dimensional curvature and the associated de Sitter universe vary as a function of cosmic time.

The concept of associated de Sitter universes will now be used to establish the relationship between the geometrical structure of space-time and the masses and spins of particles (or any other physical system). We postulate that *the masses and spins of particles in a cosmological model at cosmic time t_0 are given by the eigenvalues of the Casimir operators in the associated de Sitter universe.*

The postulate is based on the following line of thinking: Galactic observations clearly show that the four-dimensional curvature of the universe does not vanish; the universe is not flat. Nevertheless, the Casimir operators of the Poincaré group, the group of motion of flat space-time, do correspond to masses and spins of particles. This correspondence can be readily understood, because the Poincaré group is obtained from the de Sitter group, the group of motion in curved space-time with constant negative four-dimensional curvature, by the process of group contraction (see Sec. II). The associated de Sitter universe is a much closer approximation to the cosmological model; the process of group contraction is not used, and one can expect to obtain a better understanding of how fundamental properties of particles are related to the geometrical structure of space-time.

IV. MASSES AND SPINS IN AN ISOTROPIC, SPATIALLY HOMOGENEOUS UNIVERSE

It was suggested in Sec. III that a relationship between the geometrical structure of an isotropic, spatially homogeneous universe U with varying four-dimensional curvature, and the masses and spins of particles can be established by assuming that the masses and spins are obtained from the Casimir operators in the associated de Sitter universe D_t (t-cosmic time in U). Although the Casimir operator I_1 [Eq. (2.14)] includes a term which corresponds to total angular momentum, this term is numerically negligible [Eq. (2.17)], and for all practical purposes I_1 can be considered as corresponding to the mass, and $I_1^{-1}I_2 = C_1^{-1}C_2$ as corresponding to the spin.

Consider the universe U at cosmic time t_0, with its associated de Sitter universe D_0. Let the mass and spin of a particle[14] in U correspond, at the time t_0, to the eigenvalues $m_0{}^2$ and $m_0{}^2 s_0(s_0+1)$ of the operators I_1 and I_2, or, equivalently, to the eigenvalues

$$c_1^0 = -2b_0 m_0{}^2 , \qquad (4.1)$$

$$c_2^0 = -64 b_0{}^2 m_0{}^2 s_0(s_0+1) \qquad (4.2)$$

of the operators C_1 and C_2, where b_0 is the radius of curvature of the de Sitter universe D_0 [see Eqs. (2.9) and (2.10)]. The following question now arises: As the particle traces its world line in the universe U, what will be the eigenvalues of the Casimir operators C_1 and C_2 corresponding to the associated de Sitter universe D_1 at cosmic time t_1?

Considerations of continuity provide a partial answer. Since the radius b varies continuously as a function of cosmic time t, while the spectrum of the operator $C_1{}^{-1}C_2$ is discrete, the eigenvalue of $C_1{}^{-1}C_2$ does not vary, i.e.,

$$s_1(s_1+1) = s_0(s_0+1); \qquad (4.3)$$

the spin s is constant.

As far as the mass is concerned, no further assertion can be made without a further hypothesis. The following hypothesis seems, however, quite natural: *As a particle traces a world line in the universe U, the eigenvalue of the Casimir operator C_1 corresponding to the associated de Sitter universe does not vary as a function of cosmic time.* Because of Eq. (4.3) the eigenvalues of the Casimir operator C_2 will also be constant.

This hypothesis is most natural because the eigenvalues of C_1 and C_2 [in contradistinction to, say, I_1 and I_2, Eqs. (2.9) and (2.10)] are dimensionless.

By Eqs. (2.9), (2.14), and (2.17), if the constant eigenvalue of the Casimir operator C_1 associated with a particle is denoted by c_1, then the mass of the particle will vary with cosmic time according to the formula

$$m(t) = -2^{-1/2}[b(t)]^{-1}c_1 , \qquad (4.4)$$

where $b(t)$ is the radius of curvature of the de Sitter universe, and D_t is associated with the universe U at cosmic time t. As mentioned in the previous section, the universe U is completely determined by function $S(t)$ in Eq. (3.1). In terms of the function $S(t)$ the radius of curvature $b(t)$ is given by [Eq. (3.3)]

$$K(t) = -\frac{1}{b^2(t)} = -\frac{1}{2S^2(t)}[S(t)\ddot{S}(t) + \dot{S}^2(t) + k]. \quad (4.5)$$

Equations (4.4) and (4.5) imply that the mass of any physical system will vary as a function of cosmic time, unless the universe has a constant four-dimensional curvature. The exact rate of change depends, obviously, on the choice of cosmological model. For a Friedmann universe, for example,

$$S(t) = S_0 t^{2/3}, \quad (4.6)$$

the mass formula becomes

$$m(t) = m_0/t, \quad (4.7)$$

where[15]

$$m_0 = -(\sqrt{2}/9)c_1. \quad (4.8)$$

Other general-relativistic cosmological models give results which are not substantially different. If one takes for the age of the universe the current value of about 10^{10} years,[16] it follows from Eq. (4.7) that

$$\frac{\dot{m}}{m} \sim 10^{-10} \text{ per year}. \quad (4.9)$$

The observational and experimental evidence concerning a time variation of the masses of all physical systems of the order of magnitude (4.9) will be discussed in the next section. Let us note here that such a time variation of the masses in an expanding universe is certainly in line with Mach's principle: If the inertial mass of a system is due to the influence of all the other masses in the universe, one would expect the inertial mass to decrease as all the other masses get further and further away.

V. EXPERIMENTAL AND OBSERVATIONAL EVIDENCE

In a recent review article Dyson[9] summarized the evidence concerning time variation of the fundamental physical constants. In the context of the present work, there are no grounds to doubt the constancy of c (the speed of light), h (Planck's constant), e (the electron charge), g (Fermi's constant of weak interaction, or G (the constant of gravitation). We are exploring, however, the consequences of the variation in time of all masses, including the mass of the proton m_p.

Out of the six constants c, h, e, m_p, g, G, Dyson constructs three dimensionless ratios:

$$\alpha = e^2/\hbar c, \quad (5.1)$$

$$\beta = g m_p{}^2 c/\hbar^3, \quad (5.2)$$

$$\gamma = G m_p{}^2/\hbar c. \quad (5.3)$$

He also considers among the fundamental constants H (Hubble's constant) and ρ (the mean density of mass in the universe), whose numerical values are uncertain by about factors of 2 and 1000, respectively. With H and ρ included two further dimensionless ratios can be constructed:

$$\delta = H\hbar/m_p c^2, \quad (5.4)$$

$$\epsilon = G\rho/H^2. \quad (5.5)$$

A time variation of the masses of the order of magnitude given by Eq. (4.9) will correspond to the following variation in the dimensionless ratios:

$$\dot{\alpha} = 0, \quad (5.6)$$

$$\dot{\beta}/\beta \sim 10^{-10} \text{ per year}, \quad (5.7)$$

$$\dot{\gamma}/\gamma \sim 10^{-10} \text{ per year}. \quad (5.8)$$

The following is a discussion of the relevant observational and experimental evidence, based on Dyson's analysis.

A. Weak interactions

The most accurate determination of the rate of change of α is based on an analysis of experiments on the beta decay of Re^{187}.[17] They result in a very stringent upper limit on the rate of change of α:

$$\left|\frac{\dot{\alpha}}{\alpha}\right| \leq 5\times 10^{-15} \text{ per year} \quad (5.9)$$

in agreement with Eq. (5.6).

Experiments on the beta decay of K^{40}, together with the Re^{187} experiment, lead to the conclusion

$$\left|\frac{\dot{\beta}}{\beta}\right| \leq 10^{-10} \text{ per year}. \quad (5.10)$$

A similar and somewhat less precise upper limit was obtained by Wilkinson[18] from the study of ancient pleochroic halos. It seems, therefore, that such results are not inconsistent with Eq. (5.7), and an order-of-magnitude improvement in the experimental results is needed to provide a crucial test. The only theoretical objection to Eq. (5.7) is related to its consequences concerning the initial phase of a "big-bang" cosmology: With a fast rate of weak interaction, neutrons would have decayed to protons before being captured to form deuterium, and thus a big primeval helium abundance is excluded. "However, the evidence for a substantial primaeval helium abundance is still equivocal, and the validity of the 'big-bang' model at such early times[19] cannot be considered proved."[9]

B. Gravitational interactions

The acceleration of a mass m due to the gravitational influence of another mass M is given by GM/R^2. Therefore, the effect of a changing m and M according to Eq. (4.7) on planetary orbits is the same as the effect of a changing gravitational constant, $G \sim t^{-1}$, as suggested by Dirac.[20] The effect on the orbits of Mercury, Venus, and Mars is of the order of

$$\dot{r} \sim 10 \text{ meters per year}. \qquad (5.11)$$

Interplanetary ranging observations[21] are expected to be accurate enough to detect such an effect, if it exists, within the next few years.

Other attempts to test Dirac's hypothesis rely on the extreme sensitivity of stellar luminosity L as a function of the dimensionless ratio γ. For a main sequence star in which the proton-proton reaction is dominant, one obtains

$$L \sim \gamma^{7.7}. \qquad (5.12)$$

The following effects were discussed in the literature:

(a) the change in the climatic conditions and in the interior of the earth due to the changing luminosity of the sun;

(b) the change in the solar evolution;

(c) the change in the estimated ages of globular clusters due to the increased tempo of stellar evolution;

(d) the possibility of an eruption of early neutron stars because of the decrease in the gravitational forces that hold them together.

Dyson's analysis leads to the conclusion that our present knowledge of solar and stellar evolution and its effects is not inconsistent with Dirac's hypothesis. Equation (5.12) was derived on the assumption that γ is a constant; calculations by Pochada and Schwarzschild[22] show that if $\gamma \sim t^{-1}$ the luminosity L is less sensitive to variations in γ. As far as the ages of stars are concerned, they too are derived from evolutionary models which assume a constant γ. "We do not yet have a reliable way to calculate stellar ages independently of the behaviour of γ, except in the case of the sun for which the earth and the meteorites provide independent evidence."[9] In addition, our understanding of solar and stellar evolution is cast into doubt by "the mystery of the missing neutrinos,"[23] i.e., the fact that the flow of neutrinos from the sun, as implied by solar model calculations, does not seem to be there.[24]

In the context of stellar-evolution calculations it is not clear whether the effects of a changing γ ($\gamma \sim t^{-1}$) are the same whether the change is due to a changing constant of gravitation (Dirac's hypothesis) or whether it is due to changing masses [Eq. (4.7)]. Further calculations are necessary to clarify this point.

C. Cosmological considerations

A time variation of all masses will certainly have an effect on galactic luminosities and redshifts. Since the Hubble constant is uncertain to within a factor of 2, and the mean density of matter in the universe is known only in order of magnitude, cosmological considerations of the possibility of time variation of masses lead to no definite conclusions. As pointed out by Sandage[16] even the evolutionary change in luminosity of galaxies is, as yet, undetermined.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor A. O. Barut of the University of Colorado and Professor J. G. Dodd of Colgate University for valuable discussions.

---

*This work was supported in part by the Sloan Foundation and the Colgate Research Council.

[1]E. Wigner, Ann. Math. 40, 149 (1939).

[2]L. M. Thomas, Ann. Math. 42, 113 (1941); T. D. Newton, *ibid.* 51, 730 (1959); E. P. Wigner, Proc. Natl. Acad. Sci. USA 36, 184 (1950); J. Dixmier, Bull. Soc. Math. Fr. 89, 9 (1961); T. O. Philips, Ph.D. thesis, Princeton University, 1962 (unpublished); R. Takahashi, Bull. Soc. Math. Fr. 91, 289 (1963); A. Kihlberg and S. Ström, Ark. Fys. 31, 491 (1966); J. G. Kuriyan *et al.*, Commun. Math. Phys. 8, 204 (1968).

[3]For an introduction to the de Sitter group, see F. Gürsey, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, Lectures of the Istanbul Summer School of Theoretical Physics, edited by F. Gürsey (Gordon and Breach, New York, 1965).

[4]In a straightforward physical interpretation of the infinitesimal operators of the de Sitter group the quantized energy operator is not completely real [C. Fronsdal, Rev. Mod. Phys. 37, 221 (1965)]. It seems, however, that this difficulty is resolved when localizability properties of physical systems are taken into account (see Ref. 6). See also P. Roman and J. J. Aghassi, Nuovo Cimento 17A, 193 (1966).

[5]In the general case groups of motion have to be replaced

by the more general structure of quasigroups; see M. Halpern and S. Malin, J. Math. Phys. 12, 213 (1971); *Studies in Mathematical Physics*, edited by A. O. Barut (Reidel, Dordrecht, Holland, 1974).

[6]T. O. Philips and E. P. Wigner, in *Group Theory and Its Applications*, edited by E. M. Loebl (Academic, New York, 1968).

[7]I. E. Segal, Duke Math. J. 18, 221 (1951); E. Inönu and E. P. Wigner, Proc. Natl. Acad. Sci. USA 39, 510 (1953); E. Inönu, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, Lectures of the Istanbul Summer School of Theoretical Physics, edited by F. Gürsey (Gordon and Breach, New York, 1965).

[8]W. J. Holman III, J. Math. Phys. 10, 1888 (1969).

[9]F. J. Dyson, Institute of Advanced Study report, Princeton, New Jersey, 1972.

[10]F. Gürsey and T. D. Lee, Proc. Natl. Acad. Sci. USA 49, 179 (1963); F. Gürsey, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[11]The natural units $\hbar = c = 1$ are used. All Latin indices take the values $1, \ldots, 5$ and all Greek indices take the values $1, \ldots, 4$. All repeated indices are to be summed over.

[12]H. P. Robertson and T. W. Noonan, *Relativity and Cosmology* (Saunders, Philadelphia, Pa., 1968), Chaps. 14–18.

[13]H. P. Robertson, Astrophys. J. 82, 284 (1935).

[14]The term "particle" is used for brevity. Similar considerations apply to all physical systems.

[15]In fact, any mass variation entails some modification of the Friedmann model; this is a bootstrap situation. A bootstrap calculation shows that Eq. (4.7) is still valid, while the constant (4.8) is somewhat modified.

[16]A. Sandage, Astrophys. J. 178, 1 (1972).

[17]W. Herr *et al.*, Z. Naturforsch. 16A, 1053 (1961); B. Hirt *et al.*, in *Earth Science and Meteorites*, edited by J. Geiss and E. D. Goldberg (North-Holland, Amsterdam, 1963); R. L. Brodzinski and D. C. Conway, Phys. Rev. 138, B1386 (1965); R. W. P. Denver and J. A. Payne, quoted by Dyson (Ref. 9) as private communication, 1968.

[18]D. H. Wilkinson, Philos. Mag. 3, 582 (1958).

[19]$t \sim 10$ minutes.

[20]P. A. M. Dirac, Nature 139, 323 (1937); Proc. R. Soc. A165, 199 (1938).

[21]I. I. Shapiro *et al.*, Phys. Rev. Lett. 26, 27 (1971).

[22]P. Pochoda and M. Schwarzschild, Astrophys. J. 139, 587 (1964).

[23]W. A. Fowler, California Institute of Technology report, 1973 (unpublished).

[24]R. Davis, Jr. and J. C. Evans, in Proceedings of the Thirteenth International Cosmic Ray Conference, Denver, Colorado, 1973 [unpublished, quoted by Fowler (Ref. 23)].

PHYSICAL REVIEW D VOLUME 9, NUMBER 12 15 JUNE 1974

Classical-particle description of photons and phonons

W. B. Joyce
Bell Laboratories, Murray Hill, New Jersey 07974

(Received 18 December 1972; revised manuscript received 19 February 1974)

An $\hbar$-free analogy-free synthesis of classical mechanics and geometrical optics (acoustics) is presented which embraces without distinction particles of both zero and finite rest energy. The point-particle Lagrangian and other classical-particle properties of a photon (corpuscle) in gallium phosphide are found explicitly. Further brief applications are given to Newtonian mechanics, relativistic particle dynamics, fluid-immersed-body dynamics, hole-electron recombination in semiconductors, electrostatic and magnetic lenses, standardization of particle-flux units, and the still-controversial question of optical and acoustical radiation pressure and momentum.

I. INTRODUCTION

It would clearly be unreasonable to arbitrarily restrict classical mechanics (of point particles) to the description of the geometry of trajectories. Schrödinger's wave equation would then become by default the lowest-order theory of particle dynamics, and both the logical structure and practical value of physical theory would suffer from the loss of classical-particle dynamics. Nevertheless, according to the thesis of the present article, we do precisely this in the cases of light and sound by endorsing geometrical optics and acoustics as legitimate disciplines, while at the same time asserting[1-10] that a corresponding corpuscular or classical-particle theory is, at most, an erroneous historical curiosity and that the dynamics of light and sound must therefore be described by classical or quantum-mechanical field (wave) theories.

To support this thesis we reject as unphysical the usual geometrical analogy[2,4-6,8-16] between ray optics and the paths of classical mechanics and, instead, present an *analogy-free* *$\hbar$-free* set