Koba-Nielsen-Olesen scaling and eikonal (tower-graph) approximations*

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A prediction of the possible failure of Koba-Nielsen-Olesen (KNO) scaling at CERN ISR energies is given on the basis of the eikonal tower-graph approximation. For high multiplicities, the simplest Cheng-Wu impact-parameter forms yield approximate KNO scaling quantities in agreement with the NAL data. A formula for the calculation of all inclusive moments, from the knowledge of the elastic absorptive eikonal, is written for a wide class of dynamical approximations, of which the simplest is the tower-graph eikonal approximation.

Multiplicity distributions at NAL energies[1] are described surprisingly well by a semiempirical set of relations known as Koba-Nielsen-Olesen (KNO) scaling.[2] Over the range of CERN ISR energies, σ_{tot} (pp) now appears to increase[3] by 10%, an effect which some[4] have attributed to and parametrized in terms of the eikonal tower-graph approximations of field theory.[5] Alternately, the NAL multiplicity distributions may be fitted by a variety of models, in particular the two-component picture of hadronic production.[6] If one interprets the diffractive component as coming from multiple Pomeron exchange, the rise in σ_{tot} over ISR energies can then be understood in terms of the same mechanism: It simply corresponds to the rise in the diffractive dissociation cross section, and may be considered a transient effect.[7]

At this time we do not wish to address ourselves to the relative merits of these two seemingly opposite approaches. Rather, we would like to comment on the question of asymptotic KNO scaling within the context of currently popular Cheng-Wu physics. Independently of any such model, the lack of rigorous KNO scaling has already been proven,[8] and the formulas we exhibit may be regarded as a moderately realistic example of this situation, where possible KNO scaling is directly correlated with σ_{tot}. In addition, we give a simple formula, of validity greater than that of the tower-graph eikonal approximations, which permits straightforward calculation of all inclusive moments once the *elastic* eikonal is specified as a function of energy and impact parameter.

Although the retention and iteration of only the relatively simple tower graphs may be severely criticized[9] in the extreme asymptotic limit, at present ISR energies the approximation may in fact be relevant, and it becomes a matter of some interest to examine the higher-multiplicity predictions suggested by the tower-graph approximations. In general, specific expressions for the moments $\langle n^l\rangle$ or fluctuations f_l are sensitive to the detailed form of the energy and impact-parameter dependence of the (assumed absorptive) eikonal function, $i\chi = -\rho(s,b)$. However, certain less precise predictions — such as constancy with increasing s — may be made most simply (but not necessarily crucially) on the basis of the product form, $\rho = V(s)K(b)$.

Such predictions are possibly more general than, but not at variance with, recent estimates of one- and two-particle inclusive cross sections given by Cheng and Wu,[10] who find that $V(s) \sim s^a$ (all $\ln s$ dependence multiplying power dependence is here omitted) produces $\langle n\rangle \sim s^{a/(1+2a)}$, $\langle n^2\rangle \sim s^{2a/(1+2a)}$, etc., after these power exponents have been corrected to enforce energy conservation, thereby preventing kinematical impossibilities. The same type of energy conservation may be imposed below, when necessary, and for simplicity is omitted everywhere; the asymptotic ratios of Eq. (3) are unaffected by the energy conservation prescription used by Cheng and Wu. Testing KNO scaling may be a simpler matter, experimentally, than studying specific forms of the s dependence of the higher inclusive moments. By KNO scaling we mean the constancy in s of $C_l = \langle n^l\rangle/\langle n\rangle^l$ as s increases at fixed l.

Based upon an existing and simplified but typical field-theoretic eikonal model,[11] a general statement of the form of the absorptive part of the eikonal, as a function of a multiparticle fugacity, will first be written down; this includes as a special (and the simplest) case the conventional tower-graph summations. Assuming a product form for $\rho(s,b)$, the simplest situation, one may then qualitatively describe the energy dependence of the $\langle n^l\rangle$. Assuming, further, the specific $V(s)\sim s^a$ behavior of the tower graphs, it will then follow that KNO scaling is violated when σ_{tot} varies appreciably. It is amusing to note that, at NAL energies where $\sigma_{\text{tot}} \sim$ constant, use of the simplest Cheng-Wu forms suggests that KNO scaling is approximately valid, and that the multiplicity ratios $C_l = \langle n^l \rangle / \langle n \rangle^l$ are not unrelated to their experimental (constant) values.

In an absorptive eikonal context, the statement of unitarity is conventionally written as

$$\begin{aligned} \sigma_{\text{tot}} &= 2\int d^2b(1-e^{-\rho}), \\ \sigma_{\text{el}} &= \int d^2b(1-e^{-\rho})^2, \\ \sigma_{\text{in}} &= \int d^2b(1-e^{-2\rho}) \end{aligned} \tag{1}$$

for inelastic processes corresponding to the reactions $p_1+p_2 \to p_1'+p_2'+\sum_{i=1}^{n} k_i$, where the p (p') momenta denote incident (final) nucleons, and the k_i represent produced pions. In terms of the exclusive cross sections σ_n, $\sigma_{\text{in}} = \sum_{n=1}^{\infty} \sigma_n$. In useful analogy to the grand canonical partition function of statistical physics, one introduces[12] the partition function $\sigma_{\text{tot}}(\zeta) \equiv \sum_{n=0}^{\infty} \zeta^n \sigma_n$, thereby providing a convenient representation for the fluctuation coefficients f_l,

$$\sigma_{\text{tot}}(\zeta) = \sigma_{\text{tot}} \exp\left[\sum_{l=1}^{\infty} \frac{1}{l!} f_l (\zeta-1)^l\right],$$

where $\sigma_0 = \sigma_{\text{el}}$, $\sigma_{\text{tot}} = \sigma_{\text{tot}}(1)$. Inclusive moments are generated by repeated differentiation with respect to the fugacity ζ:

$$\langle n(n-1)\cdots(n-(l-1))\rangle \sigma_{\text{tot}} = \left(\frac{\partial}{\partial \zeta}\right)^l \sigma_{\text{tot}}(\zeta)\bigg|_{\zeta=1}.$$

What is the relation of $\sigma_{\text{tot}}(\zeta)$ to the forms of (1)? For a class of nontrivial dynamical models, stated below, this question has a specific and well-defined answer; it is[13]

$$\sigma_{\text{tot}}(\zeta) = \sigma_{\text{el}} + \int d^2b\, e^{-2\rho(s,b)}[e^{2\rho(s^{\zeta},b)} - 1]. \tag{2}$$

Knowledge (or the assumption) of $\rho(s,b)$ then permits the calculation of all σ_n, or of all $\langle n^l \rangle$.

Equation (2) may be derived by writing the general statement of unitarity for the absorptive part of the eikonal function in a theory constructed from multiple chains (to use the terminology of Ref. 10), linking in all possible ways a pair of scattering nucleons, with emission along each chain assumed to be of pionization form (i.e., the average multiplicity along each chain is proportional to $\ln s$), and computed in leading-log approximation. Not only tower graphs but all possible nonplanar t-channel connected amplitudes are included, and always in leading-log approximation.[11] However, these arbitrarily complicated unitarity sums which define the absorptive part of the eikonal are here constructed from "tree-graph" amplitudes, which themselves do not contain internal radiative corrections. This covers quite a large class of models, and its simplest possible realization is the familiar tower-graph eikonal.

The simplest product form, suggested by the work of Cheng and Wu, represents $\rho = V(s)K(b)$, with V and K dimensionless functions of s and b, respectively; one expects that V increases and K decreases with increasing argument. It is then a simple matter to perform the ζ differentiations of (2) and obtain moment distributions in terms of derivatives of V and integrals over powers of K. Most relevant to existing experiments is the further choice $V(s) \sim s^a$, from which follow the predictions,[14] for increasing s at fixed l,

$$C_l = \frac{\langle n^l \rangle}{\langle n \rangle^l} \simeq \frac{\kappa_l}{(\kappa_1)^l} (\sigma_{\text{tot}})^{l-1}, \tag{3}$$

where $\kappa_l = \int d^2b(K)^l$. In order to evaluate κ_l, the small b dependence of $K(b)$ must be specified, as below (Cheng-Wu analysis derives only the asymptotically decreasing $\sim e^{-\mu b}$ form). But one sees that KNO scaling, which requires C_l independent of s, will be violated in just those (ISR) regions where σ_{tot} increases.

To illustrate the sensitivity of C_l to $K(b)$, it is sufficient to consider two examples. That suggested by potential theory, $K(b) \to K_0(\mu b)$, generates $\kappa_l \simeq (2\pi/\mu^2) l!/2^{l+1}$. With $\sigma_{\text{tot}} = (2\pi/\mu^2)L$ [where, asymptotically, $L \to L(s) \sim \ln^2 s$ for the tower graph $V(s) \sim s^a$], one obtains $C_l \simeq l!(\frac{1}{2}L)^{l-1}$, a form which grows far too rapidly with l to fit the NAL data,[15] even assuming that L does not increase over this energy range. On the other hand, the replacement $K(b) \to e^{-\mu b}$, the simplest form nonsingular for small b, generates $C_l \simeq L^{l-1}/l^2$, which may be compared with the NAL data by calculating the experimental ratios

$$L_{l+1} \equiv \left(\frac{l+1}{l}\right)^2 \frac{C_{l+1}}{C_l}.$$

According to this simple picture, L_l should be a constant. That this property is so accurately

TABLE I. Experimental values for the ratios C_l and L_l.

l	C_l (exp. average)	L_l
2	1.24	4.96
3	1.81	3.28
4	2.97	2.92
5	5.36	2.82
6	10.4	2.79
7	21.6	2.83
8	47.	2.84

satisfied for high-multiplicity events (see Table I) suggests that the simplest Cheng-Wu form of $K(b)$, derived by them for large b and in a higher-energy region, is not too inaccurate at smaller impact parameters.

In summary: The possible relevance of eikonal tower-graph approximations to σ_{tot} at ISR energies suggests that KNO scaling will be violated at the same energies. Equation (2) rests upon the observation that pionization along a chain implies that each associated pion's phase-space volume effectively grows as a power of $\ln s$, in a leading-log calculation.

A simple, functional derivation of (2), and its relation to possible singularities of $\sigma_{tot}(\zeta)$ in the fugacity plane, will be given elsewhere.

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[10] H. Cheng and T. T. Wu (unpublished).

[11] See the second paper of Ref. 9 and H. M. Fried, *Functional Methods and Models in Quantum Field Theory* (MIT Press, Cambridge, Mass., 1972).

[12] For recent usage in the KNO context, see H. Moreno, Phys. Rev. D 8, 268 (1973).

[13] Equation (2) is valid for the ϕ^3-theory tower graphs of S.-J. Chang and T.-M. Yan [Phys. Rev. D 4, 537 (1971)] only in their so-called strong-coupling approximation. There, the eikonal contains an extra, essentially kinematic, factor of s^{-1} (which is absent when the sides of the towers are constructed from neutral vector mesons as in the models of Ref. 11), $\rho(s,b)\sim a_1(s^{a_2-1}/\ln s)\times e^{-\mu^2 b^2/\ln s}$, with $a_2\sim g^2$. Hence the ζ dependence of (2) should appear in the form $\rho(s^{\zeta-\zeta_0},b)$; with $a_2\zeta_0=1$. In the strong-coupling limit, $a_2\gg 1$, ζ_0 may be neglected, as in (2).

[14] Use of the Chang-Yan eikonal (see Ref. 13) in their strong-coupling limit generates a result $C_l\sim(\sigma_{tot}/\ln s)^{l-1}$ rather than the Cheng-Wu form of (3), $C_l\sim(\sigma_{tot})^{l-1}$. Such diminution of the s dependence of C_l occurs because of the $(\ln s)^{-1}$ factor in the exponent of ρ, which scales away a factor of $\ln s$, but both forms predict an increase of C_l with s. In the recent phenomenological forms of M. Le Bellac, J. L. Meunier, and G. Plaut [Nice Report, 1973 (unpublished)] the b^2 dependence of ρ is assumed scaled by σ_{in}, and the asymptotic C_l which result are independent of s. Thus, the test proposed in the present paper is relevant to the field-theoretic tower-graph models. It should also be noted that (3) has been derived under the assumption that $C_l/C_{l-1}\gg l(l-1)/2\langle n\rangle$, a condition following from our replacement of $\langle n(n-1)\cdots(n-l+1)\rangle$ by $\langle n^l\rangle$. At finite energies corrections to (3) can become important for higher l, and it may not be realistic to compare our asymptotic C_l with the nonasymptotic NAL data. It may be emphasized, and should be apparent from the text, that the experimental constancy of the L_l in Table I is without rigorous theoretical basis anyway, because of the identification $K(b)=\exp(-\mu b)$ for all b.

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