Nonlinear σ-model Padé calculation of $\pi\pi$ phase shifts

K. S. Jhung and R. S. Willey

Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

(Received 20 July 1973)

A calculation of the $\pi\pi$ scattering amplitude based on the nonlinear σ model, including pions and nucleons, through the one-loop approximation of perturbation theory is given. Unitary partial-wave amplitudes are obtained by constructing the first diagonal Padé approximant for the perturbation series. The s- and p-wave shifts are computed and compared with recent experimental analyses.

I. INTRODUCTION

The $SU_2 \times SU_2$ current algebra and the notion of the partially conserved axial-vector current (PCAC) have led to a number of low-energy theorems for processes involving pions.[1] These low-energy theorems are off-shell (pion mass shell) theorems, so it is necessary to make some smoothness assumption, or to introduce some model to extrapolate to the physical, on-shell, threshold. It is also an important problem to extend these results to energies above threshold, into the resonance region. The framework in which we approach these problems is that of field-theory (Lagrangian) models which incorporate the $SU_2 \times SU_2$ current algebra and PCAC as a (formal) canonical operator relation. In this approach, the first order calculation (Born term, or tree diagrams) gives an amplitude which satisfies the off-shell low-energy theorem, and defines an extrapolation on to the pion mass shells. The extension to energies above threshold, and some correction to the on-shell threshold amplitude, is obtained by including higher-order calculations. Ordinary perturbation theory will not do because it necessarily fails to converge in the presence of resonances. We attack this problem by use of the Padé algorithm for summation of divergent series. Field-theory models which incorporate the $SU_2 \times SU_2$ current algebra and PCAC are the linear σ model (LσM) and the nonlinear σ model (NLσM). A first diagonal Padé approximant calculation of $\pi\pi$ scattering based on the LσM, including pions and sigmas, but not nucleons, has been carried out by Basdevant and Lee[2] with fair success in predicting the $\pi\pi$ scattering amplitude (s- and p-wave phase shifts below 900 MeV c.m. energy). The main embarrasment of the LσM is the experimental nonexistence of a σ particle. We have therefore carried out a Padé calculation of $\pi\pi$ scattering based on the NLσM, including pions and nucleons.

II. KINEMATICS

The momentum and isospin labels are shown in Fig. 1. The invariant matrix element for $\pi\pi$ scattering, defined as

$$\begin{aligned}S_{abcd}(p,q,p',q') &= 1_{abcd}(p,q,p',q') \\ &\quad + i(2\pi)^4\delta^4(p+q-p'-q') \\ &\quad \times M_{abcd}(p,q,p',q')\,, \end{aligned} \tag{2.1}$$

has the isospin decomposition

$$\begin{aligned}M_{abcd}(p,q,p',q') &= \delta_{ab}\delta_{cd}A(p,q,p',q') \\ &\quad + \delta_{ac}\delta_{bd}B(p,q,p',q') \\ &\quad + \delta_{ad}\delta_{bc}C(p,q,p',q')\,. \end{aligned} \tag{2.2}$$

The invariant functions $A, B, C,$ are, for arbitrary off-shell external pions, functions of the usual scalar variables

$$s=(p+q)^2, \quad t=(p-p')^2, \quad u=(p-q')^2\,, \tag{2.3a}$$

$$s+t+u=p^2+q^2+p'^2+q'^2 \tag{2.3b}$$

and the squares of the pion four-momenta,

$$A(p,q,p',q')=A(s,t,u;p^2,q^2,p'^2,q'^2), \quad \text{etc.} \tag{2.4}$$

We will use these two sets of variables interchangeably. When the pions are on-shell we write simply $A(s,t,u)$, or $A(s,t)$, etc.

The crossing relations are

$$\begin{aligned}A(p,q,p',q') &= A(p,q,q',p') \\ &= A(q,p,p',q')\,, \\ B(p,q,p',q') &= A(-q',q,p',-p)\,, \\ C(p,q,p',q') &= A(-p',q,-p,q')\,. \end{aligned} \tag{2.5}$$

III. THE NONLINEAR σ MODEL

The Lagrangian which formally defines the nonlinear σ model[3] (NLσM) is

$$\mathcal{L} = \mathcal{L}_{\rm inv} + \mathcal{L}_{\rm SB}\,, \tag{3.1a}$$

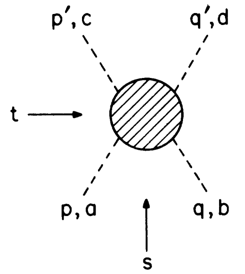

FIG. 1. Kinematics of $\pi\pi$ scattering.

$$\mathcal{L}_{\text{inv}}=\tfrac{1}{2}(\partial\vec{\phi})^2+\bar{\psi}i\gamma\cdot\partial\psi+\frac{1}{2(f_0^2-\vec{\phi}^2)}(\vec{\phi}\cdot\partial\vec{\phi})^2+ig_0\bar{\psi}\gamma_5\vec{\tau}\psi\cdot\vec{\phi}-g_0\bar{\psi}\psi(f_0^2-\vec{\phi}^2)^{1/2}\,, \tag{3.1b}$$

$$\mathcal{L}_{\text{SB}}=f_0\mu_1^2[(f_0^2-\vec{\phi}^2)^{1/2}-f_0]\,. \tag{3.1c}$$

$\mathcal{L}_{\text{inv}}$ is invariant under ordinary isospin transformations of the fields and also invariant under chiral transformations of the fields

$$\vec{\phi}\to\vec{\phi}+\vec{\alpha}(f_0^2-\phi^2)^{1/2}\,, \tag{3.2a}$$

$$\psi\to\psi+i\vec{\alpha}\cdot\tfrac{1}{2}\vec{\tau}\gamma_5\psi\,. \tag{3.2b}$$

The nonlinear chiral transformation (3.2a) is a special case of the general nonlinear chiral transformations discussed by Weinberg.[4] The axial-vector current formally derived from this Lagrangian is

$$A_a^\mu=\bar{\psi}\gamma^\mu\gamma_5\tfrac{1}{2}\tau_a\psi-(f_0^2-\vec{\phi}^2)^{1/2}\partial^\mu\phi_a-\phi_a\frac{1}{(f_0^2-\vec{\phi}^2)^{1/2}}(\vec{\phi}\cdot\partial^\mu\vec{\phi})\,. \tag{3.3}$$

Again formally, i.e., ignoring operator product problems, the divergence of this current satisfies

$$\partial_\mu A_a^\mu=f_0\mu_1^2\phi_a \tag{3.4}$$

which identifies f_0 as the "bare" pion decay constant. (The role of the renormalized version of this PCAC relation in constructing the renormalized perturbation series will be discussed later.)

After expansion of the denominator factor and square roots in (3.1), and identification of

$$m_1=g_0f_0\,, \tag{3.5}$$

the Lagrangian of (3.1) takes on the form

$$\tfrac{1}{2}[(\partial\vec{\phi})^2-\mu_1^2\vec{\phi}^2]+\bar{\psi}(i\gamma\cdot\partial-m_1)\psi+\frac{1}{2f_0^2}(\vec{\phi}\cdot\partial\vec{\phi})^2-\frac{\mu_1^2}{8f_0^2}(\vec{\phi}^2)^2+ig_0\bar{\psi}\gamma_5\vec{\tau}\psi\cdot\vec{\phi}+\frac{g_0^2}{2m_1}\bar{\psi}\psi\vec{\phi}^2+\cdots\,. \tag{3.6}$$

The Lagrangian (3.1), or (3.6), is not renormalizable in the conventional perturbation theory sense. Eventually we must face this problem, but we may proceed a while without doing so. Given a Lagrangian, renormalizable or not, we can compute the first-order approximation (Born term, or tree approximation) to the invariant matrix element for $\pi\pi$ scattering.

$$M^{(1)}_{abcd}=\frac{1}{f^2}[\delta_{ab}\delta_{cd}(s-\mu^2)+\delta_{ac}\delta_{bd}(t-\mu^2)+\delta_{ad}\delta_{bc}(u-\mu^2)]\,. \tag{3.7}$$

The momentum-dependent terms arise from the derivative pion coupling in (3.6). If we identify $f=f_\pi$, this is just the $\pi\pi$ invariant matrix element constructed by Weinberg[5] from current algebra, PCAC, and an assumption about the chiral transformation property of the $SU_2\times SU_2$ symmetry breaking. The Lagrangian (3.1), or (3.6), incorporates all of these features. This matrix element is purely real and increases rapidly with energy, i.e., it violates unitarity as the energy increases above threshold.

Next, we may think of introducing unitarity corrections by iteration of (3.7) in the unitarity equations

$$\text{disc}M^{(2)}_{abcd}=i\sum_{(\gamma)}(2\pi)^4\delta^4(p+q-p_{(\gamma)})M^{(1)*}_{cd(\gamma)}M^{(1)}_{ab(\gamma)}\,. \tag{3.8}$$

In this iteration the sum includes $(\gamma)=\pi\pi$ and $(\gamma)=N\bar{N}$. The $\pi\pi$ contributions are computed by substitution of (3.7) into (3.8). In order to compute the $N\bar{N}$ contributions we need the first-order invariant matrix element for $\pi\pi\to N\bar{N}$ computed with the Lagrangian (3.6). The corresponding first-order Feynman diagrams are shown in Fig. 2.

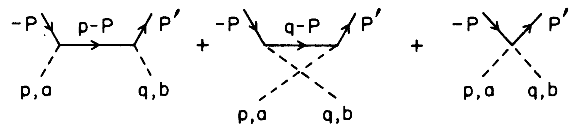

FIG. 2. First-order Feynman diagrams for $\pi\pi\to N\bar{N}$.

$$M^{(1)}_{ab}(p,q,P,P')=g'^2\bar{u}(p')\left\{\frac{1}{m}\delta_{ab}+\tfrac{1}{2}\gamma(p-q)\left[\delta_{ab}\left(\frac{-1}{(P-p)^2-m^2}+\frac{1}{(P-q)^2-m^2}\right)+\tfrac{1}{2}[\tau_b,\tau_a]\left(\frac{-1}{(P-p)^2-m^2}-\frac{1}{(P-q)^2-m^2}\right)\right]\right\}v(p)\,. \tag{3.9}$$

Here g' is an effective πN coupling constant which may differ from the conventionally defined πN coupling constant g by a finite renormalization. (See Appendix B for details.) Then, substituting (3.7) and (3.9) into (3.8), and computing the discontinuities in all three (s, t, u) channels, we obtain

$$
\begin{aligned}
\operatorname{Im}A^{(2)}(s,t,u;p^2,q^2,p'^2,q'^2)=\frac{1}{f^4}\{&-\tfrac{1}{2}(s-\mu^2)(2s+t+u-3\mu^2)\operatorname{Im}I_{\pi\pi}(s)\\
&+\tfrac{1}{6}[st+2tu+2\mu^2(s-u)+(2\mu^2/t)(p^2-p'^2)(q^2-q'^2)\\
&\quad-2(p^2q^2+p'^2q'^2)-(p^2q'^2+p'^2q^2)]\operatorname{Im}I_{\pi\pi}(t)\\
&+\tfrac{1}{6}[su+2ut+2\mu^2(s-t)+(2\mu^2/u)(p^2-q'^2)(q^2-p'^2)\\
&\quad-2(p^2q^2+p'^2q'^2)-(p^2p'^2+q^2q'^2)]\operatorname{Im}I_{\pi\pi}(u)\}\\
+g'^4\{&-4(s/m^2)\operatorname{Im}I_{NN}(s)+4(p^2q'^2+p'^2q^2-st)\operatorname{Im}H_N(s,t,u;p^2,q^2,p'^2,q'^2)\\
&+4(p^2p'^2+q^2q'^2-su)\operatorname{Im}H_N(s,u,t;p^2,q^2,q'^2,p'^2)\\
&-4(p^2q^2+p'^2q'^2-ut)\operatorname{Im}H_N(u,t,s;p^2,q'^2,p'^2,q^2)\}\ ,
\end{aligned}
\tag{3.10}
$$

where

$$2i\operatorname{Im}M_{abcd}=\operatorname{disc}M_{abcd}$$

and $A(s,t,u;p^2,q^2,p'^2,q'^2)$ is the invariant function defined by the isospin decomposition (2.2). The functions $\operatorname{Im}I_{\pi\pi}$, $\operatorname{Im}I_{NN}$, and $\operatorname{Im}H_N$ are defined as integrals over $\pi\pi$ or $N\bar{N}$ phase space in Appendix A where the calculations leading to (3.10) are outlined. Explicit formulas for these functions and their relation to the Feynman integrals $I_{\pi\pi}$, I_{NN}, and H_N are given in Appendix C. The relation of the result (3.10) to ordinary perturbation theory with the Lagrangian (3.6) is that if we write the (divergent) Feynman integrals corresponding to the one-loop Feynman diagrams of Fig. 3 and use the Cutkosky rules[6] to compute the (finite) discontinuities, we reproduce exactly (3.10).

To compute the real parts of $A^{(2)}$, we can either substitute (3.10) into the appropriate dispersion relations, or compute directly the Feynman integrals corresponding to the Feynman diagrams of Fig. 3. In either case subtractions are required. If we were dealing with a conventionally renormalizable theory, the subtraction constants introduced would be determined in terms of the renormalized coupling constants of the theory and no new constants would be required in higher orders. In the present case, in the one-loop approximation, one more subtraction is required than can be fixed by mass and coupling-constant renormalizations. However, we remark that the LσM includes the σ mass, which is an arbitrary parameter, so on the one-loop level the number of parameters, comparing the renormalizable LσM and the nonrenormalizable NLσM, is the same. The nonrenormalizability of the NLσM, as reflected in the highly divergent integrals, does lead to certain technical problems, even on the one-loop level. This is particularly so in the approach through Feynman integrals which suffer from the well-known ambiguities with respect to choice of integration variables, lack of covariance, etc., when the divergences are worse than logarithmic. This was a primary motivation for our dispersive approach to the one-loop approximation. In fact, analyticity, combined with (3.10) and the preceding equations, determines

$$
\begin{aligned}
A^{(2)}(s,t,u;p^2,q^2,p'^2,q'^2)=\frac{1}{f^4}[&\alpha(s,t,u;p^2,q^2,p'^2,q'^2)\bar{I}_{\pi\pi}(s)+\beta(s,t,u;p^2,q^2,p'^2,q'^2)\bar{I}_{\pi\pi}(t)\\
&+\beta(s,u,t;p^2,q^2,q'^2,p'^2)\bar{I}_{\pi\pi}(u)]\\
+g'^4[&-4(s/m^2)\bar{I}_{NN}(s)+4(p^2q'^2+p'^2q^2-st)H_N(s,t,u;p^2,q^2,p'^2,q'^2)\\
&+4(p^2p'^2+q^2q'^2-su)H_N(s,u,t;p^2,q^2,q'^2,p'^2)\\
&-4(p^2q^2+p'^2q'^2-ut)H_N(u,t,s;p^2,q'^2,p'^2,q^2)]\\
+\bar{P}&(s,t,u;p^2,q^2,p'^2,q'^2)\ ,
\end{aligned}
\tag{3.11}
$$

where

$$\alpha(s,t,u;p^2,q^2,p'^2,q'^2)=-\tfrac{1}{2}(s-\mu^2)(2s+t+u-3\mu^2)\ , \tag{3.12a}$$

$$\beta(s,t,u;p^2,q^2,p'^2,q'^2)=\tfrac{1}{6}[st+2tu+2\mu^2(s-u)+(2\mu^2/t)(p^2-p'^2)(q^2-q'^2)-2(p^2q^2+p'^2q'^2)-(p^2q'^2+p'^2q^2)]\ , \tag{3.12b}$$

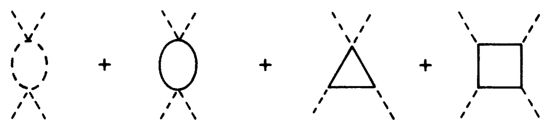

FIG. 3. Feynman diagrams for $\pi\pi$ scattering in the one-loop approximation of the nonlinear σ model.

and $\bar{P}(s, t, u; p^2, q^2, p'^2, q'^2)$ is a function which has no discontinuities in any of its variables. In this equation $\bar{I}_{\pi\pi}$ and $\bar{I}_{NN}$ are subtracted "bubble" integrals, and H_N is the standard Mandelstam scalar "box" integral. (Detailed formulas are given in Appendix C.) The function $\bar{P}$ depends on how one specifies the subtractions, i.e., subtractions at different values of $s, t, u, \ldots$ lead to different subtraction functions. We make the additional restriction on $\bar{P}$, specific to the one-loop perturbation approximation, that it should be at most quartic in the momenta (quadratic in $s, t, u, \ldots$). Then the most general polynomial consistent with Lorentz invariance, four-momentum conservation, and the symmetry properties of $A(p, q, p', q')$ may be written

$$\begin{aligned}\bar{P}(p, q, p', q') = \frac{1}{f^4}\,[&\bar{A}s^2 + \bar{B}(t^2+u^2) + \bar{C}s(t+u)\\ &+ \bar{D}tu + \bar{E}\mu^2 s + \bar{F}\mu^2(t+u)\\ &+ \bar{G}\mu^4 + \bar{H}(p^2q^2 + p'^2q'^2)\\ &+ \bar{I}(p^2+q^2)(p'^2+q'^2)]\,. \end{aligned} \quad (3.13)$$

The polynomial (3.13) contains nine subtraction constants. Even when the kinematic variables are restricted to the mass shell, this only reduces to four subtraction constants. Thus further analysis is required to show that only one of these constants is undetermined by conventional renormalization prescriptions. Finally, in listing the difficulties associated with the renormalization of the NLσM, we are faced with the possibility that new arbitrary constants may have to be added in each order of calculation.

IV. REGULARIZATION OF THE NONLINEAR σ MODEL

To deal with the renormalization problems described at the end of the previous section we adopt an approach suggested by Bessis and Zinn-Justin.[7] Most simply stated, their suggestion is to use the renormalizable LσM as a regularization of the nonrenormalizable NLσM. Their prescription is to first compute all one-loop diagrams in the LσM and then expand the resulting invariant matrix element in powers of $1/M^2$ ($M = m_\sigma$) and drop all terms which vanish in the limit $M \to \infty$. There will remain finite terms, independent of M, and also terms, dependent on M, of order $\ln M^2$, M^2, $M^2\ln M^2$, etc. For these latter terms, M plays the role of a cutoff. Their presence indicates the nonrenormalizability of the limit theory, but all the arbitrariness of the nonrenormalizable theory is now contained in its dependence on a single arbitrary parameter M. Furthermore, we will see that in the one-loop approximation, the $\pi\pi$ scattering amplitude depends only on $\ln M^2$, i.e., is not so sensitive to that parameter.

In their paper Bessis and Zinn-Justin justified this procedure by showing that in this limit the generating functional for the Green's functions of the LσM goes to the generating functional for the Green's functions of the NLσM. Additional insight into the significance of this prescription can be gained by considering it in the context of our dispersive approach to the NLσM. First we recall the well-known fact that in the tree approximation the LσM $\pi\pi$ matrix element goes to the Weinberg matrix element (3.7) in the limit $m_\sigma \to \infty$ [see (B27b)]

$$\lim_{m_\sigma\to\infty} M^{(1)}_{\mathrm{L}\sigma\mathrm{M}} = M^{(1)}_{\mathrm{NL}\sigma\mathrm{M}}\,. \quad (4.1)$$

Next, take this same limit in the iterated unitarity equation for the LσM

$$\begin{aligned}\lim_{m_\sigma\to\infty} \mathrm{disc} M^{(2)}_{\mathrm{L}\sigma\mathrm{M}} &= \lim_{m_\sigma\to\infty} i\sum M^{(1)*}_{\mathrm{L}\sigma\mathrm{M}} M^{(1)}_{\mathrm{L}\sigma\mathrm{M}}\\ &= i\sum M^{(1)*}_{\mathrm{NL}\sigma\mathrm{M}}\, M^{(1)}_{\mathrm{NL}\sigma\mathrm{M}}\\ &= \mathrm{disc} M^{(2)}_{\mathrm{NL}\sigma\mathrm{M}}\,. \end{aligned} \quad (4.2)$$

We can take the limit $m_\sigma \to \infty$ inside the unitarity sum because it only involves integrals over finite regions of phase space. So we have demonstrated that the proposed prescription does reproduce correctly the Born term and absorptive part of the one-loop matrix element, which we were able to compute unambiguously starting from the NLσM Lagrangian in Sec. III. Then analyticity determines the complete one loop-matrix element (3.11) up to an undetermined subtraction polynomial of form (3.13). Thus we see that the role of the limit prescription is simply to fix the subtraction polynomial (3.13), in terms of the one arbitrary constant M.

The foregoing discussion has been oversimplified in one respect. The LσM contains an additional particle relative to the NLσM and hence involves additional renormalized coupling constants. If these finite constants survive the limit $M \to \infty$, then the subtraction polynomial (3.13) will depend on these additional constants also. One has to carry through the renormalization of the LσM in accordance with all the Ward identities which follow from the chiral structure of that model, and see to what extent these Ward identities fix these additional constants. It turns out that this analysis leaves one constant, in addition to M, undetermined. We then consider the $\mu \to 0$ chiral-invariant limit. It is known that using only chiral invariance, the chiral invariant one-loop NLσM $\pi\pi$ matrix element admits two arbitrary constants. However, there have also been advanced more or less plausible heuristic arguments[8] that fix the ratio of those two constants. If we accept these arguments, then consideration of the $\mu \to 0$ limit (after the $M\to\infty$ limit) fixes the additional constant and we obtain the result described above, that the subtraction polynomial (3.13) is determined in terms of one arbitrary parameter M. The details of the analysis of the LσM are lengthy; first, because there are many more diagrams than in the NLσM, so simply constructing the one-loop matrix element consistent with all the Ward identities is a more lengthy task; second, because there is considerable cancellation of singular terms in the limit $M\to\infty$, one has to keep several terms in the asymptotic expansions of the individual Feynman integrals, which also become quite lengthy. We have therefore relegated these details to Appendix B and here give only the resulting NLσM matrix element (on-shell)

$$\begin{aligned} A(s,t,u) &= \frac{1}{\mathfrak{F}^2}(s-\mu^2) \\ &+ \frac{1}{\mathfrak{F}^4}[\,\alpha(s,t,u)I^{(0)}_{\pi\pi}(s)+\beta(s,t,u)I^{(0)}_{\pi\pi}(t)+\beta(s,u,t)I^{(0)}_{\pi\pi}(u)-4sm^2I^{(\pi)}_{NN}(s)+4(2\mu^4-st)m^4H_N(s,t,u) \\ &+4(2\mu^4-su)m^4H_N(s,u,t)-4(2\mu^4-ut)m^4H_N(u,t,s)+4\mu^2(s-\mu^2)m^2I'_{NN}(\mu^2) \\ &+\mathfrak{A}s^2+\mathfrak{B}tu+\mathfrak{C}\mu^2 s+\mathfrak{D}\mu^4]\ , \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\alpha(s,t,u) = -\tfrac{1}{2}(s^2-\mu^4)\ , \\ &\beta(s,t,u)=\tfrac{1}{6}[-t(t-u)+2\mu^2(t-2u)+2\mu^4]\ , \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\mathfrak{A}=\frac{1}{16\pi^2}(\tfrac{2}{3}L-\tfrac{16}{9})=-\mathfrak{B} \\ &\mathfrak{C}=\frac{1}{16\pi^2}(-\tfrac{2}{3}L+\tfrac{22}{9}) \\ &\mathfrak{D}=\frac{1}{16\pi^2}(\tfrac{3}{2}L-\tfrac{14}{3})\ , \\ &L=\ln(M^2/\mu^2)\ . \end{aligned} \tag{4.5}$$

The subtracted integrals are

$$\begin{aligned} &I^{(0)}_{\pi\pi}(s)=I_{\pi\pi}(s)-I_{\pi\pi}(0)\ , \\ &I^{(\pi)}_{NN}(s)=I_{NN}(s)-I_{NN}(\mu^2)\ , \\ &I'_{NN}(\mu^2)=\left(\frac{d}{ds}I_{NN}(s)\right)_{s=\mu^2}\ , \end{aligned} \tag{4.6}$$

and I and H are the integrals which were introduced in Sec. III. Detailed formulas are given in Appendix C. The constant $\mathfrak{F}$ is defined in Appendix B. Although it is not identical to f_π, in the one-loop approximation, it differs from f_π by something on the order of 1%; so in numerical calculation we simply use $\mathfrak{F}=f_\pi$.

V. ISOSPIN AND PARTIAL-WAVE PROJECTIONS

The $I=0,1,2$ isospin amplitudes are

$$M_0(s,t,u)=3A(s,t,u)+A(t,s,u)+A(u,t,s)\ , \tag{5.1a}$$

$$M_1(s,t,u)=A(t,s,u)-A(u,t,s)\ , \tag{5.1b}$$

$$M_2(s,t,u)=A(t,s,u)+A(u,t,s)\ , \tag{5.1c}$$

where $A(s,t,u)=A(s,u,t)$ is the amplitude given in (4.3). The partial-wave amplitudes are

$$a_{IJ}(s)=\tfrac{1}{2}\int_{-1}^{1} d\cos\theta\, P_J(\cos\theta)M_I(s,t)\ , \tag{5.2}$$

$$\begin{aligned} &s=4(\vec{p}^{\,2}+\mu^2)\ , \\ &t=-2\vec{p}^{\,2}(1-\cos\theta)\ , \\ &u=-2\vec{p}^{\,2}(1+\cos\theta)\ . \end{aligned} \tag{5.3}$$

The normalization is such that the elastic unitarity relation for two identical particles is

$$\mathrm{Im}\,a_{IJ}(s)=\frac{1}{32\pi}\left(\frac{s-4\mu^2}{s}\right)^{1/2}|a_{IJ}(s)|^2\ . \tag{5.4}$$

Hence

$$a_{IJ}(s)=32\pi\left(\frac{s}{s-4\mu^2}\right)^{1/2}e^{i\delta_{IJ}(s)}\sin\delta_{IJ}(s)\ . \tag{5.5}$$

The partial-wave amplitudes projected out of the matrix element (4.3), which was constructed to satisfy the perturbative unitarity equation (3.8), will satisfy a perturbative partial-wave elastic unitarity equation

$$\mathrm{Im}\,a^{(2)}_{IJ}(s)=\frac{1}{32\pi}\left(\frac{s-4\mu^2}{s}\right)^{1/2}|a^{(1)}_{IJ}(s)|^2\ . \tag{5.6}$$

The partial-wave amplitudes projected out of (5.1) and (4.3) are

$$a_{00}(s)=\frac{1}{\mathfrak{F}^2}(2s-\mu^2)+\frac{1}{16\pi^2\mathfrak{F}^4}\Big\{-\tfrac{50}{27}(s-4\mu^2)^2-\tfrac{128}{9}\mu^2(s-4\mu^2)-27\mu^4+L[\tfrac{25}{9}(s-4\mu^2)^2+\tfrac{52}{3}\mu^2(s-4\mu^2)+\tfrac{63}{2}\mu^4]$$
$$-\tfrac{1}{2}(2s-\mu^2)^2I_0(s)+\frac{1}{12p^2}[(8\mu^2 s-37\mu^4)\mu^2G_0(s)+(-2s+32\mu^2)\mu^4G_1(s)-10\mu^6G_2(s)]$$
$$+28m^2\mu^2\left[1-\left(\frac{4m^2-\mu^2}{\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}-\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\right]$$
$$+8m^2(s-4\mu^2)\left[1-2\left(\frac{4m^2-\mu^2}{\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}-\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\right]$$
$$-12(16\pi^2)m^2sI_{NN}^{(\pi)}(s)-\frac{2m^6}{p^2}G_{1N}(s)$$
$$+16\pi^2\left(\frac{6m^4}{\pi}\right)\int_{4m^2}^{\infty}ds'h(s',s)\left[4s+\frac{-ss'+2\mu^4}{p^2}\ln\left(\frac{s'+4p^2}{s'}\right)\right]$$
$$+16\pi^2\left(-\frac{2m^4}{\pi}\right)\int_{4m^2}^{\infty}ds'h(s',4\mu^2-s-s')\left[-4s'-8p^2+\frac{s'^2+4p^2s'+2\mu^4}{p^2}\ln\left(\frac{s'+4p^2}{s'}\right)\right]\Big\}, \tag{5.7a}$$

$$a_{11}(s)=\frac{1}{\mathfrak{F}^2}(\tfrac{4}{3}p^2)+\frac{1}{16\pi^2\mathfrak{F}^4}\Big\{-\tfrac{8}{9}\mu^2(s-4\mu^2)+\tfrac{2}{3}\mu^2(s-4\mu^2)L-\tfrac{1}{18}(s-4\mu^2)^2I_0(s)$$
$$+\frac{1}{12p^2}\left[(4\mu^2s+\mu^4)\mu^2\left(G_0(s)+\frac{\mu^2}{2p^2}G_1(s)\right)+(-\mu^2s-2\mu^4)\mu^2\left(G_1(s)+\frac{\mu^2}{2p^2}G_2(s)\right)-2\mu^6\left(G_2(s)+\frac{\mu^2}{2p^2}G_3(s)\right)\right]$$
$$+\tfrac{4}{3}m^2(s-4\mu^2)\left[1+\left(\frac{4m^2-\mu^2}{\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}-\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\right]$$
$$-\frac{2m^6}{p^2}\left[G_{1N}(s)+\frac{m^2}{2p^2}G_{2N}(s)\right]$$
$$+16\pi^2\left(\frac{2m^4}{\pi}\right)\int_{4m^2}^{\infty}ds'h(s',s)\frac{1}{p^2}\left[4ss'-8\mu^4+\frac{(-ss'+2\mu^4)(s'+2p^2)}{p^2}\ln\left(\frac{s'+4p^2}{s'}\right)\right]\Big\}, \tag{5.7b}$$

$$a_{20}(s)=\frac{1}{\mathfrak{F}^2}(-s+2\mu^2)+\frac{1}{16\pi^2\mathfrak{F}^4}\Big\{-\tfrac{20}{27}(s-4\mu^2)^2-\tfrac{8}{9}\mu^2(s-4\mu^2)+2\mu^4+L[\tfrac{10}{9}(s-4\mu^2)^2+\tfrac{10}{3}\mu^2(s-4\mu^2)+3\mu^4]$$
$$-\tfrac{1}{2}(s-2\mu^2)^2I_0(s)+\frac{1}{12p^2}[(-4\mu^2s+5\mu^4)\mu^2G_0(s)+(s+2\mu^2)\mu^4G_1(s)-4\mu^6G_2(s)]$$
$$-8m^2\mu^2\left[1-\left(\frac{4m^2-\mu^2}{\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}-\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\right]$$
$$-4m^2(s-4\mu^2)\left[1+\left(\frac{4m^2-\mu^2}{\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}-\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\tan^{-1}\left(\frac{\mu^2}{4m^2-\mu^2}\right)^{1/2}\right]$$
$$-\frac{2m^6}{p^2}G_{1N}(s)+16\pi^2\left(\frac{4m^4}{\pi}\right)\int_{4m^2}^{\infty}ds'h(s',4\mu^2-s-s')$$
$$\times\left[-4s'-8p^2+\frac{s'^2+4p^2s'+2\mu^4}{p^2}\ln\left(\frac{s'+4p^2}{s'}\right)\right]\Big\}, \tag{5.7c}$$

In these formulas the functions $I_0(s)$, $G_{0,1,2,3}(s)$, $G_{1N,2N}(s)$ are the results of doing the angular integrations over polynomials times $I^{(0)}_{\pi\pi}(s)$, $I^{(0)}_{\pi\pi}(t)$, and $I^{(\pi)}_{NN}(t)$. Explicit formulas are given in Appendixes C and D, both for the physical range $s \geq 4\mu^2$ and for the continuation below threshold to the range $0 \leq s \leq 4\mu^2$. The functions h are discontinuities of the Mandelstam box integral. Again, explicit formulas are given in Appendixes C and D. The integrals involving h, i.e., the partial-wave projections of the nucleon box integral, could only be done numerically by computer. However, in the region below threshold, and also for a small range above threshold in which $4\mu^2 \leq s \ll 4m^2$, these integrals are very well approximated by polynomials in s/m^2 and μ^2/m^2 which provides a check on the numerical integration.

For numerical calculation we use

$$\begin{aligned} \mathcal{F} = f_\pi &= \frac{1}{\sqrt{2}}(0.96\mu^\pm) = 0.68\mu^\pm = 949 \text{ MeV} , \\ \mu &= 138.1 \text{ MeV} , \\ m &= 938.9 \text{ MeV} , \\ (m/\mu &= 6.80) . \end{aligned} \tag{5.8}$$

VI. PADÉ APPROXIMANTS

Even after achieving a regularization of the individual terms in the perturbation expansion of the $\pi\pi$ scattering amplitude, we are left with the first few terms of a series which cannot converge for energies in the resonance region and may not converge for any energy. In recent years it has been suggested by a number of people that one might hope to overcome these problems by application of the Padé algorithm[9] to the perturbation series. Given a perturbation series

$$f(z;\lambda) = f^{(0)}(z) + \lambda f^{(1)}(z) + \lambda^2 f^{(2)}(z) + \cdots \tag{6.1}$$

the N, M Padé approximant is defined as that rational fraction approximation to $f(\lambda)$ which agrees with its Taylor series expansion to order λ^{N+M+1},

$$f^{[N,M]}(z;\lambda) = \frac{P_N(z;\lambda)}{Q_M(z;\lambda)} , \tag{6.2a}$$

$$f(z;\lambda) - f^{[N,M]}(z;\lambda) = O(\lambda^{N+M+1}) . \tag{6.2b}$$

These conditions have a unique solution for any N, M; we make use of only the first nontrivial approximant

$$f^{[1,1]}(z;\lambda) = f^{(0)}(z) + \lambda f^{(1)}(z) \Big/ \left(1 - \lambda \frac{f^{(2)}(z)}{f^{(1)}(z)}\right) . \tag{6.3}$$

There does not exist any proof of the convergence of the sequence of Padé approximants in a real field theory, but it has been proved that the diagonal ($N=M$) approximants converge to the scattering amplitude in potential theory. This is a strong indication that at least the bound state (resonance) problem (poles in f as function of λ) is handled by the Padé algorithm. Some indication of the power of the Padé algorithm to sum series which fail to converge for any value of λ has been obtained from consideration of the anharmonic oscillator.

$$H = p^2 + x^2 + \lambda x^4 .$$

It is known that this Hamiltonian possesses eigenvalues $E_n(\lambda)$ which are analytic functions of λ in the λ plane cut along the negative real axis, i.e., $\lambda = 0$ is a branch point, so the perturbation series for the energy levels must diverge. Nevertheless, Loeffel *et al.*[10] proved that the diagonal Padé approximants formed from the coefficients of the divergent perturbation expansion do converge to the correct $E_n(\lambda)$.

An important feature of the diagonal Padé approximants, applied to the partial-wave amplitudes, is that they satisfy elastic unitarity exactly.

$$a^{[1,1]}_{IJ}(s) = a^{(1)}_{IJ}(s) \Big/ \left(1 - \frac{a^{(2)}_{IJ}(s)}{a^{(1)}_{IJ}(s)}\right) \tag{6.4}$$

satisfies

$$\text{Im}\,a^{[1,1]}_{IJ}(s) = \frac{1}{32\pi}\left(\frac{s-4\mu^2}{s}\right)^{1/2} |a^{[1,1]}_{IJ}(s)|^2 \tag{6.5}$$

which follows directly from substitution of (5.6) into (6.4). Then the phase shifts are given by

$$\tan\delta^{[1,1]}_{IJ}(s) = \frac{\text{Im}\,a^{[1,1]}_{IJ}(s)}{\text{Re}\,a^{[1,1]}_{IJ}(s)} . \tag{6.6}$$

VII. CALCULATIONS, EFFECTIVE πNN COUPLING, RESULTS

A computer program was written to evaluate the functions and integrals appearing in (5.7a)–(5.7c), substitute the resulting $a^{(1)}_{IJ}(s)$, $a^{(2)}_{IJ}(s)$ into the [1, 1] Padé formula (6.4), and compute the phase shifts from (6.6). The results are that one can get $I=0$ and 2 s-wave phase shifts that agree reasonably well with experiment, and one gets an $I=1$, $J=1$ resonance (ρ) as output of the calculation, but there is not enough p-wave binding to get the ρ mass below 1000 MeV for any value of L which gives reasonable s-wave phase shifts. The conclusion is that the complete one-loop calculation treated by the first diagonal Padé approximant is in qualitative, but not quantitative, accord with experimental information on low-energy (below 900 MeV) $\pi\pi$ scattering.

We have made some conjectures about the effects of higher-order (two or more loops) calculations and found a simple two-parameter model which gives a very good account of the $\pi\pi$ scattering up to 900 MeV. In particular, we observed that the p wave is not so sensitive to the parameter L, but is quite sensitive to the effective πNN and $\pi\pi NN$ coupling constants because it is the nucleon loop terms which are providing the p-wave binding in this calculation.[11] There are a set of two-loop diagrams, some of which are illustrated in Fig. 4, whose effect at low energy is to modify the effective πNN and $\pi\pi NN$ coupling. Of course these diagrams have momentum dependence as well, and there are other two-loop diagrams which are not πN vertex corrections; but at the moment we are not contemplating a full two-loop calculation, we only seek a crude but simple parametrization of

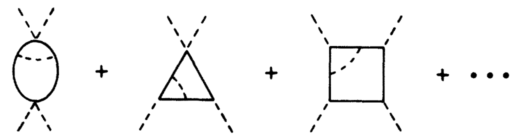

FIG. 4. Some two-loop diagrams which modify the effective πN couplings.

the modifications of the one-loop calculation by higher-order calculations. We can get some idea of what to expect by comparing the first-order perturbative $\pi N \to \pi N$ matrix element with the nonperturbative threshold matrix element determined by current algebra and PCAC.[12]

In terms of the conventional decomposition

$$M_{ab}(q_1, q_2, P_1, P_2) = \bar{u}(P_2)\{\delta_{ab}[A^{(+)}(s,t,u) + \tfrac{1}{2}\gamma(q_1+q_2)B^{(+)}(s,t,u)] + \tfrac{1}{2}[\tau_b, \tau_a][A^{(-)}(s,t,u) + \tfrac{1}{2}\gamma(q_1+q_2)B^{(-)}(s,t,u)]\}u(P_1)\,, \tag{7.1}$$

the Born term from the Lagrangian (3.6) gives

$$A^{(+)} = \frac{g'^2}{m}\,, \qquad B^{(+)} = g'^2\left(\frac{-1}{s-m^2} + \frac{1}{u-m^2}\right)\,, \tag{7.2a}$$

$$A^{(-)} = 0\,, \qquad B^{(-)} = g'^2\left(\frac{-1}{s-m^2} - \frac{1}{u-m^2}\right)\,. \tag{7.2b}$$

The current-algebra plus PCAC (two-soft-pion) calculation gives in the limit

$$q_1^\mu = q_2^\mu = (\mu, \vec{0}) \approx 0\,, \tag{7.3}$$

$$A^{(+)} = m\frac{g_A^2}{f_\pi^2}\,, \qquad B^{(+)} = m^2\frac{g_A^2}{f_\pi^2}\left(-\frac{1}{m\mu} - \frac{\mu}{4m^3} + \cdots\right)\,, \tag{7.4a}$$

$$A^{(-)} = 0\,, \qquad B^{(-)} = \frac{1}{2f_\pi^2}\,. \tag{7.4b}$$

The odd-isospin term $B^{(-)}$ is the current-algebra term; the even-isospin terms come from the nucleon-pole contribution to the matrix element of two axial-vector currents, which brings in the factor g_A^2. Taking the same limit (7.3) in (7.2), we see that the first-order perturbation theory matrix element has the same isospin-odd terms as the nonperturbative threshold matrix element if

$$g'^2 = \frac{m^2}{f_\pi^2} \tag{7.5a}$$

while the isospin-even terms are the same if

$$g'^2 = g_A^2\frac{m^2}{f_\pi^2}\,. \tag{7.5b}$$

We therefore introduce a second parameter into the calculation by multiplying all nucleon-loop terms by a factor $(g_A')^4$ and treating g_A' as a variable parameter. The value $g_A' = 1$ gives the original one-parameter one-loop calculation, and we have argued that use of a value of g_A' greater than one is a way of taking into account some of the effect of higher-order calculations.[13]

In Appendix E we consider a simplified model in which the pion mass is set equal to zero and the nucleon-loop integrals are approximated by neglecting terms of order s/m^2. In this case the rather complicated formulas (5.7) and (D1)–(D7) simplify enormously, and the dependence of the various amplitudes and phase shifts on the two parameters L, g_A' becomes transparent. In particular, in this limit, the p wave is independent of L. Thus we expect the complete p-wave amplitude (5.7b) to be only weakly dependent on L. So our procedure is to choose g_A' to get the right amount of binding from the nucleon loops to get the observed value of m_ρ, and then to vary L to get the best possible fit to the $I=0$ and $I=2$ s-wave phase shifts. One problem is that although there is general experimental agreement on the qualitative behavior of the s-wave $\pi\pi$ phase shifts below one GeV, there is still considerable uncertainty as to the precise details. In particular, the scattering lengths are not at all well determined experimentally; so it is not feasible to fix L from any combination of the experimental scattering lengths. Therefore, to determine an allowed range for the parameter L, we have considered the difference of the $I=0$ and $I=2$ phase shifts at center-of-mass energy equal to the kaon mass. The reason for this choice is that this is the only combination of phase shifts and energy for which one has a hope of getting a reasonably reliable experimental value by two completely independent methods. First is the Chew-Low extrapolation to the pion pole in the reaction $\pi+N\to\pi+\pi+N(\Delta)$, the source of almost all of the experimental information about the $\pi\pi$ phase shifts. Second is through the dependence of the branching ratios of the various $K\to 2\pi$ decays on the final-state $\pi\pi$ s-wave phase shifts. The validity of the second method depends on the assumptions that the observed $\Delta I\neq\frac{1}{2}$ component comes from an $I=\frac{3}{2}$ piece of H_{wk}, rather than from electromagnetic corrections, and that there is no (or negligible) $I=\frac{5}{2}$ piece of H_{wk}. If these assumptions are correct, then the determination of the phase shift difference is rather clean because there is no background from additional hadrons in the final state. Using 1972 Particle Data Group values,[14] we find

$$\delta_{00}-\delta_{20}\Big|_{M_{\pi\pi}=m_K} = 50°\pm 6° \text{ (from } K\to 2\pi \text{ branching ratios)}. \tag{7.6}$$

It is encouraging that values from recent high-statistics Chew-Low extrapolation analyses appear to be consistent with this value although there is still considerable uncertainty in δ_{20} (which gives the *smaller* part of the difference).

These considerations determine $g_A'\approx 1.3$ and $L\approx 4$ to 5. In Figs. 5 and 6 we plot the computed $I=0$ and 2 s-wave and $I=1$ p-wave phase shifts, from threshold up to 900-MeV center-of-mass energy for $g_A'=1.31$ and $L=4.0$, 4.5, and 5.0. We have also plotted some representative experimental data for comparison. We have included the

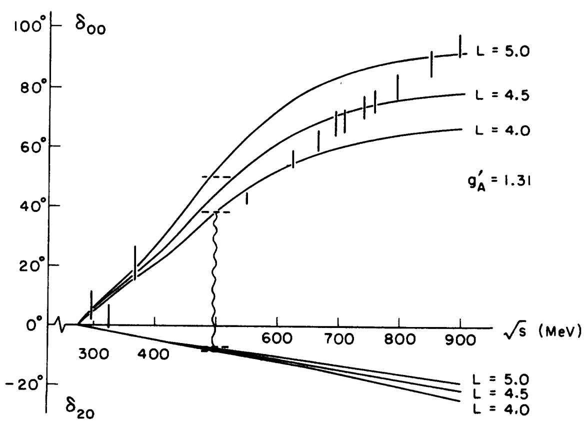

FIG. 5. $I=0$ and $I=2$ s-wave phase shifts.

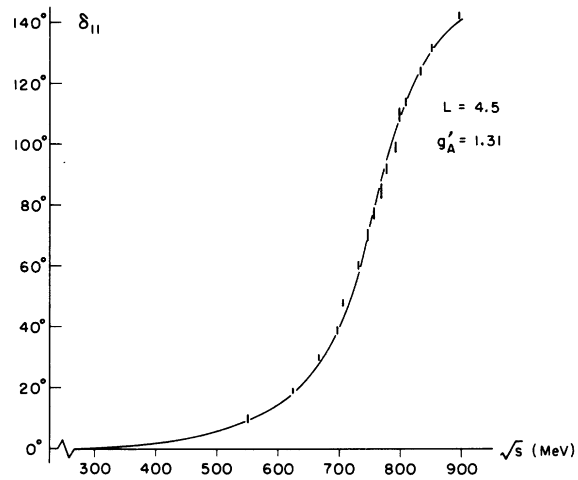

FIG. 6. p-wave phase shift.

three low-energy δ_{00} points given by the K_{e4} experiment of Beier *et al.*[15] We have included as a wavy line with "gates" the constraint (7.6) obtained from the $K\to 2\pi$ data. And we have included points from the Chew-Low extrapolation analysis of the LBL group.[16] This experiment and analysis gives values for δ_{00} and δ_{11} from 500 MeV to 1100 MeV. Results of an even higher statistic Chew-Low extrapolation analysis have been reported[17] by a Cern-Munich group. At the time of writing of this article, they had not published as detailed a phase-shift analysis as the LBL group, but published figures indicate that their "down" solution for δ_{00} is essentially the same as the LBL δ_{00} in the range 500 to 900 MeV. We have not plotted any experimental δ_{20} points. A review of recent experimental work is given by Poirier.[18] All experiments are agreed that δ_{20} is small and negative, but some experiments give only $-10°$ to $-15°$ between 700 and 900 MeV while others may be as large as $-25°$ or $-30°$ in this range.

In comparing the calculated curves with the experimental data described above it is seen that we fit almost exactly the experimentally rather well determined p-wave phase shift up to 900 MeV. An empirical fit would require at least two parameters—m_ρ and Γ_ρ for an elastic Breit-Wigner form—plus specification of the background function over the entire energy range. In our calculation, only one parameter, g_A', is adjusted to fit the p wave and furthermore that parameter is not completely arbitrary; the sign and order of magnitude are determined by the arguments we gave to introduce it. The calculated $I=0$ and 2 s-wave phase shifts are at least in qualitative accord with the experimental phase shifts up to 900 MeV. The primary deviation is that the calculated δ_{00} is flattening out above 700 MeV while the experimental

TABLE I. Some properties of the partial-wave amplitudes calculated for various values of the parameters L, g_A'.

L	g_A'	$a_0/a_0^{(1)}$	$a_2/a_2^{(1)}$	$\delta_{00}-\delta_{20}$ ($\sqrt{s}=m_K$)	m_ρ (MeV)	Γ_ρ (MeV)	δ_{00} ($\sqrt{s}=m_\rho$)
4.0	1.29	1.34	0.98	46°	799	165	64°
4.5	1.29	1.40	0.97	51°	794	160	75°
5.0	1.29	1.47	0.96	59°	788	164	88°
4.0	1.30	1.35	0.98	46°	786	157	64°
4.5	1.30	1.41	0.97	52°	780	149	75°
5.0	1.30	1.47	0.96	59°	776	153	88°
4.0	1.31	1.35	0.98	48°	775	150	64°
4.5	1.31	1.41	0.97	52°	770	144	75°
5.0	1.31	1.48	0.96	60°	764	150	88°

phase shift continues to rise, roughly linearly. We have cut off our calculation at 900 MeV because experiment indicates a very rapid rise of δ_{00} through 180° just below one GeV, presumably associated with the opening of the $K\bar{K}$ channel, which our calculation, based on an $SU_2 \times SU_2$ model with no kaons cannot account for.

We have also computed the correction factors to the Weinberg (Born terms) scattering lengths implied by our Padé calculation. The scattering lengths are defined as

$$\delta_{I0}(s)_{\,(p\to 0)}\, p\, a_I \; . \tag{7.7}$$

So (5.5) and (6.4) give

$$a_I = a_I^{(1)} \left[1 - \frac{a_{I0}^{(2)}(4\mu^2)}{a_{I0}^{(1)}(4\mu^2)}\right]^{-1} , \tag{7.8}$$

where

$$a_I^{(1)} = a_I^{(\mathrm{Weinberg})} = \frac{1}{32\pi\mu} M_I^{(1)}(\mathrm{threshold}) . \tag{7.9}$$

From (3.7)

$$a_0^{(1)} = \frac{1}{32\pi}\left(\frac{\mu^2}{f_\pi^2}\right) 7\mu^{-1} \simeq 0.15\mu^{-1} , \tag{7.10a}$$

$$a_2^{(1)} = \frac{1}{32\pi}\left(\frac{\mu^2}{f_\pi^2}\right) (-2\mu^{-1}) \simeq -0.04\mu^{-1} . \tag{7.10b}$$

The computed correction factors (7.8) are given in Table I.

We have considered the question of the violation of crossing symmetry which is induced by the application of the Padé algorithm to the partial-wave amplitudes, resulting in exactly unitary (elastic) partial-wave amplitudes. Because our calculation gives the partial-wave amplitudes in the unphysical region $0 \le s \le 4\mu^2$ as well as in the physical region, we can use the relations derived by Roskies.[19] For s and p waves Roskies gives five integral relations which are necessary and sufficient conditions for crossing to be exactly satisfied by these partial waves. These integral relations are

$$\begin{aligned} I_0 &= I_2 ,\\ J_0 &= J_2 ,\\ S &= P ,\\ S' &= P' ,\\ S'' &= P'' , \end{aligned} \tag{7.11}$$

where

$$\begin{pmatrix} I_0 \\ I_2 \end{pmatrix} = \int_0^4 dx(4-x) \begin{pmatrix} 2a_{00}(x) \\ 5a_{20}(x) \end{pmatrix} ,$$

$$\begin{pmatrix} J_0 \\ J_2 \end{pmatrix} = \int_0^4 dx(4-x)(3x-4) \begin{pmatrix} a_{00}(x) \\ -2a_{20}(x) \end{pmatrix} , \tag{7.12}$$

$$\begin{pmatrix} S \\ S' \\ S'' \end{pmatrix} = \int_0^4 dx\, x(4-x) \begin{pmatrix} 1 \\ 4-x \\ (4-x)^2 \end{pmatrix} [2a_{00}(x) - 5a_{20}(x)] ,$$

$$\begin{pmatrix} P \\ P' \\ P'' \end{pmatrix} = -3\int_0^4 dx(4-x)^2 \begin{pmatrix} 1 \\ x \\ x(3x-4) \end{pmatrix} a_{11}(x) \quad (x = s/\mu^2).$$

The values of these integrals are given in Table II. It is seen that four of the five relations (7.11) are very well satisfied (agreement to the order of one percent), but the fifth is not. However, the magnitudes of the two integrals involved in the fifth relation are very much smaller than the magnitudes of the integrals in the four well-satisfied relations. The *difference* between the two integrals S'' and P'', although somewhat larger than the differences between the other integrals, is small compared to the magnitudes of those integrals or compared to the integrals of the absolute values of the integrands of the S'', P'' integrals (which integrals are about 200). The polynomials in the S'', P'' integrands have the property that they annihilate the Born terms in the respective integrals. The par-

TABLE II. Values of the integrals (7.12) involved in the test of crossing symmetry.

L	g_A'	I_0	I_2	J_0	J_2	S	P	S'	P'	S''	P''
4.0	1.29	67.4	67.9	107	106	159	159	128	129	0.7	3.8
4.5	1.29	69.7	70.4	109	108	162	162	130	130	0.4	3.9
5.0	1.29	72.2	72.9	112	111	165	165	132	133	0.9	4.0
4.0	1.30	67.5	68.0	107	107	160	160	128	129	0.9	4.0
4.5	1.30	69.8	70.5	110	109	163	162	130	131	0.5	4.1
5.0	1.30	72.3	73.0	112	111	166	165	133	133	0.9	4.2
4.0	1.31	67.6	68.0	108	107	160	160	128	129	1.0	4.2
4.5	1.31	69.9	70.5	110	109	163	163	130	131	0.6	4.3
5.0	1.31	72.3	73.1	112	111	166	165	133	134	0.9	4.5
Born term		57.6	57.6	92	92	138	138	111	111	0	0

tial-wave amplitudes projected out of the first-order invariant matrix element satisfy all five conditions exactly, but only for the fifth relation does the equality reduce to $0=0$. We have also listed in Table II the contributions of the Born term to each of the ten integrals. One can see that even after subtracting out the contributions from the crossing symmetric Born terms, the first four relations are still satisfied to within 5% accuracy.

In investigating the properties of the partial-wave amplitudes below threshold, $0 \leq s < 4\mu^2$, we have discovered that both the $I=0$ and $I=2$ s-wave amplitudes have a weak pole in this region. A little reflection indicates that this should be a general feature of the [1, 1] Padé approximation in any theory in which the Born terms have a zero (implied by current algebra and PCAC) in this region, and the second-order terms are small corrections to the first-order terms (also in the below threshold region, not necessarily at higher energies). This follows simply from inspection of the form of the [1, 1] Padé approximant (6.4)

$$a^{[1,1]}(s) = \frac{[a_1(s)]^2}{a_1(s) - a_2(s)} \ . \tag{7.13}$$

If $a_1(s)$ has a zero in $0 \leq s \leq 4\mu^2$, and $|a_2(s)| \ll |a_1(s)|$ in this region [except for a small subinterval which includes the zero of $a_1(s)$], then the [1, 1] Padé denominator should have a zero close to the zero of the amplitude. The "weakness" of the corresponding pole in $a^{[1,1]}(s)$ follows from the fact that the residue is proportional to $[a_1(s)]^2$, which is very close to its zero. In our particular calculation, the $I=0$ pole at $s \approx \frac{1}{2}\mu^2$ is a ghost while the residue of the $I=2$ pole at $s \approx 2\mu^2$ is of the correct sign for a particle pole (call the particle E for exotic), but very weak, i.e., expressed in terms of an effective coupling constant λ'

$$\mathcal{L}' = -\lambda' E_{ab}\phi_a\phi_b$$

we find $(\lambda')^2 \approx 10^{-5}$. However, we believe that both of these poles are specific to the [1, 1] Padé approximant. The mechanism of its appearance in the [1, 1] approximation—a zero in the dominant term in the [1, 1] denominator—is absent in the next diagonal approximation.

$$a^{[2,2]} = \frac{(a_1a_3 - a_2^2)a_1 + (a_1a_3 - a_2^2)a_2 + (a_1a_2a_3 - a_1^2a_4)}{a_1a_3 - a_2^2 + a_2a_3 - a_1a_4 + a_2a_4 - a_3^2} \ . \tag{7.14}$$

In this case a_1a_3 and a_2^2 are of the same (fourth) order with the other terms being of higher order, and presumably smaller for small s; so that the zero of $a_1(s)$ does not imply the vanishing of the denominator, $a_1a_3 - a_2^2 +$smaller, nearby.

Finally, we give a brief comparison with the earlier one-loop Padé calculation by Basdevant and Lee based on the LσM. First, the LσM calculation did not include any contribution from nucleon loops and it did start with an elementary σ-particle as input (in the s-channel the σ-pole is displaced into the second sheet by the Padé approximant). In the strict one-loop approximation the LσM has one parameter, m_σ or equivalently the coupling constant $\lambda = (m_\sigma^2 - m_\pi^2)/2\mathfrak{F}^2$. However, Basdevant and Lee also treated $\mathfrak{F} \approx f_\pi$ as a variable parameter to approximate the effects of higher-order calculations and to improve the fit to the experimental data, so the two calculations have the same number of free parameters. The calculated s-wave phase shifts are quite similar, but the flattening out of the $I=0$ s-wave phase shift above 700 MeV, in disagreement with recent experimental determinations, is more pronounced in the LσM calculation than in the NLσM calculation. In each calculation the parameters may be adjusted to get the correct ρ mass but the width is not adjustable. The NLσM calculation gets the correct ρ width while the LσM gets much too small a value for Γ_ρ (35 MeV). The only phenomenological advantage of the LσM calculation is that the σ exchange terms in the t and u channels generate all partial waves so that Basdevant and Lee could also apply the [1, 1] Padé approximant to the d waves and generate an f^0 resonance of reasonable mass. In the NLσM the Born term contains only s and p waves so the d waves cannot be treated in the Padé scheme until higher orders are calculated. However, this advantage of the LσM is offset by the fact that the LσM calculation also gives an $I=2$ (exotic) d-wave resonance 200 MeV above the f^0, and one must appeal to higher-order approximation for its elimination. Finally, the s- and p-wave amplitudes coming from the NLσM calculation satisfy the Roskies relations required by crossing symmetry significantly better than do the partial-wave amplitudes from the LσM calculation.

VIII. DISCUSSION

We have given a complete one-loop calculation of the $\pi\pi$ scattering amplitude based on the NLσM which incorporates all the features implied by the $SU_2 \times SU_2$ current algebra and PCAC in a model which starts with a Lagrangian including only the stable pions and nucleons. We have adopted and slightly refined a regularization procedure for the NLσM based on the $m_\sigma \to \infty$ limit of the LσM, so that the complete one-loop $\pi\pi$ scattering matrix element depends on only one undetermined parameter. We have then applied the Padé algorithm to the divergent perturbation series, in order to have an approximation scheme which has some hope of converging and which is capable of predicting the spectrum of resonances. Thus the input is a Lagrangian, a regularization scheme which involves one undetermined constant in addition to the experimental values of the pion and nucleon masses and the pion decay constant, and a systematic approximation scheme. In the first order of approximation—the [1, 1] Padé approximant constructed from the zero- and one-loop diagrams we can get reasonable $I=0, 2$ s-wave phase shifts and an $I=1$, $J=1$ resonance but there is not enough binding in the p wave to get the ρ resonance near its experimental value. We then made a handwaving argument that higher-order approximations would give increased p-wave binding and further, that the increased binding could be simply taken into account by the introduction of one additional parameter into the scattering matrix element constructed in the first approximation. This led to a two-parameter model which gives a rather good account of the known experimental features of $\pi\pi$ scattering up to 900 MeV including the ρ resonance and no narrow $\epsilon(\sigma)$ resonance.

There are a number of aspects to be considered in assessing the significance of these results. First, one cannot hope to calculate $\pi\pi$ scattering above 900 MeV in a purely $SU_2 \times SU_2$ framework because of the great influence the opening of the $K\bar{K}$ channel is observed to have on δ_{00}. We are currently considering how to include the kaons in the scheme described above, in order to open the way to calculations at higher energies, and also to see if inclusion of the kaons will lead to a calculated δ_{00} which does not flatten out below the $K\bar{K}$ threshold. The second general area of concern is the approximation scheme used. It is basically a low-energy approximation and to go higher in energy it may well be necessary to go to higher-order Padé approximants. These will almost surely be necessary to produce higher resonances and it may also be that the flattening out of the calculated δ_{00} between 700 and 900 MeV is a feature of [1, 1] Padé approximant. It would also be very desirable to have an honest, no additional parameter, calculation of the second diagonal Padé approximant ([2, 2]) to check that the claimed additional p-wave binding actually appears, and that the weak $I=0$ ghost and $I=2$ particle poles below threshold disappear. Unfortunately such a detailed calculation may be out of reach, because the [2, 2] Padé requires two more orders of perturbation theory, i.e., through three loops which is probably prohibitive[20] for the LσM which is needed to regularize the NLσM in the present scheme. What may be feasible, and is currently under investigation, is a two-loop perturbation calculation in the simplified chiral-invariant (zero pion mass) model discussed in Appendix E. This would be sufficient to construct the [1, 2] approximant, which is also unitary, and to get some feeling for the stability of the successive steps in the approximation scheme and whether the changes are improvements. A two-loop perturbation calculation would also be valuable to verify that the regularization scheme, the $M \to \infty$ limit of the LσM, does provide a finite matrix element in higher orders with no additional undetermined parameters. We speculate that the two-loop matrix element would include terms proportional to $[\ln(M^2/\mu^2)]^2$ as well as $\ln(M^2/\mu^2)$, but this does not introduce any new parameter. A third general area of concern is one common to all model calculations of $\pi\pi$ scattering. That is the $\pi\pi$ system at low energy is highly constrained by crossing, unitarity, and analyticity; one can obtain nontrivial confrontations with experiment proceeding just from the "axioms".[21] Thus one has the feeling that any model which satisfies some of these properties exactly and the others to a good approximation and has the feeling that any model which satisfies some of these properties exactly and the others to a good approximation and has one or two adjustable parameters should give a reasonable account of

$\pi\pi$ scattering over some greater or lesser energy range. In fact, there do exist a plethora of model calculations of $\pi\pi$ scattering, too numerous to list. One attractive point in favor of a calculational scheme starting from a Lagrangian is that it is not in any way specific to just one process. In particular, if there is any wider significance to the calculations reported here, the same scheme should work also in the kinematically more complicated case of πN scattering (and possibly also NN scattering and $\pi N \to \pi\pi N$, etc.). In fact, Filkov and Palyushev[22] have already published a πN calculation which starts from the NLσM Born term (7.2), computes the second-order s- and u-channel discontinuities by iteration of the Born term in the unitarity equation, parametrizes the second-order t-channel discontinuities by a subtraction in the fixed-t dispersion relation, projects out partial-wave amplitudes, and computes the [1, 1] Padé approximant s-, p-, and d-wave $I=\frac{1}{2}$ and $\frac{3}{2}$ phase shifts from threshold to 1800 MeV center-of-mass energy. The agreement with the experimentally well-determined πN phase shifts is very good. The lowest-lying resonance is found correctly in each channel and no resonance is found in those channels which do not have low-lying resonances. However, Bergere and Drouffe[23] have done similar calculations and report poor agreement. We (and presumably others) are currently engaged in Padé calculations with the NLσM for the πN system in the hope of resolving this discrepancy. If the Filkov and Palyushev results, or something like them, hold up, those πN results together with the $\pi\pi$ results reported here constitute some impressive success for Padé calculations based on the regularized NLσM.

APPENDIX A: CALCULATION OF THE IMAGINARY PART OF THE ONE-LOOP AMPLITUDE IN THE NONLINEAR σ MODEL

The sum in (3.8), for $(\gamma)=\pi\pi$, includes an integral over two-pion phase space,

$$\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}(2\pi)^2\delta_+((k+p)^2-\mu^2)\,\delta_+((k-q)^2-\mu^2)=\frac{1}{16\pi}\left(\frac{s-4\mu^2}{s}\right)^{1/2}\theta(s-4\mu^2)\equiv -\mathrm{Im}I_{\pi\pi}(s)\;. \tag{A1}$$

In terms of this integral, the contribution of the $\pi\pi$ intermediate states in the s channel is

$$\mathrm{disc}_s M^{(2)}_{abcd} \underset{(\pi\pi)}{=} \frac{1}{f^4}\left[\delta_{ab}\delta_{cd}A_{\pi\pi(s)}(p,q,p',q')+\delta_{ac}\delta_{bd}B_{\pi\pi(s)}(p,q,p',q')+\delta_{ad}\delta_{bc}C_{\pi\pi(s)}(p,q,p',q')\right]\left[-i\,\mathrm{Im}I_{\pi\pi}(s)\right], \tag{A2}$$

where

$$A_{\pi\pi(s)}(p,q,p',q')=(s-\mu^2)(2s+t+u-3\mu^2)\;, \tag{A3a}$$

$$B_{\pi\pi(s)}(p,q,p',q')=\tfrac{1}{2}s^2+\tfrac{1}{6}s(t-u)-\tfrac{1}{2}s(s+t+u)-\tfrac{2}{3}\mu^2(t-u) +\tfrac{1}{6}\left(1-4\frac{\mu^2}{s}\right)(p^2-q^2)(p'^2-q'^2)+\tfrac{1}{2}(p^2+q^2)(p'^2+q'^2)\;, \tag{A3b}$$

$$C_{\pi\pi(s)}(p,q,p',q')=\tfrac{1}{2}s^2+\tfrac{1}{6}s(u-t)-\tfrac{1}{2}s(s+t+u)-\tfrac{2}{3}\mu^2(u-t) -\tfrac{1}{6}\left(1-4\frac{\mu^2}{s}\right)(p^2-q^2)(p'^2-q'^2)+\tfrac{1}{2}(p^2+q^2)(p'^2+q'^2)\;. \tag{A3c}$$

We have computed this discontinuity for arbitrary off-shell *external* pions, because we want eventually to impose off-shell current-algebra and PCAC conditions on the corresponding real part of the matrix element. There are similar discontinuities in the t and u variables from the contributions of the 2π intermediate states to the unitarity equations for the t- and u-channel reactions. Defining

$$\mathrm{disc}M=2i\,\mathrm{Im}M\;, \tag{A4}$$

and including the t- and u-channel discontinuities, we obtain

$$\mathrm{Im}_{\pi\pi}A^{(2)}(s,t,u;p^2,q^2,p'^2,q'^2)=\frac{1}{f^4}\big[\alpha(s,t,u;p^2,q^2,p'^2,q'^2)\mathrm{Im}I_{\pi\pi}(s) +\beta(s,t,u;p^2,q^2,p'^2,q'^2)\mathrm{Im}I_{\pi\pi}(t)+\beta(s,u,t;p^2,q^2,q'^2,p'^2)\mathrm{Im}I_{\pi\pi}(u)\big]\;, \tag{A5}$$

where $\alpha(s,t,u;p^2,q^2,p'^2,q'^2)$ and $\beta(s,t,u;p^2,q^2,p'^2,q'^2)$ are the functions given in (3.12a) and (3.12b). The imaginary parts of the invariant functions $B^{(2)}$ and $C^{(2)}$ follow from (A5) by the substitution rules (2.5).

To compute the $(\gamma)=N\bar{N}$ contributions to (3.8) we have to substitute the first-order $\pi\pi\to N\bar{N}$ invariant matrix element (3.9) into (3.8). Then for the s-channel discontinuity we require the following integrals over the $N\bar{N}$ phase space

$$\int \frac{d^4k}{(2\pi)^4}(2\pi)^2\delta_+((k+p)^2-m^2)\delta_+((k-q)^2-m^2)=\frac{1}{8\pi}\left(\frac{s-4m^2}{s}\right)^{1/2}\theta(s-4m^2)\equiv -2i\,\mathrm{Im}I_{NN}(s)\,, \tag{A6a}$$

$$\int \frac{d^4k}{(2\pi)^4}(2\pi)^2\delta_+((k+p)^2-m^2)\delta_+((k-q)^2-m^2)\frac{1}{k^2-m^2}\equiv -2\,\mathrm{Im}K_N(s;p^2,q^2)\equiv -2\,\mathrm{Im}K_N(p,q)\,, \tag{A6b}$$

$$\int \frac{d^4k}{(2\pi)^4}(2\pi)^2\delta_+((k+p)^2-m^2)\delta_+((k-q)^2-m^2)\frac{1}{(k^2-m^2)[(k+p-p')^2-m^2]}\equiv -2\,\mathrm{Im}_s H_N(s,t,u;p^2,q^2,p'^2,q'^2)$$
$$\equiv -2\,\mathrm{Im}_s H_N(p,q,p',q')\,. \tag{A6c}$$

In terms of these integrals, the contribution of the $N\bar{N}$ intermediate states is

$$\begin{aligned}\mathrm{disc}_s M^{(2)}_{abcd\,(N\bar{N})} = g'^4\Big\{ &\delta_{ab}\delta_{cd}\Big[-8\frac{s}{m^2}\,i\,\mathrm{Im}I_{NN}(s)+8(p^2q'^2+p'^2q^2-st)i\,\mathrm{Im}_s H_N(p,q,p',q')\\ &\quad +8(p^2p'^2+q^2q'^2-su)i\,\mathrm{Im}_s H_N(p,q,q',p')\Big]\\ &+\delta_{ac}\delta_{bd}[8(p^2q'^2+p'^2q^2-st)i\,\mathrm{Im}_s H_N(p,q,p',q')-8(p^2p'^2+q^2q'^2-su)i\,\mathrm{Im}_s H_N(p,q,q',p')]\\ &+\delta_{ad}\delta_{bc}[-8(p^2q'^2+p'^2q^2-st)i\,\mathrm{Im}_s H_N(p,q,p',q')+8(p^2p'^2+q^2q'^2-su)i\,\mathrm{Im}_s H_N(p,q,q',p')]\Big\}.\end{aligned} \tag{A7}$$

The K integrals (A6b) appear at intermediate stages, but cancel out in the result.

Again, there are discontinuities in the t and u variables from the contributions of the $N\bar{N}$ states to the t- and u-channel unitarity equations. Taking these into account we obtain

$$\begin{aligned}\mathrm{Im}_{N\bar{N}}A^{(2)}(s,t,u;p^2,q^2,p'^2,q'^2)=g'^4[&-4(s/m^2)\mathrm{Im}I_{NN}(s)+4(p^2q'^2+p'^2q^2-st)\mathrm{Im}H_N(s,t,u;p^2,q^2,p'^2,q'^2)\\ &+4(p^2p'^2+q^2q'^2-su)\mathrm{Im}H_N(s,u,t;p^2,q^2,q'^2,p'^2)\\ &-4(p^2q^2+p'^2q'^2-ut)\,\mathrm{Im}H_N(u,t,s;p^2,q'^2,p'^2,q^2)]\,.\end{aligned} \tag{A8}$$

Combining (A5) and (A8) gives (3.10) of the text.

APPENDIX B: THE LINEAR σ MODEL, RENORMALIZATION, AND THE $m_\sigma\to\infty$ LIMIT

Renormalization of the linear σ model, consistent with broken chiral symmetry, has been carried out by Lee,[24] by Symanzik,[25] and by Bessis and Turchetti.[26] The $m_\sigma\to\infty$ limit has been proposed and discussed by Bessis and Zinn-Justin[7] (for the LσM without nucleons). This appendix is included partly for completeness, partly to give the straightforward generalization required to include the nucleons in the $m_\sigma\to\infty$ limit discussion, and partly because we differ from the treatment of Bessis and Zinn-Justin in regard to the details of the $\mu\to 0$ chiral-invariant limit.

Although the renormalization of the LσM has been more then adequately treated in the references cited, it is necessary to include at least a summary of the definitions and formulas involved in order to be able to discuss the $m_\sigma\to\infty$ limit, which is the point of interest for the calculations of this paper. The Lagrangian for the LσM is

$$\mathcal{L}_{\rm inv}=\tfrac{1}{2}[(\partial\vec{\phi})^2+(\partial\chi)^2]-\tfrac{1}{2}\mu_0^2(\vec{\phi}^{\,2}+\chi^2)+i\bar{\psi}\gamma\cdot\partial\psi$$
$$-\tfrac{1}{4}\lambda_0(\vec{\phi}^{\,2}+\chi^2)^2-g_0\bar{\psi}(\chi-i\gamma_5\vec{\tau}\cdot\vec{\phi})\psi\,, \tag{B1a}$$

$$\mathcal{L}_{\rm SB}=c_0\chi\,. \tag{B1b}$$

Here χ is the unrenormalized canonical σ field, whose inclusion admits linear chiral transformations for the π field. The axial-vector current is

$$A_{\mu,a}=\phi_a\partial_\mu\chi-\chi\partial_\mu\phi_a+\bar{\psi}\gamma_\mu\gamma_5\tfrac{1}{2}\tau_a\psi\,, \tag{B2}$$

and satisfies

$$\partial^\mu A_{\mu,a}=c_0\phi_a\,. \tag{B3}$$

The scalar field χ may have a nonzero vacuum expectation value

$$\langle\chi\rangle=\mathfrak{F}_0\,, \tag{B4}$$

so we define a translated field

$$\hat{\chi}=\chi-\mathfrak{F}_0\,. \tag{B5}$$

In terms of this field the LσM Lagrangian is

$$\mathcal{L}=\tfrac{1}{2}((\partial\vec{\phi})^2-\mu_1{}^2\vec{\phi}^2)+\tfrac{1}{2}((\partial\hat{\chi})^2-M_1{}^2\hat{\chi}^2)$$
$$+\bar{\psi}(i\gamma\cdot\partial-m_1)\psi-\lambda_0\mathfrak{F}_0\hat{\chi}(\vec{\phi}^2+\hat{\chi}^2)-\tfrac{1}{4}\lambda_0(\vec{\phi}^2+\hat{\chi}^2)^2$$
$$+ig_0\bar{\psi}\gamma_5\vec{\tau}\psi\cdot\vec{\phi}-g_0\bar{\psi}\psi\hat{\chi}\ . \tag{B6}$$

The relations

$$\mu_1{}^2=\mu_0{}^2+\lambda_0\mathfrak{F}_0{}^2\ ,$$
$$M_1{}^2=\mu_0{}^2+3\lambda_0\mathfrak{F}_0{}^2\ , \tag{B7}$$
$$m_1=g_0\mathfrak{F}_0$$

come out directly from the algebra of going from (B1) to (B6) by the substitution of (B5). In order to ensure the consistency of (B5), i.e., to ensure that $\langle\hat{\chi}\rangle=0$, it is also necessary to impose the condition

$$c_0=\mathfrak{F}_0\mu_0{}^2+\lambda_0\mathfrak{F}_0{}^3=\mathfrak{F}_0\mu_1{}^2\ . \tag{B8}$$

As we shall see, these conditions have their counterparts in renormalized perturbation theory.

The renormalized field operators are

$$\pi_a=\frac{1}{\sqrt{Z_\pi}}\phi_a,\quad \sigma=\frac{1}{\sqrt{Z_\pi}}\hat{\chi},\quad N=\frac{1}{\sqrt{Z_N}}\psi\ . \tag{B9}$$

Also,

$$\mathfrak{F}=\frac{1}{\sqrt{Z_\pi}}\mathfrak{F}_0\ ,\quad c=\sqrt{Z_\pi}\,c_0\ ; \tag{B10}$$

so

$$\partial^\mu A_{\mu,a}=c\pi_a\ , \tag{B11a}$$

$$c=f_\pi\mu^2\ . \tag{B11b}$$

Note that the renormalization of the σ field is not the conventional one; i.e., in the presence of symmetry breaking $Z_\sigma\neq Z_\pi$, so the renormalization (B9) gives $\langle\Omega|\sigma|1\sigma\rangle\neq1$. Since we are interested in the limit $m_\sigma\to\infty$ in which there is no asymptotic one-sigma state, this unconventional normalization does not create any problems, and it does lead to simple Ward identities. Since the commutation relations of the canonical fields with the generators of chiral transformations are linear in the fields, the renormalized boson fields, renormalized according to (B9), satisfy the same chiral commutation relations,

$$\delta(x_0-y_0)[A_a^0(x),\pi_b(y)]=i\delta(x-y)\delta_{ab}(\sigma(y)+\mathfrak{F})\ , \tag{B12a}$$

$$\delta(x_0-y_0)[A_a^0(x),\sigma(y)]=-i\delta(x-y)\pi_a(y)\ . \tag{B12b}$$

Using (B11) and (B12), one can derive Ward identities relating the two-point, three-point, and four-point renormalized Green's function.

The functions with which we are concerned are

$$\langle\Omega|(\pi_a(x)\pi_b(y))_+|\Omega\rangle=iD_{ab}(x-y)$$
$$=\delta_{ab}iD_\pi(x-y)\ , \tag{B13a}$$

$$\langle\Omega|(\sigma(x)\sigma(y))_+|\Omega\rangle=iD_\sigma(x-y)\ , \tag{B13b}$$

$$\langle\Omega|(\pi_a(x)\pi_b(y)\sigma(z))_+|\Omega\rangle=\int dx'dy'dz'iD_\sigma(z-z')iV_{ab}(x'y',z')iD_\pi(x'-x)iD_\pi(y'-y)\ , \tag{B13c}$$

$$\langle\Omega|(\pi_a(x)\pi_b(y)\pi_c(z)\pi_d(w))_+|\Omega\rangle=\int dx'dy'dz'dw'iD_\pi(z-z')iD_\pi(w-w')iV_{abcd}(x'y'z'w')iD_\pi(x'-x)iD_\pi(y'-y)$$
$$+\langle\Omega|(\pi_a(x)\pi_b(y))_+|\Omega\rangle\langle\Omega|(\pi_c(z)\pi_d(w))_+|\Omega\rangle+\langle ac\rangle\langle bd\rangle+\langle ad\rangle\langle bc\rangle\ .$$

The Fourier transforms of these functions satisfy the Ward identities.

$$\text{(I)}\ D_\pi(q=0)^{-1}=-\mu^2f_\pi/\mathfrak{F}\ , \tag{B14a}$$

$$\text{(II)}\ \mathfrak{F}V_{\pi\pi\sigma}(p=0,q)=D_\sigma(q)^{-1}-D_\pi(q)^{-1}\ , \tag{B14b}$$

$$\text{(III)}\ \mathfrak{F}A(p=0,q,p',q')D_\sigma(q)^{-1}=D_\pi(q)^{-1}V_{\pi\pi\sigma}(p',q')\ , \tag{B14c}$$

where

$$V_{ab}(p,q)=\delta_{ab}V_{\pi\pi\sigma}(p,q)\ , \tag{B15a}$$

$$V_{abcd}(p,q,p',q')=\delta_{ab}\delta_{cd}A(p,q,p',q')+\delta_{ac}\delta_{bd}B(p,q,p',q')+\delta_{ad}\delta_{bc}C(p,q,p',q')\ . \tag{B15b}$$

The third Ward identity, combined with the conventional pion renormalization

$$D_\pi(q)^{-1}|_{q^2=\mu^2}=0\ , \tag{B16a}$$

$$\left(\frac{d}{dq^2}D_\pi(q)^{-1}\right)_{q^2=\mu^2}=1\ , \tag{B16b}$$

gives both the Adler zero (one soft pion)

$$A(p=0,q,p',q')|_{q^2=\mu^2;\ p',q'\ \text{arbitrary}}=0 \tag{B17}$$

and the Adler-Weisberger-Weinberg low-energy theorem (two soft pions)

$$\left(\frac{d}{dq^2}A(p=0,q,p'=0,q)\right)_{q^2=\mu^2}=\frac{1}{\mathfrak{F}^2}\ . \tag{B18}$$

These results are independent of perturbation theory, and independent of m_σ.

The next task is to construct these functions through the one-loop approximation in perturbation theory. We follow the perturbation treatment of Symanzik, in which the Ward identities play a central role. This approach is based on the Bogoliubov-Parasiuk-Hepp-Zimmerman (BPHZ) formulation[27] of perturbation theory.

$$\text{Def: } \langle\Omega|(\pi_a\cdots\sigma\cdots\overline{N}\cdots N\cdots)_+|\Omega\rangle = \text{F.P.}\left\langle 0\left|\left\{\pi_{a_{\text{in}}}\cdots\sigma_{\text{in}}\cdots\overline{N}_{\text{in}}\cdots N_{\text{in}}\cdots\exp\left[i\int\mathcal{L}_{\text{eff}}(\pi_{\text{in}},\sigma_{\text{in}},\overline{N}_{\text{in}},N_{\text{in}})\right]\right\}_+\right|0\right\rangle . \tag{B19}$$

The F.P. stands for the BPHZ finite part which in general is a complicated but systematic subtraction procedure which renders finite the Feynman integrals generated by the Wick reduction of the right-hand side, but in the simple one-loop case merely instructs one to expand the integrand in a Taylor series about zero external momenta and drop the divergent terms. The prescription also includes the instruction to treat the ordered product of in-fields as a T^* product, i.e., to drop noncovariant terms. To make up for these mutilations of the canonical, unrenormalized perturbation expansion, one adds to the Lagrangian all finite counterterms of dimension less than or equal to four, consistent with the symmetry of the Lagrangian. Then one uses the conventional renormalization conditions and the Ward identities to fix these finite counterterms in terms of the renormalized masses and coupling constants of the theory.

$$\begin{aligned}\mathcal{L}_{\text{eff}} = {}&\tfrac{1}{2}(1-\beta_\pi)(\partial\vec{\pi})^2 - \tfrac{1}{2}(\mu^2+\alpha_\pi)\vec{\pi}^2 + \tfrac{1}{2}(1-\beta_\sigma)(\partial\sigma)^2\\ &-\tfrac{1}{2}(M^2+\alpha_\sigma)\sigma^2 + (1-\beta_N)\overline{N}i\gamma\cdot\partial N\\ &-(m+\alpha_N)\overline{N}N - (\lambda-\gamma)\mathfrak{F}\sigma(\vec{\pi}^2+\sigma^2)\\ &-\tfrac{1}{4}(\lambda-\gamma)(\vec{\pi}^2+\sigma^2)^2 + i(g-\delta)\overline{N}\gamma_5\vec{\tau}N\cdot\vec{\pi}\\ &-(g-\delta)\overline{N}N\sigma\ .\end{aligned} \tag{B20}$$

The constants μ, M, m are the renormalized pion, σ,[28] and nucleon masses; λ and g are conventionally renormalized coupling constants, e.g., the value of the $\pi\pi$ matrix element at the symmetry point, or the residue of the nucleon pole terms in the πN matrix element, etc. We define effective coupling constants[29]

$$\begin{aligned}\lambda' &= \lambda - \gamma\ ,\\ g' &= g-\delta\ .\end{aligned} \tag{B21}$$

We first compute the functions through the one-loop approximation in terms of the constants appearing in (B20) and (B21). If we impose the condition that $\alpha_\sigma - \alpha_\pi$ vanish in zeroth order, then the second Ward identity gives the following relations among the constants:

$$\begin{aligned}\lambda &= (M^2-\mu^2)/2\mathfrak{F}^2\ ,\\ \alpha_\sigma &= \alpha_\pi - 2\gamma\mathfrak{F}^2\ ,\\ \beta_\sigma &= \beta_\pi\ ,\\ g' &= m/\mathfrak{F}\ ,\end{aligned} \tag{B22}$$

with γ an undetermined constant except that it is of order $\mathfrak{F}^{-4}$. The constants α_π and β_π are determined by the renormalization conditions for $D_\pi(q)$, Eqs. (B16). The first Ward identity gives an equation for $\mathfrak{F}$ in terms of the other constants, so the functions depend on two constants, M and γ. We do not impose any renormalization condition on D_σ or $V_{\pi\pi\sigma}$ because in the limit $m_\sigma\to\infty$, we do not treat the σ as a physical particle. Thus if the constant γ is to be fixed it will have to be through a condition on the physical $\pi\pi$ matrix element, i.e., on the invariant function A. Substituting the results (B22) and reexpressing the perturbation expansion in powers of $\mathfrak{F}^{-1}$, we obtain the results

$$\begin{aligned}D_\pi(q)^{-1} &= q^2-\mu^2-\Sigma_\pi(q)\\ &= q^2-\mu^2-\beta_\pi q^2-\alpha_\pi-\frac{(M^2-\mu^2)}{\mathfrak{F}^2}I^{(0)}_{\sigma\pi}(q^2)\\ &\quad -4\frac{m^2}{\mathfrak{F}^2}q^2 I^{(0)}_{NN}(q^2)\ ,\end{aligned} \tag{B23}$$

$$\begin{aligned}\alpha_\pi = {}&\frac{(M^2-\mu^2)^2}{\mathfrak{F}^2}[-I^{(0)}_{\sigma\pi}(\mu^2)+\mu^2 I'_{\sigma\pi}(\mu^2)]\\ &+4\frac{m^2}{\mathfrak{F}^2}\mu^4 I'_{NN}(\mu^2)\ ,\end{aligned} \tag{B24a}$$

$$\begin{aligned}\beta_\pi = {}&-\frac{(M^2-\mu^2)^2}{\mathfrak{F}^2}I'_{\sigma\pi}(\mu^2)\\ &-4\frac{m^2}{\mathfrak{F}^2}[I^{(0)}_{NN}(\mu^2)+\mu^2 I'_{NN}(\mu^2)]\ ,\end{aligned} \tag{B24b}$$

$$D_\sigma(q)^{-1} = q^2 - M^2 - \bar{\Sigma}_\sigma(q) + 2\gamma \mathfrak{F}^2$$
$$= q^2 - M^2 - \beta_\pi q^2 - \alpha_\pi - \tfrac{3}{2}\frac{(M^2-\mu^2)^2}{\mathfrak{F}^2} I^{(0)}_{\pi\pi}(q^2) - \tfrac{9}{2}\frac{(M^2-\mu^2)^2}{\mathfrak{F}^2} I^{(0)}_{\sigma\sigma}(q^2) - 4\frac{m^2}{\mathfrak{F}^2}(q^2-4m^2) I^{(0)}_{NN}(q^2) + 2\gamma\mathfrak{F}^2 , \tag{B25}$$

$$V_{\pi\pi\sigma}(p,q) = -\frac{M^2-\mu^2}{\mathfrak{F}} + 2\gamma\mathfrak{F} - \frac{(M^2-\mu^2)^2}{\mathfrak{F}^3}\{(M^2-\mu^2)[K^{(0)}_{\sigma\pi\pi}(p,q) + 3K^{(0)}_{\pi\sigma\sigma}(p,q)] + \tfrac{5}{2} I^{(0)}_{\pi\pi}(s)$$
$$+ \tfrac{3}{2} I^{(0)}_{\sigma\sigma}(s) + I^{(0)}_{\sigma\pi}(p^2) + I^{(0)}_{\sigma\pi}(q^2)\}$$
$$+ 8\frac{m^4}{\mathfrak{F}^3}[2I^{(0)}_{NN}(s) + (s-p^2-q^2)K_{NNN}(p,q)]$$
$$= -\frac{M^2-\mu^2}{\mathfrak{F}} + \bar{V}^{(2)}_{\pi\pi\sigma}(p,q) + 2\gamma\mathfrak{F} , \tag{B26}$$

$$A(p,q,p',q') = A^{(1)} + A^{(2)}_p + A^{(2)}_{\rm IR}$$
$$= \bar{A} + 2\gamma\left(\frac{s-\mu^2}{s-M^2}\right)^2 , \tag{B27a}$$

$$A^{(1)} = -\frac{m^2-\mu^2}{\mathfrak{F}^2}\left(\frac{s-\mu^2}{s-M^2}\right) , \tag{B27b}$$

$$A^{(2)}_p = \frac{M^2-\mu^2}{\mathfrak{F}}\ \frac{1}{s-M^2}[\bar{V}^{(2)}_{\pi\pi\sigma}(p,q) + \bar{V}^{(2)}_{\pi\pi\sigma}(p',q')] - \frac{(M^2-\mu^2)^2}{\mathfrak{F}^2}\ \frac{1}{(s-M^2)^2}\bar{\Sigma}^{(2)}_\sigma(s) + 2\gamma\left(\frac{s-\mu^2}{s-M^2}\right)^2 - 2\gamma , \tag{B27c}$$

$$A^{(2)}_{\rm IR} = 2\gamma - \frac{(M^2-\mu^2)^2}{\mathfrak{F}^4}\{\tfrac{7}{2} I^{(0)}_{\pi\pi}(s) + I^{(0)}_{\pi\pi}(t) + I^{(0)}_{\pi\pi}(u) + \tfrac{1}{2} I^{(0)}_{\sigma\sigma}(s)$$
$$+ (M^2-\mu^2)[K^{(0)}_{\pi\sigma\sigma}(p,q) + K^{(0)}_{\pi\sigma\sigma}(p',q') + K^{(0)}_{\sigma\pi\pi}(p,q) + K^{(0)}_{\sigma\pi\pi}(p',q')$$
$$+ K^{(0)}_{\sigma\pi\pi}(-q',q) + K^{(0)}_{\sigma\pi\pi}(p,-p') + K^{(0)}_{\sigma\pi\pi}(p,-q') + K^{(0)}_{\sigma\pi\pi}(-p',q)]$$
$$+ (M^2-\mu^2)^2[H^{(0)}_{\pi\sigma\pi\sigma}(p,q,p',q') + H^{(0)}_{\pi\sigma\pi\sigma}(p,q,q',p')]\}$$
$$+ 4\frac{m^4}{\mathfrak{F}^4}[4I^{(0)}_{NN}(s) + 2(s-p^2-q^2)K_{NNN}(p,q) + 2(s-p'^2-q'^2)K_{NNN}(p',q')$$
$$+ (p^2q'^2 + p'^2q^2 - st)H_{NNNN}(p,q,p',q') + (p^2p'^2 + q^2q'^2 - su)H_{NNNN}(p,q,q',p')$$
$$- (p^2q^2 + p'^2q'^2 - ut)H_{NNNN}(p,-q',p',-q)] . \tag{B27d}$$

The integrals I, K, H are the integrals introduced in Sec. III. ($K_{NNN} \equiv K_N$ and $H_{NNNN} \equiv H_N$.) The superscript indicates one subtraction at the point where all external four-momenta vanish.

$$K^{(0)}(p,q) = K(p,q) - K(0,0), \quad \text{etc} . \tag{B28}$$

Detailed formulas are given in Appendix C. Using the identities

$$K_{abc}(0,q) = \frac{1}{m_a^2 - m_b^2}[I_{ac}(q^2) - I_{bc}(q^2)] ,$$

$$H_{\pi\sigma\pi\sigma}(0,q,p',q') = \frac{1}{M^2-\mu^2}[K_{\pi\sigma\sigma}(p',q') - K_{\sigma\pi\pi}(-q',q)],$$

one can check explicitly that the functions of (B23–B27) satisfy the second and third Ward identities (B14b) and (B14c). The first Ward identity (B14a) gives an equation for $\mathfrak{F}$

$$D_\pi(q=0)^{-1} = -\mu^2 - \alpha_\pi$$
$$= -\mu^2\frac{f_\pi}{\mathfrak{F}} ,$$

or

$$\mathfrak{F} = f_\pi\left\{1 + \frac{(M^2-\mu^2)^2}{\mu^2\mathfrak{F}^2}[-I^{(0)}_{\sigma\pi}(\mu^2) + \mu^2 I'_{\sigma\pi}(\mu^2)] + 4\frac{m^2}{\mathfrak{F}^2}\mu^2 I'_{NN}(\mu^2)\right\}^{-1} . \tag{B29}$$

In the $M\to\infty$ limit, the $\sigma\pi$ term goes to $-(\mu^2/6\mathfrak{F}^2)/16\pi^2$. (See Appendix C for expansions of the integrals). To estimate the magnitude of the NN term, we may consider the leading term in an expansion in powers of μ^2/m^2. Using that result, (B29) becomes

$$\mathfrak{F} \simeq f_\pi\left[1 + \frac{1}{16\pi^2}\left(-\frac{5}{6}\frac{\mu^2}{\mathfrak{F}^2}\right)\right]^{-1} . \tag{B29'}$$

Because $1/16\pi^2$ is a small number, the solutions of (B29′) are approximately

$$\mathfrak{F} \simeq f_\pi, \; -\frac{5}{96\pi^2}\frac{\mu^2}{f_\pi} \; . \tag{B30}$$

Only the first solution leads to a Born term (B27b) which, in the limit $M\to\infty$, agrees with the current-algebra and PCAC result (Weinberg); so we accept it and reject the second solution. Since the estimate shows that the actual solution of (B29) differs from f_π by the order of one percent, we simply use $\mathfrak{F}=f_\pi$ in numerical calculations.

The $M\to\infty$ limit of the Born term (B27b) is trivial

$$A^{(1)} \underset{M\to\infty}{\longrightarrow} \frac{1}{\mathfrak{F}^2}(s-\mu^2) \; . \tag{B31}$$

Comparing this with (3.7) we identify

$$f=\mathfrak{F} \; . \tag{B32}$$

The $M\to\infty$ limit is also simple for the nucleon-loop contributions to $A^{(2)}$. The only M dependence comes from the σ-pole terms and considerable simplification ensues when we combine the nucleon-loop contributions to A_p and $A_{\rm IR}$ in this limit. Note that we include the nucleon contributions to α_π, β_π [(B24a) and (B24b)].

$$A_N = 4\frac{m^2}{\mathfrak{F}^2}[-sI_{NN}^{(\pi)}(s)+(s-\mu^2)\mu^2 I_{NN}'(\mu^2)+(p^2q'^2+p'^2q^2-st)m^2H_N(p,q,p',q')$$
$$+(p^2p'^2+q^2q'^2-su)m^2H_N(p,q,q',p')-(p^2q^2+p'^2q'^2-ut)m^2H_N(p,-q',p',-q)] \; , \tag{B33}$$

$$I_{NN}^{(\pi)}(s)=I_{NN}(s)-I_{NN}(\mu^2) \; . \tag{B34}$$

We will use a tilde to denote the functions (constants) of (B23)–(27) with the nucleon contributions taken out

$$\tilde{\Sigma}_\pi(q)=\tilde{\beta}_\pi q^2+\tilde{\alpha}_\pi+\frac{(M^2-\mu^2)^2}{\mathfrak{F}^2}I_{\sigma\pi}^{(0)}(q^2) \; , \tag{B23′}$$

$$\tilde{\alpha}_\pi=\frac{(M^2-\mu^2)^2}{\mathfrak{F}^2}[-I_{\sigma\pi}^{(0)}(\mu^2)+\mu^2 I_{\sigma\pi}'(\mu^2)] \; , \tag{B24a′}$$

etc.

In comparing (B27) with (3.11), we note that our use of the BPHZ perturbation theory has led us to subtract the integrals at zero external momenta, i.e., the $\bar{I}_{\pi\pi}$ integrals in (3.11), for which the subtraction point was not specified, are now determined to be $I_{\pi\pi}^{(0)}$ integrals. Also note that the nucleon contributions to (B27), collected in (B33), contribute the term $4(m^2/\mathfrak{F}^4)\mu^2 I_{NN}'(\mu^2)(s-\mu^2)$ to the polynomial (3.13). Let us denote by P (without the overbar), and constants A, B, etc., the polynomial associated with subtraction at zero external momenta, and not including the nucleon contribution. Our first step in determining these constants is to use the results (B17) and (B18) which follow from the Ward identities, independent of the σ mass M. Since

$$\alpha(p=0,q,p',q')_{q^2=\mu^2}=0 \; ,$$
$$\beta(p=0,q,p',q')_{q^2=\mu^2}=0 \; ,$$

see (3.12); condition (B17) implies that

$$P(p=0,q,p',q')_{q^2=\mu^2;\; p',q'\,\text{arbitrary}}=0 \; . \tag{B35}$$

This gives the relations

$$A+E+G=0 \; , \quad C+F+I=0 \; ,$$
$$D+H=0 \; , \quad B=0 \; . \tag{B36}$$

Similarly,

$$\left(\frac{d}{dq^2}\alpha(p=0,q,p'=0,q)\right)_{q^2=\mu^2}=0 \; ,$$
$$\left(\frac{d}{dq^2}\beta(p=0,q,p'=0,q)\right)_{q^2=\mu^2}=0 \; ;$$

thus (B18) implies that

$$\left(\frac{d}{dq^2}P(p=0,q,p'=0,q)\right)_{q^2=\mu^2}=0 \; . \tag{B37}$$

This gives one additional relation,

$$A-F-G=0 \; . \tag{B38}$$

To obtain additional relations, it is necessary to take the limit $M\to\infty$. In principle, one could just do this for all the terms in (B27), but the asymptotic expansions of the box integrals are quite complicated for general momenta, so it is easier to work out a sequence of special cases. In working these out we use the $M\to\infty$ expansions of the I, K, H integrals given in Appendix C. We find

$$\tilde{A}^{(2)}(o,q,o,q) \underset{M\to\infty}{\longrightarrow} \frac{1}{\mathfrak{F}^4}\left\{(q^2-\mu^2)^2 \times\left[-\tfrac{3}{2}I_{\pi\pi}^{(0)}(q^2)+\frac{1}{16\pi^2}(\tfrac{1}{6}+2\Gamma)\right]\right\}, \tag{B39}$$

where we have defined a new constant Γ; related to γ by

$$\gamma=\frac{M^4}{16\pi^2\mathfrak{F}^4}\Gamma \; . \tag{B40}$$

Comparing with (3.11), we obtain three conditions—the coefficients of q^4, $q^2\mu^2$, and μ^4—but only one is independent of the conditions already obtained (B36), (B38). The new condition is

$$G=\frac{1}{16\pi^2}\left(\tfrac{1}{6}+2\Gamma\right). \tag{B41}$$

The next case we consider is

$$\bar{A}^{(2)}(o,o,-q',q')_{\overrightarrow{M\to\infty}}\ \frac{1}{16\pi^2\mathfrak{F}^4}\left(-\mu^2q'^2L+2\mu^2q'^2+\tfrac{1}{6}\mu^4+2\mu^4\Gamma\right) \tag{B42}$$

$$L\equiv\ln(M^2/\mu^2) \tag{B43}$$

Comparison with (3.11) gives

$$F=\frac{1}{16\pi^2}\left(-\tfrac{1}{2}L+1\right). \tag{B44}$$

In computing (B39) and (B42), considerable labor was saved by using the Ward identities. It was possible to do this because each case involved at least one zero four-momentum. It is clear that we cannot obtain any more independent conditions by considering kinematic situations with any zero four-momentum because if any one four-momentum is equal to zero, then only the combinations $D+H$ and $C+I$ enter in (3.13); i.e., there must be two conditions which can only be obtained by considering all nonzero four-momenta. We work out

$$\bar{A}^{(2)}(-q,q,-q,q)_{\overrightarrow{M\to\infty}}\ \frac{1}{\mathfrak{F}^4}\left\{-q^4I_{\pi\pi}^{(0)}(u)+\frac{1}{16\pi^2}\left[(2q^4-2q^2\mu^2)L-\tfrac{16}{3}q^4+4q^2\mu^2+\tfrac{1}{6}\mu^4+2\mu^4\Gamma\right]\right\},\quad u=4q^2\,. \tag{B45}$$

Comparison with (3.11) gives two conditions consistent with previous conditions and one new (independent) condition

$$H+2I=\frac{1}{16\pi^2}\left(L-\tfrac{8}{3}\right). \tag{B46}$$

The last special case worked out is

$$\begin{aligned}\bar{A}^{(2)}(s,t,u;o,o,o,o)_{\overrightarrow{M\to\infty}}\ \frac{1}{\mathfrak{F}^4}\Big\{&I_{\pi\pi}^{(0)}(s)\left(-\tfrac{1}{2}s^2+2\mu^2s-\tfrac{3}{2}\mu^4\right)\\&+I_{\pi\pi}^{(0)}(t)\left(-\tfrac{1}{3}t^2-\tfrac{1}{6}st+\tfrac{2}{3}\mu^2s+\tfrac{1}{3}\mu^2t\right)+I_{\pi\pi}^{(0)}(u)\left(-\tfrac{1}{3}u^2-\tfrac{1}{6}su+\tfrac{2}{3}\mu^2s+\tfrac{1}{3}\mu^2u\right)\\&+\frac{1}{16\pi^2}\left[\left(-\tfrac{5}{6}s^2-\tfrac{2}{3}tu+\mu^2s\right)L+\tfrac{31}{18}s^2+\tfrac{16}{9}tu-\tfrac{7}{3}\mu^2s+\tfrac{1}{6}\mu^4\right]\Big\}\,.\end{aligned} \tag{B47}$$

Comparison with (3.11) gives three conditions consistent with previous conditions, and one new condition,

$$D=\frac{1}{16\pi^2}\left(-\tfrac{2}{3}L+\tfrac{16}{9}\right). \tag{B48}$$

We note that in the calculations leading to (B45) and (B47), one encounters terms of order $M^2\ln M^2$ and M^2, as well as $\ln M^2$ and 1, but these cancel out in the final results.

Now we solve the nine equations given in (B36), (B38), (B41), (B44), (B46), and (B48) for the nine constants $A, B, \ldots, I$, in terms of the two undetermined constants M, Γ.

$$\begin{aligned}&A=\frac{1}{16\pi^2}\left(-\tfrac{1}{2}L+\tfrac{7}{6}+2\Gamma\right),\\&B=0,\quad C=\frac{1}{16\pi^2}\left(\tfrac{1}{3}L-\tfrac{5}{9}\right),\\&D=\frac{1}{16\pi^2}\left(-\tfrac{2}{3}L+\tfrac{16}{9}\right),\quad E=\frac{1}{16\pi^2}\left(\tfrac{1}{2}L-\tfrac{4}{3}-4\Gamma\right),\\&F=\frac{1}{16\pi^2}\left(-\tfrac{1}{2}L+1\right),\quad G=\frac{1}{16\pi^2}\left(\tfrac{1}{6}+2\Gamma\right),\\&H=\frac{1}{16\pi^2}\left(\tfrac{2}{3}L-\tfrac{16}{9}\right),\quad I=\frac{1}{16\pi^2}\left(\tfrac{1}{6}L-\tfrac{4}{9}\right).\end{aligned} \tag{B49}$$

We now fix the constant Γ by consideration of the chiral invariant limit. The NLσM Lagrangian (3.1), or (3.6), is chiral invariant if $\mu_1=0$. The one-loop amplitude for the chiral-invariant theory (pions only, no nucleons) is given by Allen and Willey[8] or Lehmann and Trute[8]

$$\begin{aligned}\bar{A}^{(2)}(s,t,u)_{m_\pi=0}=\frac{1}{\mathfrak{F}^4}\Big[&-\tfrac{1}{2}s^2\frac{1}{16\pi^2}\ln\left(\frac{-s}{c}\right)\\&-\tfrac{1}{6}t(t-u)\frac{1}{16\pi^2}\ln\left(\frac{-t}{c}\right)\\&-\tfrac{1}{6}u(u-t)\frac{1}{16\pi^2}\ln\left(\frac{-u}{c}\right)\Big],\end{aligned} \tag{B50}$$

with c an arbitrary constant. In fact, chiral invariance alone would allow one constant in the t and u logarithms and a different constant in the s logarithm. The two papers cited gave different heuristic arguments for equality of these two constants. Allen and Willey observed that in ordinary canonical unrenormalized perturbation theory the *same* divergent integral arises from pion "bubble" diagrams in the s, t, and u channels. This was taken as heuristic justification for introducing only one subtracted integral, i.e., the same divergent integrals were all regularized (subtracted) the same way. Lehmann and Trute, who constructed the chiral-invariant amplitude dispersively, arrived at the same result by invoking a principle of minimal growth for large s; they observed that the real part of the $I=1$ amplitude is less singular as $s\to\infty$ if the two constants are the same than it is if the two constants are different.

To compare the pionic part of (3.11) with (B50) we use the formula from Appendix C (C12)

$$I^{(0)}_{\pi\pi}(s)\underset{\mu^2\to 0}{\simeq}\frac{1}{16\pi^2}\left[\ln\left(\frac{-s}{\mu^2}\right)-2\right] . \quad \text{(B51)}$$

Then the pionic part of (3.11) reduces, in the $\mu\to 0$ limit, to

$$\tilde{A}^{(2)}(s,t,u)\underset{\mu\to 0}{\simeq}\frac{1}{\mathfrak{F}^4}\left\{-\tfrac{1}{2}s^2\frac{1}{16\pi^2}\left[\ln\left(\frac{-s}{\mu^2}\right)-2\right]\right.$$
$$-\tfrac{1}{6}t(t-u)\frac{1}{16\pi^2}\left[\ln\left(\frac{-t}{\mu^2}\right)-2\right]$$
$$-\tfrac{1}{6}u(u-t)\frac{1}{16\pi^2}\left[\ln\left(\frac{-u}{\mu^2}\right)-2\right]$$
$$\left.+(A+B-C)s^2+(-2B+D)tu\right\} . \quad \text{(B52)}$$

The requirement that (B52) and (B50) are the same gives the relations[30]

$$\ln\frac{c}{\mu^2}=\ln\frac{M^2}{\mu^2}-\tfrac{2}{3} , \quad \text{(B53)}$$

$$\Gamma=\tfrac{3}{4}L-\tfrac{7}{4} . \quad \text{(B54)}$$

Equation (B53) only says that we may use either c or M as the single undetermined constant, but (B54) fixes Γ in terms of L and enables us to eliminate it from (B49). The final form of these constants is

$$A=\frac{1}{16\pi^2}(L-\tfrac{7}{3}) ,$$
$$B=0 , \quad C=\frac{1}{16\pi^2}(\tfrac{1}{3}L-\tfrac{5}{9}) ,$$
$$D=\frac{1}{16\pi^2}(-\tfrac{2}{3}L+\tfrac{16}{9}) , \quad E=\frac{1}{16\pi^2}(-\tfrac{5}{2}L+\tfrac{17}{3}) ,$$
$$F=\frac{1}{16\pi^2}(-\tfrac{1}{2}L+1) , \quad G=\frac{1}{16\pi^2}(\tfrac{3}{2}L-\tfrac{10}{3}) , \quad \text{(B55)}$$
$$H=\frac{1}{16\pi^2}(\tfrac{2}{3}L-\tfrac{16}{9}) , \quad I=\frac{1}{16\pi^2}(\tfrac{1}{6}L-\tfrac{4}{9}) .$$

On the pion mass-shell, the polynomial (3.13) reduces to

$$P(s,t,u)=(A+B-C)s^2+(D-2B)tu$$
$$+(E-8B+4C-F)\mu^2 s$$
$$+(G+16B+4F+2H+4I)\mu^4 . \quad \text{(B56)}$$

$$A+B-C=\frac{1}{16\pi^2}(\tfrac{2}{3}L-\tfrac{16}{9})\equiv\mathfrak{A} ,$$
$$D-2B=\frac{1}{16\pi^2}(-\tfrac{2}{3}L+\tfrac{16}{9})\equiv\mathfrak{B} ,$$
$$E-8B+4C-F=\frac{1}{16\pi^2}(-\tfrac{2}{3}L+\tfrac{22}{9})\equiv\mathfrak{C} , \quad \text{(B57)}$$
$$G+16B+4F+2H+4I=\frac{1}{16\pi^2}(\tfrac{3}{2}L-\tfrac{14}{3})\equiv\mathfrak{D} ,$$

Substitution of these results into (3.13) along with the polynomial part of A_N, [i.e., $4(m^2/\mathfrak{F}^4)\mu^2 I'_{NN}\times(\mu^2)(s-\mu^2)$] then gives the result (4.3).

APPENDIX C: DEFINITIONS AND FORMULAS FOR INTEGRALS

We start with the scalar "bubble" integral

$$I_{ab}(s)=i\int(dk)\frac{1}{[(k+p)^2-m_a{}^2][(k-q)^2-m_b{}^2]} , \quad \int(dk)\equiv\int\frac{d^4k}{(2\pi)^4} , \quad s=(p+q)^2 . \quad \text{(C1)}$$

The Feynman parametric representation is

$$I_{ab}(s)=\frac{1}{16\pi^2}\left[-\ln\frac{\Lambda^2}{s_1}+\int_0^1 dx\ln\frac{a^2(1-x)+b^2x-sx(1-x)}{s_1}+1\right] , \quad \Lambda^2=\text{cutoff}, \quad s_1=\text{arbitrary mass squared}; \quad \text{(C2)}$$

$$J_{ab}(s)=\int_0^1 dx\ln[\alpha(1-x)+\beta x-\xi x(1-x)], \quad \alpha=a^2/s_1, \quad \beta=b^2/s_1, \quad \xi=s/s_1 ; \quad \text{(C3)}$$

$$J_{ab}(s)=-2+\tfrac{1}{2}\ln(\alpha\beta)+\frac{\beta-\alpha}{2\xi}\ln\frac{\beta}{\alpha}+\frac{\sqrt{\lambda_\xi}}{2\xi}\ln\left(\frac{\beta+\alpha-\xi-\sqrt{\lambda_\xi}}{\beta+\alpha-\xi+\sqrt{\lambda_\xi}}\right) , \quad s<(m_a-m_b)^2,$$
$$\lambda_\xi=\xi^2+\alpha^2+\beta^2-2\alpha\xi-2\beta\xi-2\alpha\beta \quad \text{(C4)}$$
$$=[\xi-(\sqrt{\alpha}+\sqrt{\beta})^2][\xi-(\sqrt{\alpha}-\sqrt{\beta})^2] ;$$

$$J_{ab}(0)=-1+\tfrac{1}{2}\ln(\alpha\beta)+\frac{\beta+\alpha}{2(\beta-\alpha)}\ln\frac{\beta}{\alpha}\quad . \tag{C5}$$

The integral subtracted at $s=0$ is

$$I_{ab}^{(0)}(s)=\frac{1}{16\pi^2}\left[-1+\left(\frac{b^2-a^2}{2s}-\frac{b^2+a^2}{2(b^2-a^2)}\right)\ln\frac{b^2}{a^2}+\frac{\sqrt{\lambda}}{2s}\ln\left(\frac{a^2+b^2-s-\sqrt{\lambda}}{a^2+b^2-s+\sqrt{\lambda}}\right)\right],$$
$$s<(m_a-m_b)^2,\quad \lambda=s^2+a^4+b^4-2a^2s-2b^2s-2a^2b^2\ . \tag{C6}$$

The expansion for one large mass, $M^2\gg\mu^2$, s is

$$I_{\sigma\pi}^{(0)}(s)=\frac{1}{16\pi^2}\left(-\frac{s}{2M^2}+\frac{\mu^2 s}{M^4}\ln\frac{M^2}{\mu^2}-\frac{s^2+9\mu^2 s}{6M^4}+\cdots\right)\quad . \tag{C7}$$

The equal-mass case, $m_a=m_b=\mu$ (or M, or m), is

$$I_{\pi\pi}^{(0)}(s)=\frac{1}{16\pi^2}\left\{-2+\left(\frac{s-4\mu^2}{s}\right)^{1/2}\ln\left[\frac{[(s-4\mu^2)/s]^{1/2}+1}{[(s-4\mu^2)/s]^{1/2}-1}\right]\right\},\quad s<0\ . \tag{C8a}$$

The branches of the square roots and logarithm may be chosen such that

$$I_{\pi\pi}^{(0)}(s)=\frac{1}{16\pi^2}\left\{-2+2\left(\frac{4\mu^2-s}{s}\right)^{1/2}\tan^{-1}\left[\frac{1}{[(4\mu^2-s)/s]^{1/2}}\right]\right\},\quad 0<s<4\mu^2\ , \tag{C8b}$$

$$I_{\pi\pi}^{(0)}(s)=\frac{1}{16\pi^2}\left\{-2+\left(\frac{s-4\mu^2}{s}\right)^{1/2}\ln\left[\frac{1+[(s-4\mu^2)/s]^{1/2}}{1-[(s-4\mu^2)/s]^{1/2}}\right]-i\pi\left(\frac{s-4\mu^2}{s}\right)^{1/2}\right\}\quad ,\quad 4\mu^2<s\ . \tag{C8c}$$

The imaginary part is given in (C8c); alternatively it may be computed by using the Cutkosky rules to compute the discontinuity of (C1)

$$\mathrm{disc}I_{\pi\pi}(s)=2i\,\mathrm{Im}I_{\pi\pi}(s)=i\int(dk)2\pi i\delta_+((k+p)^2-\mu^2)2\pi i\delta_+((k-q)^2-\mu^2)$$
$$=-\frac{i}{8\pi}\left(\frac{s-4\mu^2}{s}\right)^{1/2}\theta(s-4\mu^2)\ . \tag{C9}$$

Then

$$I_{\pi\pi}^{(0)}(s)=I_{\pi\pi}(s)-I_{\pi\pi}(0)=\frac{s}{\pi}\int_{4\mu^2}^{\infty}ds'\,\frac{\mathrm{Im}I_{\pi\pi}(s')}{s'(s'-s)}\quad . \tag{C10}$$

We also need, derived from (C8b),

$$I_{\sigma\sigma}^{(0)}(s)=\frac{1}{16\pi^2}\left(-\frac{1}{6}\frac{s}{M^2}-\frac{1}{60}\frac{s^2}{M^4}+\cdots\right),\quad M^2\gg s \tag{C11}$$

and, from (B8c),

$$I_{\pi\pi}^{(0)}(s)\sim\frac{1}{16\pi^2}\left[\ln\left(\frac{s}{\mu^2}\right)-2-i\pi\right],\quad \mu^2\to0\quad(s>0)$$
$$=\frac{1}{16\pi^2}\left\{\ln\left[\frac{-(s+i\epsilon)}{\mu^2}\right]-2\right\},\quad \mu^2\to0\quad(\text{any }s)\ . \tag{C12}$$

For partial-wave projections, we need the integrals

$$\int_{-4p^2}^{0}dt\,t^n I_{\pi\pi}^{(0)}(t)=\frac{1}{16\pi^2}\left[\frac{2}{n+1}(4\mu^2-s)^{n+1}+(\mu^2)^{n+1}G_n(s)\right], \tag{C13}$$

$$(\mu^2)^{n+1}G_n(s)=\int_{-4p^2}^{0}dt\,t^n\left(\frac{t-4\mu^2}{t}\right)^{1/2}\ln\left[\frac{[(t-4\mu^2)/t]^{1/2}+1}{[(t-4\mu^2)/t]^{1/2}-1}\right]$$
$$=-2(-4\mu^2)^{n+1}\int_{z_0}^{\infty}dz\,\frac{z^2}{(z^2-1)^{n+2}}\ln\left(\frac{z+1}{z-1}\right), \tag{C14}$$

$$z_0=\left(\frac{s}{s-4\mu^2}\right)^{1/2}\quad(4\mu^2<s)\ . \tag{C14a}$$

The integrals are

$$G_0(s)=\left[\ln\left(\frac{z_0+1}{z_0-1}\right)\right]^2+\frac{4z_0}{z_0^2-1}\ln\left(\frac{z_0+1}{z_0-1}\right)-\frac{4}{z_0^2-1}\ , \qquad (C15a)$$

$$G_1(s)=\left[\ln\left(\frac{z_0+1}{z_0-1}\right)\right]^2-\frac{4z_0(z_0^2+1)}{(z_0^2-1)^2}\ln\left(\frac{z_0+1}{z_0-1}\right)+\frac{4z_0^2}{(z_0^2-1)^2}\ , \qquad (C15b)$$

$$G_2(s)=2\left[\ln\left(\frac{z_0+1}{z_0-1}\right)\right]^2-\frac{8z_0(32z_0^4-8z_0^2-3)}{3(z_0^2-1)^3}\ln\left(\frac{z_0+1}{z_0-1}\right)+\frac{8(9z_0^4-21z_0^2+4)}{9(z_0^2-1)^3}\ , \qquad (C15c)$$

$$G_3(s)=5\left[\ln\left(\frac{z_0+1}{z_0-1}\right)\right]^2+\frac{4z_0(-15z_0^6+55z_0^4-73z_0^2-15)}{3(z_0^2-1)^4}\ln\left(\frac{z_0+1}{z_0-1}\right)+\frac{4(45z_0^6-150z_0^4+173z_0^2-32)}{9(z_0^2-1)^4}\ . \qquad (C15d)$$

Substitution of (C14a) into these formulas gives the formulas (D7b)–(D7e), Appendix D. The continuation below threshold is made by means of the relations

$$\ln\left(\frac{z_0+1}{z_0-1}\right)=\ln\left[\frac{1+[(s-4\mu^2)/s]^{1/2}}{1-[(s-4\mu^2)/s]^{1/2}}\right]=2i\tan^{-1}\left(\frac{4\mu^2-s}{s}\right)^{1/2}\text{ for } 0\le s\le 4\mu^2\ . \qquad (C16a)$$

$$\left(\frac{s-4\mu^2}{s}\right)^{1/2}=i\left(\frac{4\mu^2-s}{s}\right)^{1/2}\text{ for } 0\le s\le 4\mu^2\ , \qquad (C16b)$$

but note

$$\left(\frac{s-4\mu^2}{s}\right)^{1/2}\ln\left[\frac{1+[(s-4\mu^2)/s]^{1/2}}{1-[(s-4\mu^2)/s]^{1/2}}\right]-i\pi\left(\frac{s-4\mu^2}{s}\right)^{1/2}=2\left(\frac{4\mu^2-s}{s}\right)^{1/2}\left[-\tan^{-1}\left(\frac{4\mu^2-s}{s}\right)^{1/2}-\tfrac{1}{2}\pi\right]$$
$$=2\left(\frac{4\mu^2-s}{s}\right)^{1/2}\tan^{-1}\left(\frac{s}{4\mu^2-s}\right)^{1/2}\ . \qquad (C16c)$$

For the partial-wave projection from the nucleon bubble integral, we have

$$(m^2)^{n+1}G_{nN}(s)=\int_{-4p^2}^{0}dt\,t^n\left(\frac{t-4m^2}{t}\right)^{1/2}\ln\left[\frac{[(t-4m^2)/t]^{1/2}+1}{[(t-4m^2)/t]^{1/2}-1}\right]$$
$$=-2(-4m^2)^{n+1}\int_{\bar z_0}^{\infty}dz\,\frac{z^2}{(z^2-1)^{n+1}}\ln\frac{z+1}{z-1} \qquad (C17)$$
$$=(m^2/\mu^2)^{n+1}(\mu^2)^{n+1}\tilde G_n(\bar z_0)\ , \qquad (C18)$$

where

$$\bar z_0=\left(\frac{4m^2+s-4\mu^2}{s-4\mu^2}\right)^{1/2} \qquad (C18a)$$

and

$$\tilde G_n(z_0)=G_n(s)\ , \qquad (C18b)$$

i.e., the functions $\tilde G_n(\bar z_0)$ are obtained by substituting $\bar z_0$ for z_0 in formulas (B15). Substitution of (C18a) into (C15b) and (C15c) gives formulas (D7f) and (D7g) of Appendix D.

The triangle integral is

$$K_{abc}(p,q)=i\int(dk)\frac{1}{[k^2-m_a^2][(k+p)^2-m_b^2][(k-q)^2-m_c^2]}\ . \qquad (C19)$$

The Feynman parametric representation, for the case of two equal masses, is

$$K_{abb}(p,q)=\frac{1}{16\pi^2}\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\,\frac{\delta(1-\sum x_i)}{a^2x_1+b^2(x_2+x_3)-p^2x_2(1-x_2-x_3)-q^2x_3(1-x_2-x_3)-sx_2x_3}\ . \qquad (C20)$$

Because of the constraint, one integration is trivial after a change of variable. To obtain the expansions for $m_a^2\to\infty$ or $m_b^2\to\infty$, it is necessary to do a second integration before expanding the integrand. The results are[7]

$$K^{(0)}_{\pi\sigma\sigma}(s;p^2,q^2)=\frac{1}{16\pi^2}\left\{\frac{1}{M^4}\left[\tfrac{1}{12}s+\tfrac{1}{4}(p^2+q^2)\right]+\frac{L}{M^6}\left[-\mu^2(p^2+q^2)\right]\right.$$
$$\left.+\frac{1}{M^6}\left[\tfrac{1}{9}(p^4+q^4+p^2q^2+\tfrac{1}{4}s(p^2+q^2)+\tfrac{1}{10}s^2)+2\mu^2(p^2+q^2)-\tfrac{1}{6}\mu^2 s\right]+\cdots\right\}\ , \qquad (C21)$$

$$K^{(0)}_{\sigma\pi\pi}(s;p^2,q^2)=I^{(0)}_{\pi\pi}(s)\left\{\frac{1}{M^2}[-1]+\frac{1}{M^4}[\tfrac{1}{2}s-\tfrac{1}{2}(p^2+q^2)-\mu^2]\right.$$
$$\left.+\frac{1}{M^6}[-\tfrac{1}{3}s^2+\tfrac{2}{3}s(p^2+q^2)+\tfrac{4}{3}\mu^2 s-\tfrac{5}{3}\mu^2(p^2+q^2)-\tfrac{1}{3}(p^4+q^4+p^2q^2)-\mu^4+\frac{\mu^2}{3s}(p^2-q^2)^2]+\cdots\right\},$$
$$+\frac{1}{16\pi^2}\left\{\frac{L}{M^4}[-\tfrac{1}{2}(s-p^2-q^2)]+\frac{1}{M^4}[\tfrac{3}{4}s-\tfrac{5}{4}(p^2+q^2)]\right.$$
$$+\frac{L}{M^6}[\tfrac{1}{3}s^2-\tfrac{2}{3}s(p^2+q^2+3\mu^2)+3\mu^2(p^2+q^2)+\tfrac{1}{3}(p^4+q^4+p^2q^2)]$$
$$\left.+\frac{1}{M^6}[-\tfrac{5}{9}s^2+s(\tfrac{13}{9}(p^2+q^2)+2\mu^2)-4\mu^2(p^2+q^2)-\tfrac{1}{18}(19p^4+19q^4+22p^2q^2)]+\cdots\right\},$$
$$s=(p+q)^2,\quad L=\ln(M^2/\mu^2)\ . \tag{C22}$$

The box integral is

$$H_{abcd}(p,q,p',q')=i\int(dk)\frac{1}{(k^2-m_a{}^2)[(k+p)^2-m_b{}^2][(k+p-p')^2-m_c{}^2][(k-q)^2-m_d{}^2]}\ . \tag{C23}$$

We need the $M^2\gg\mu^2$, s, t expansion of $H^{(0)}_{\pi\sigma\pi\sigma}$. This is[31]

$$H^{(0)}_{\pi\sigma\pi\sigma}(s,t,u;p^2,q^2,p'^2,q'^2)=I^{(0)}_{\pi\pi}(t)\left\{\frac{1}{M^4}[1]+\frac{1}{M^6}[\tfrac{1}{2}\rho-t+2\mu^2]\right.$$
$$\left.+\frac{1}{M^8}\left[t^2+\tfrac{1}{6}st-t(\rho+4\mu^2)-\tfrac{2}{3}\mu^2 s+3\mu^4+\tfrac{5}{2}\mu^2\rho+\tfrac{1}{3}\bar\rho-\frac{\mu^2}{3t}\bar\rho'\right]+\cdots\right\}$$
$$+\frac{1}{16\pi^2}\left\{\frac{L}{M^6}[-\tfrac{1}{2}\rho+t]+\frac{1}{M^6}[-\tfrac{1}{6}s-2t+\tfrac{3}{2}\rho]\right.$$
$$+\frac{L}{M^8}[-t^2-\tfrac{1}{6}st+t(\rho+6\mu^2)+\mu^2 s-\tfrac{9}{2}\mu^2\rho-\tfrac{1}{3}\bar\rho]$$
$$\left.+\frac{1}{M^8}[-\tfrac{1}{60}s^2+\tfrac{1}{2}st+2t^2-s(\tfrac{1}{12}\rho+\tfrac{7}{3}\mu^2)-t(\tfrac{5}{2}\rho+8\mu^2)+\tfrac{15}{2}\mu^2\rho+\tfrac{11}{9}\bar\rho-\tfrac{1}{18}\bar\rho']+\cdots\right\}, \tag{C24}$$

where

$$\rho=p^2+q^2+p'^2+q'^2\ ,$$
$$\bar\rho=p^4+q^4+p'^4+q'^4+p^2q^2+p'^2q'^2+p^2p'^2+q^2q'^2+\tfrac{1}{2}p^2q'^2+\tfrac{1}{2}q^2p'^2\ ,$$
$$\bar\rho'=(p^2-p'^2)^2+(q^2-q'^2)^2+(p^2-p'^2)(q^2-q'^2)\ .$$

We also make use of some properties of the integral[32]

$$\begin{aligned}H_N(p,q,p',q')&=H_{NNNN}(p,q,p',q')\\&=H_N(s,t,u)\text{ for all }p_i{}^2=\mu^2\\&=H_N(s,t)\text{ for }u=4\mu^2-s-t\ .\end{aligned}\tag{C25}$$

The Feynman parametric representation is

$$H_N(s,t)=-\frac{1}{16\pi^2}\int_0^1dx_1\cdots\int_0^1dx_4\frac{\delta(1-\sum x_i)}{[m^2-\mu^2(x_1+x_3)(x_2+x_4)-sx_2x_4-tx_1x_3]^2}\ . \tag{C26}$$

From this we have

$$H_N(s,t)=H_N(t,s) \tag{C27}$$

and

$$H_N(s,t)=\frac{1}{16\pi^2}\left(-\frac{1}{6m^4}+\cdots\right)\ \text{for }\mu=0,\quad |s/m^2|,\ |t/m^2|\ll1\ . \tag{C28}$$

For the projection of partial-wave amplitudes we make use of some manipulations with the single and double dispersion relations satisfied by H_N.

$$d(s,t)=\mathrm{disc}_s H_N(s,t)=i\int(dk)[-i2\pi\delta_+((k+p)^2-m^2)][-i2\pi\delta_+((k-q)^2-m^2)]\frac{1}{[k^2-m^2][(k+p-p')^2-m^2]}$$

$$=-\frac{i}{8\pi}\frac{1}{\{(-st)[(4m^2-t)(s-4m^2)+4(2m^2-\mu^2)^2]\}^{1/2}}\ln\left(\frac{A+\sqrt{B}}{A-\sqrt{B}}\right)\theta(s-4m^2)\,,$$

where

$$A=(s-4m^2)(2m^2-t)+2(2m^2-\mu^2)^2\,,$$
$$B=-t(s-4m^2)[(s-4m^2)(4m^2-t)+4(2m^2-\mu^2)^2]\,. \tag{C29}$$

It is useful to rewrite this as

$$d(s,t)=\frac{-i}{16\pi k_s}\frac{1}{[st(t-t_1)]^{1/2}}\ln\left[\frac{-t+\frac{1}{2}t_1+[t(t-t_1)]^{1/2}}{-t+\frac{1}{2}t_1-[t(t-t_1)]^{1/2}}\right]\theta(s-4m^2)\,, \tag{C30a}$$

where

$$t_1=t_1(s)=\frac{4m^2s-4\mu^2(4m^2-\mu^2)}{s-4m^2}\,,\quad s-4m^2=4k_s^{\,2}\,. \tag{C30b}$$

We choose the square-root branches as

$$[t(t-t_1)]^{1/2}=-[-t(-t+t_1)]^{1/2}\text{ for }t<0$$
$$=i[t(t_1-t)]^{1/2}\text{ for }0<t<t_1$$
$$=[t(t-t_1)]^{1/2}\text{ for }t_1<t\,. \tag{C31}$$

Then

$$d(s,t)=-\frac{1}{8\pi k_s}\frac{1}{[st(t_1-t)]^{1/2}}\tan^{-1}\left[\frac{[t(t_1-t)]^{1/2}}{-t+\frac{1}{2}t_1}\right]\text{ for }s>4m^2,\quad 0<t<t_1\,. \tag{C32}$$

With the definition

$$h(s,t)=\frac{1}{2i}d(s,t)\,, \tag{C33}$$

(C32) gives (D5).

The double spectral function is

$$\rho(s,t)=\mathrm{Im}_t h(s,t)$$
$$=\frac{-1}{32\pi k_s}\frac{1}{[st(t-t_1)]^{1/2}}2\pi\theta(t-t_1)\theta(s-4m^2)$$
$$=\frac{-1}{8\{st[(s-4m^2)(t-4m^2)-4(2m^2-\mu^2)^2]\}^{1/2}}\Theta(s,t)=\rho(t,s)\,, \tag{C34}$$

where $\Theta(s,t)$ gives the boundary of the double spectral region, determined by (C30b). The single and double dispersion relations are

$$H_N(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{h(s',t)}{s'-s}$$
$$=\frac{1}{\pi^2}\iint_{D(s',t')}ds'dt'\frac{\rho(s',t')}{(s'-s)(t'-t)}\,. \tag{C35}$$

The s, t symmetry of $H_N(s,t)$ enables us to write

$$H_N(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{h(s',t)}{s'-s}$$
$$=\frac{1}{\pi}\int_{4m^2}^{\infty}dt'\frac{h(t',s)}{t'-t}\,, \tag{C36}$$

$$h(s', t) = \frac{1}{\pi}\int_{t_1(s')}^{\infty} dt' \frac{\rho(s', t')}{t' - t},$$
$$h(t', s) = \frac{1}{\pi}\int_{s_1(t')}^{\infty} ds' \frac{\rho(s', t')}{s' - s}. \tag{C37}$$

We use these results to write a dispersion relation for $H_N(u, t)$, suitable[33] for partial-wave projection and subsequent numerical integration

$$\begin{aligned} H_N(u, t) &= \frac{1}{\pi^2}\iint du'dt'\Theta(u', t')\frac{\rho(u', t')}{(u'-u)(t'-t)} \\ &= \frac{1}{\pi^2}\iint du'dt'\Theta(u', t')\rho(u', t')\frac{1}{u'+t'+s-4\mu^2}\left(\frac{1}{t'-t}+\frac{1}{u'-u}\right) \\ &= \frac{1}{\pi}\int_{4m^2}^{\infty} dv\, h(v, 4\mu^2 - s - v)\left(\frac{1}{v-t}+\frac{1}{v-u}\right). \end{aligned} \tag{C38}$$

APPENDIX D: THE PARTIAL-WAVE PROJECTIONS

The partial-wave amplitudes projected out of (5.1) and (4.3) are

$$\begin{aligned} a_{00}(s) = \frac{1}{\mathfrak{F}^2}(2s-\mu^2) + \frac{1}{\mathfrak{F}^4}\Bigg\{ & -\tfrac{1}{2}(2s-\mu^2)^2 I_{\pi\pi}^{(0)}(s) + \frac{1}{12p^2}\int_{-4p^2}^{0} dt[8\mu^2 s - 37\mu^4 + (-2s+32\mu^2)t - 10t^2] I_{\pi\pi}^{(0)}(t) \\ & -12sm^2 I_{NN}^{(\pi)}(s) - \frac{2}{p^2}\int_{-4p^2}^{0} dt\, tm^2 I_{NN}^{(\pi)}(t) \\ & +6\frac{m^4}{\pi}\int_{4m^2}^{\infty} ds' h(s', s)\left[4s + \frac{-ss'+2\mu^4}{p^2}\ln\left(\frac{s'+4p^2}{s'}\right)\right] \\ & -2\frac{m^4}{\pi}\int_{4m^2}^{\infty} ds' h(s', 4\mu^2 - s - s')\left[-4s' - 8p^2 + \left(\frac{s'^2+4p^2s'+2\mu^4}{p^2}\right)\ln\left(\frac{s'+4p^2}{s'}\right)\right] \\ & +4\mu^2(2s-\mu^2)m^2 I'_{NN}(\mu^2) + \mathfrak{A}_0 s^2 + \mathfrak{B}_0 \tfrac{8}{3}p^4 + \mathfrak{C}_0\mu^2 s + \mathfrak{D}_0\mu^4 \Bigg\}, \end{aligned} \tag{D1a}$$

$$\begin{aligned} a_{11}(s) = \frac{1}{\mathfrak{F}^2}\tfrac{4}{3}p^2 + \frac{1}{\mathfrak{F}^4}\Bigg\{ & -\tfrac{8}{9}p^4 I_{\pi\pi}^{(0)}(s) + \frac{1}{12p^2}\int_{-4p^2}^{0} dt\left(1+\frac{t}{2p^2}\right)[4\mu^2 s + \mu^4 + (-s-2\mu^2)t - 2t^2] I_{\pi\pi}^{(0)}(t) \\ & -\frac{2}{p^2}\int_{-4p^2}^{0} dt\left(1+\frac{t}{2p^2}\right) tm^2 I_{NN}^{(\pi)}(t) \\ & +2\frac{m^4}{\pi}\int_{4m^2}^{\infty} ds' h(s's)\frac{1}{p^2}\left[4ss' - 8\mu^4 + \left(\frac{(-ss'+2\mu^4)(s'+2p^2)}{p^2}\ln\left(\frac{s'+4p^2}{s'}\right)\right)\right] \\ & +\tfrac{16}{3}\mu^2 p^2 m^2 I'_{NN}(\mu^2) + \tfrac{4}{3}p^2\mu^2(4\mathfrak{A}+\mathfrak{C}) \Bigg\}, \end{aligned} \tag{D1b}$$

$$\begin{aligned} a_{20}(s) = \frac{1}{\mathfrak{F}^2}(-s+2\mu^2) + \frac{1}{\mathfrak{F}^4}\Bigg\{ & -\tfrac{1}{2}(s-2\mu^2)^2 I_{\pi\pi}^{(0)}(s) + \frac{1}{12p^2}\int_{-4p^2}^{0} dt[-4\mu^2 s + 5\mu^4 + (s+2\mu^2)t - 4t^2] I_{\pi\pi}^{(0)}(t) \\ & -\frac{2}{p^2}\int_{-4p^2}^{0} dt\, tm^2 I_{NN}^{(\pi)}(t) \\ & +4\frac{m^4}{\pi}\int_{4m^2}^{\infty} ds' h(s', 4\mu^2 - s - s')\left[-4s' - 8p^2 + \frac{s'^2+4p^2s'+2\mu^4}{p^2}\ln\left(\frac{s'+4p^2}{s'}\right)\right] \\ & -4\mu^2(s-2\mu^2)m^2 I'_{NN}(\mu^2) + \mathfrak{A}_2 s^2 + \mathfrak{B}_2\tfrac{8}{3}p^4 + \mathfrak{C}_2\mu^2 s + \mathfrak{D}_2\mu^4 \Bigg\}, \end{aligned} \tag{D1c}$$

where

$$p^2 = \frac{s-4\mu^2}{4}, \tag{D2}$$

$$\mathcal{A}_0 = \frac{1}{16\pi^2}\left(\tfrac{10}{3}L - \tfrac{80}{9}\right) = -\mathcal{B}_0, \quad \mathcal{C}_0 = \frac{1}{16\pi^2}\left(-\tfrac{28}{3}L + \tfrac{236}{9}\right), \quad \mathcal{D}_0 = \frac{1}{16\pi^2}\left(\tfrac{31}{2}L - 42\right), \tag{D3}$$

$$\mathcal{A}_2 = \frac{1}{16\pi^2}\left(\tfrac{4}{3}L - \tfrac{32}{9}\right) = -\mathcal{B}_2, \quad \mathcal{C}_2 = \frac{1}{16\pi^2}\left(-\tfrac{22}{3}L + \tfrac{170}{9}\right), \quad \mathcal{D}_2 = \frac{1}{16\pi^2}\left(11L - \tfrac{252}{9}\right), \tag{D4}$$

$$h(s', s) = -\frac{1}{8\pi}\frac{1}{\{ss'[(s'-4m^2)(4m^2-s)+4(2m^2-\mu^2)^2]\}^{1/2}} \times \tan^{-1}\left[\frac{\{s(s'-4m^2)[(s'-4m^2)(4m^2-s)+4(2m^2-\mu^2)^2]\}^{1/2}}{(s'-4m^2)(2m^2-s)+2(2m^2-\mu^2)^2}\right], \tag{D5a}$$

$$h(s', 4\mu^2-s-s') = -\frac{1}{16\pi}\frac{1}{\{s'(s'+s-4\mu^2)[(s'-4m^2)(s'+s+4m^2-4\mu^2)+4(2m^2-\mu^2)^2]\}^{1/2}}\ln\left(\frac{U+\sqrt{V}}{U-\sqrt{V}}\right),$$

$$U = (s'-4m^2)(s'+s+2m^2-4\mu^2)+2(2m^2-\mu^2)^2, \tag{D5b}$$

$$V = (s'-4m^2)(s'+s-4\mu^2)[(s'-4m^2)(s'+s+4m^2-4\mu^2)+4(2m^2-\mu^2)^2].$$

The functions h are discontinuities of the Mandelstam box integral, i.e.,

$$H_N(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{h(s',s)}{s'-t} \quad (s>4\mu^2,\ t<0), \tag{D6a}$$

$$H_N(u,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' h(s', 4\mu^2-s-s')\left[\frac{1}{s'-t}+\frac{1}{s'-u}\right] \quad (u,\ t<0). \tag{D6b}$$

Appendix C contains a summary of definitions and formulas for the Mandelstam box integral.

The t integrations for the functions $I_{\pi\pi}^{(0)}(t)$ and $I_{NN}^{(\pi)}(t)$ can be done in terms of elementary functions (Appendix C). After these integrations, and some combination of terms, the partial-wave amplitudes take the form given in (5.7a)–(5.7c). The functions $I_0(s)$, $G_{0,1,2,3}(s)$, and $G_{1N,2N}(s)$ appearing in those equations are given here:

$$I_0(s) = \left(\frac{s-4\mu^2}{s}\right)^{1/2}\ln w - i\pi\left(\frac{s-4\mu^2}{s}\right)^{1/2}, \tag{D7a}$$

$$G_0(s) = (\ln w)^2 + \frac{[s(s-4\mu^2)]^{1/2}}{\mu^2}\ln w - \left(\frac{s-4\mu^2}{\mu^2}\right), \tag{D7b}$$

$$G_1(s) = (\ln w)^2 - \frac{(s-2\mu^2)[s(s-4\mu^2)]^{1/2}}{2\mu^4}\ln w + \frac{1}{4}\frac{s(s-4\mu^2)}{\mu^2}, \tag{D7c}$$

$$G_2(s) = 2(\ln w)^2 + \frac{(s^2-7\mu^2 s+6\mu^4)[s(s-4\mu^2)]^{1/2}}{3\mu^6}\ln w + \frac{(s-4\mu^2)(-2s^2+13\mu^2 s+16\mu^4)}{18\mu^6}, \tag{D7d}$$

$$G_3(s) = 5(\ln w)^2 + \frac{(-3s^3+34\mu^2 s^2-118\mu^4 s+60\mu^6)[s(s-4\mu^2)]^{1/2}}{12\mu^8}\ln w + \frac{(s-4\mu^2)(9s^3-100\mu^2 s^2+308\mu^4 s+512\mu^6)}{144\mu^8}, \tag{D7e}$$

$$G_{1N}(s) = \left[(\ln\bar{w})^2 - \frac{(2m^2+s-4\mu^2)[(s-4\mu^2)(4m^2+s-4\mu^2)]^{1/2}}{2m^4}\ln\bar{w} + \frac{(s-4\mu^2)(4m^2+s-4\mu^2)}{4m^4}\right], \tag{D7f}$$

$$G_{2N}(s) = \left[2(\ln\bar{w})^2 - \frac{[6m^4-m^2(s-4\mu^2)-(s-4\mu^2)^2][(s-4\mu^2)(4m^2+s-4\mu^2)]^{1/2}}{3m^6}\ln\bar{w} + \frac{(s-4\mu^2)[36m^4-3m^2(s-4\mu^2)-2(s-4\mu^2)^2]}{18m^6}\right], \tag{D7g}$$

where

$$w = \left[1+\left(\frac{s-4\mu^2}{s}\right)^{1/2}\right]\Big/\left[1-\left(\frac{s-4\mu^2}{s}\right)^{1/2}\right],$$

$$\bar{w} = \left[1+\left(\frac{s-4\mu^2}{4m^2+s-4\mu^2}\right)^{1/2}\right]\Big/\left[1-\left(\frac{s-4\mu^2}{4m^2+s-4\mu^2}\right)^{1/2}\right]. \tag{D7h}$$

The formulas are all valid for $s>4\mu^2$. The continuations to the range $0\leq s\leq 4\mu^2$ are given in Appendix C.

APPENDIX E: CHIRAL-INVARIANT LIMIT AND EFFECTIVE-RANGE APPROXIMATION

The partial-wave amplitudes of the complete one-loop calculation, given in Sec. V, Eqs. (5.7), are rather complicated functions. In particular, one cannot by inspection of these formulas determine how the partial-wave amplitudes or phase shifts will respond to variations in the parameter L, or the parameter g_A' introduced by replacing g' by $(m/F)g_A'$ in (3.11). On the other hand, if one sets the pion mass equal to zero, $\mu\to 0$, and in the nucleon terms keeps only the leading terms in an expansion in powers of $1/m^2$, i.e., s/m^2, $|t|/m^2\ll 1$; then enormous simplification of the formulas ensues, and the dependence on the parameters becomes transparent. The $\mu\to 0$ limit is the exactly chiral-invariant limit, as may be seen by inspection of the Lagrangian (3.1) or (B1) and (B8). The expansion of the nucleon-loop integrals in powers of $1/m^2$ corresponds to an effective-range approximation.[34,35]

Using the integral formulas (C11), (C12), and (C28), and inserting the g_A' factors, the amplitude (4.3) reduces in the chiral-invariant and effective-range limits to

$$A(s,t,u)=\frac{1}{\mathfrak{F}^2}s+\frac{1}{16\pi^2\mathfrak{F}^2}\left\{-\tfrac{1}{2}s^2\left[\ln\left(\frac{-s}{\mu^2}\right)-2\right]-\tfrac{1}{6}t(t-u)\left[\ln\left(\frac{-t}{\mu^2}\right)-2\right]-\tfrac{1}{6}u(u-t)\left[\ln\left(\frac{-u}{\mu^2}\right)-2\right]+(g_A')^4\tfrac{1}{3}(-s^2+t^2+u^2)+\tfrac{2}{3}(s^2-tu)\left(\ln\frac{M^2}{\mu^2}-\tfrac{8}{3}\right)\right\} \tag{E1}$$

$$=\frac{1}{\mathfrak{F}^2}s+\frac{1}{16\pi^2\mathfrak{F}^2}\left[-\tfrac{1}{2}s^2\ln\left(\frac{-s}{c}\right)-\tfrac{1}{6}t(t-u)\ln\left(\frac{-t}{c}\right)-\tfrac{1}{6}u(u-t)\ln\left(\frac{-u}{c}\right)+\tfrac{1}{3}(g_A')^4(-s^2+t^2+u^2)\right], \tag{E1$'$}$$

where

$$L-\tfrac{2}{3}=\ln\frac{M^2}{\mu^2}-\tfrac{2}{3}=\ln\frac{c}{\mu^2}=L_c\ . \tag{E2}$$

The second form (E1′) just makes explicit the fact that the first form (E1) is independent of μ after the logarithms are combined.

The isospin and partial-wave amplitudes projected out of (E1) are

$$a_{00}(p^2)=\frac{\xi}{\mathfrak{F}^2}(2)+\frac{\xi^2}{16\pi^2\mathfrak{F}^4}\left[-\tfrac{25}{9}\ln\left(\frac{\xi}{\mu^2}\right)+\tfrac{11}{54}+\tfrac{1}{3}(g_A')^4+\tfrac{25}{9}L_c+2\pi i\theta(\xi)\right], \tag{E3a}$$

$$a_{11}(p^2)=\frac{\xi}{\mathfrak{F}^2}(\tfrac{1}{3})+\frac{\xi^2}{16\pi^2\mathfrak{F}^4}\left[-\tfrac{1}{54}+\tfrac{2}{9}(g_A')^4+\frac{i\pi}{18}\theta(\xi)\right], \tag{E3b}$$

$$a_{20}(p^2)=\frac{\xi}{\mathfrak{F}^2}(-1)+\frac{\xi^2}{16\pi^2\mathfrak{F}^4}\left[-\tfrac{10}{9}\ln\left(\frac{\xi}{\mu^2}\right)+\tfrac{25}{108}+\tfrac{2}{3}(g_A')^4+\tfrac{10}{9}L_c+\frac{i\pi}{2}\theta(\xi)\right], \tag{E3c}$$

where

$$\xi=4p^2\ . \tag{E4}$$

These are independent of μ, and the same as the $\mu\to 0$ and effective-range limits of the partial-wave amplitudes (D1a)–(D1c). These partial-wave amplitudes satisfy the $\mu\to 0$ limit of the perturbative partial-wave amplitude elastic unitarity equation (5.6)

$$\mathrm{Im}\,a_{IJ}^{(2)}(p^2)=\frac{1}{32\pi}|a_{IJ}^{(1)}(p^2)|^2;\quad(\mu=0) \tag{E5}$$

and the [1, 1] Padé approximants formed from (E3a)–(E3c)

$$a_{IJ}^{[1,1]}(p^2)=a_{IJ}^{(1)}(p^2)\Big/\left(1-\frac{a_{IJ}^{(2)}(p^2)}{a_{IJ}^{(1)}(p^2)}\right) \tag{E6}$$

will be unitary, and we can compute the phase shifts from

$$\tan\delta_{IJ}^{[1,1]}(p^2)=\frac{\mathrm{Im}\,a_{IJ}^{[1,1]}(p^2)}{\mathrm{Re}\,a_{IJ}^{[1,1]}(p^2)}\ . \tag{E7}$$

The results are

$$\tan\delta_{00}=\frac{\xi}{16\pi\mathfrak{F}^2}\left\{1-\frac{\xi}{16\pi^2\mathfrak{F}^2}\left[-\tfrac{25}{18}\ln\frac{\xi}{\mu^2}+\tfrac{11}{108}+\tfrac{1}{6}(g_A')^4+\tfrac{25}{18}L_c\right]\right\}^{-1}, \tag{E8a}$$

$$\tan\delta_{11}=\frac{\xi}{96\pi\mathfrak{F}^2}\left\{1-\frac{\xi}{16\pi^2\mathfrak{F}^2}\left[-\tfrac{1}{18}+\tfrac{2}{3}(g_A')^4\right]\right\}^{-1}, \tag{E8b}$$

$$\tan\delta_{20}=-\frac{\xi}{32\pi\mathfrak{F}^2}\left\{1-\frac{\xi}{16\pi^2\mathfrak{F}^2}\left[\tfrac{10}{9}\ln\frac{\xi}{\mu^2}-\tfrac{25}{108}-\tfrac{2}{3}(g_A')^4-\tfrac{10}{9}L_c\right]\right\}^{-1}, \tag{E8c}$$

The reciprocals of these formulas agree with the effective-range formulas for $\cot\delta_{IJ}$ given by Lehmann.[35] We can see that for small values of $L_c(L)$, δ_{00} has a maximum value less than $\frac{1}{2}\pi$, while for large values of $L_c(L)$, δ_{00} passes through $\frac{1}{2}\pi$ before flattening out. From (E8b) we see that the p wave is independent of L in this approximation, and we see how the ρ resonance comes out of the calculation, with its position determined by the value of g_A'

$$m_\rho^2-4\mu^2\approx\frac{24\pi^2\mathfrak{F}^2}{(g_A')^4},$$

chiral-invariant effective-range approximation . (E9)

For $g_A'=1$, the value for the original one-loop calculation, this gives $m_\rho\approx1400$ MeV. For $g_A'=1.30$ this gives $m_\rho\approx900$ MeV. This indicates that the combination of finite-pion-mass corrections and exact treatment of the nucleon loop integrals increases the p-wave binding, since we have already determined that $g_A'\approx1.30$ gives the correct position of the ρ in that case. The $I=0$ and 2 s-wave phase shifts computed from (E8a) and (E8c) are also smaller than those computed in the complete one-loop Padé calculation based on Eqs. (5.7) and (6.4) and (6.6) for the same values of L and g_A'. In Fig. 7 we have plotted the results of the two different calculations for δ_{00} and δ_{20} for $L=4.5$ and $g_A'=1.31$.

Finally, we remark that Lehmann has applied the superpropagator technique for nonpolynomial Lagrangians to obtain a definite value for the constant L_c. The value obtained was $L_c=5.73$ which corresponds to $L=6.4$, compared to $L=4$ to 5 which we found as the best empirical value in the full one-loop calculation. Ecker and Honerkamp[36] have done an alternate superpropagator calculation, covariant with respect to the choice of pion field, which gives different fixed values for the two constants in the chiral-invariant one-loop matrix element (see the discussion at the end of Appendix A) and gives phase shifts in the chiral-invariant effective-range limit in qualitative agreement with experiment.

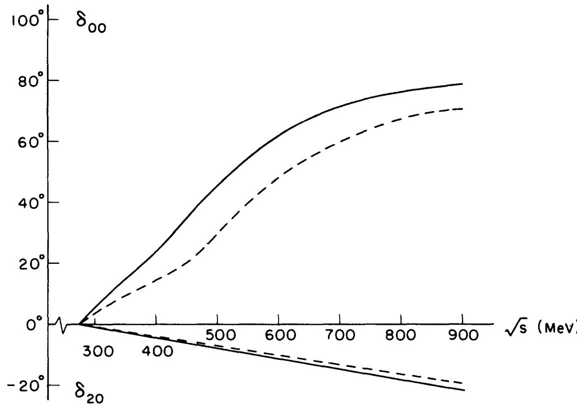

FIG. 7. Calculated $I=0$ and $I=2$ s-wave phase shifts: full one-loop calculation (—), chiral-invariant effective-range limit (---). In both cases the parameters are $L=4.5$, $g_A'=1.31$.

[1]These developments are reviewed by S. L. Adler and R. F. Dashen in their book *Current Algebras* (Benjamin, New York, 1968).
[2]J. L. Basdevant and B. W. Lee, Phys. Rev. D 2, 1680 (1970).
[3]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960); F. Gürsey, *ibid.* 16, 230 (1960).
[4]S. Weinberg, Phys. Rev. 166, 1568 (1968).
[5]S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).
[6]R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).
[7]D. Bessis and J. Zinn-Justin, Phys. Rev. D 5, 1313 (1972).
[8]H. Lehmann and H. Trute, Nucl. Phys. B52, 280 (1973); L. Allen and R. S. Willey, Phys. Rev. D 7, 1825 (1973).
[9]Some early papers applying the Padé algorithm to particle physics are J. L. Gammel and F. A. McDonald, Phys. Rev. 142, 1245 (1966); A. A. Copley and D. Masson, *ibid.* 164, 2059 (1967); D. Bessis and M. Pusterla, Nuovo Cimento 54A, 243 (1968). A recent review with extensive references is given by J. L. Basdevant, Fortsch. Phys. 20, 283 (1972).
[10]J. J. Loeffel, A. Martin, B. Simon, and A. S. Wightman, Phys. Lett. 30B, 656 (1969).
[11]See Appendix E.

[12]A similar observation has been made by H. Lehmann in his chiral-invariant, effective-range treatment of $\pi\pi$ scattering, DESY Report No. DESY 72/54 (unpublished).

[13]Note that the form of the axial-vector current (3.3) or (A2) implies that $g_A - 1$ is a higher-order effect in perturbation theory in either the LσM or the NLσM. Bessis and Turchetti in the *Cargèse Lectures in Physics*, edited by D. Bessis (Gordon and Breach, New York, 1972), Vol. 5, outline a perturbation calculation of $g_A - 1$ in the LσM.

[14]Particle Data Group, Phys. Lett. 39B, 1 (1972). However, two recent experiments reported at the XVI International Conference on High Energy Physics (Chicago-Batavia) give a value for τ_{K_S} some 10% larger than the previously accepted world average. Thus (7.6) may be subject to revision, probably upward, favoring our $L = 4.5$ solution over $L = 4.0$.

[15]E. W. Beier, D. A. Buchholz, A. K. Mann, S. H. Parker, and J. B. Roberts, Phys. Rev. Lett. 30, 399 (1973).

[16]S. D. Protopopescu *et al.*, Phys. Rev. D 7, 1279 (1973).

[17]G. Grayer *et al.*, in *Experimental Meson Spectroscopy-1972* (AIP, New York, 1972).

[18]J. Poirier, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 1, p. 9.

[19]R. Roskies, Phys. Lett. 30B, 42 (1969). These relations are based on the two-variable expansion of the scattering amplitude in the Mandelstam triangle suggested by A. P. Balachandran and J. Nuyts, Phys. Rev. 172, 1821 (1968).

[20]T. F. Wong and G. S. Guralnik, Phys. Rev. D 3, 3028 (1971) have computed the two-point functions of the LσM in the two-loop approximation. There do not exist any calculations of the four-point function in even the two-loop approximation.

[21]A detailed example is the application of the Roy equations by Pennington and Protopopescu to test the consistency of the LBL experimental $\pi\pi$ phase shifts (Ref. 16) with general principles. S. M. Roy, Phys. Lett. 36B, 353 (1971); M. R. Pennington and S. D. Protopopescu, Phys. Rev. D 7, 1429 (1973).

[22]L. V. Filkov and B. B. Palyushev, Nucl. Phys. B42, 541 (1972).

[23]M. C. Bergere and J. M. Drouffe, Nucl. Phys. B53, 191 (1973).

[24]B. W. Lee, Nucl. Phys. B9, 649 (1969).

[25]K. Symanzik, Commun. Math. Phys. 16, 48 (1970).

[26]See reference cited in footnote 13.

[27]A detailed treatment is given by W. Zimmermann in *Lectures on Elementary Particles and Quantum Field Theory* (M.I.T. Press, Cambridge, Massachusetts, 1970), Vol. I.

[28]Since we are considering the case in which there is no asymptotic one-σ state, there is no physical significance attached to the renormalized σ mass, or the σ in-field. By σ_{in} we mean simply a free field constructed with mass M, such that

$$\langle 0|(\sigma_{\text{in}}(x)\sigma_{\text{in}}(y))_+|0\rangle = i\int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \frac{1}{k^2-M^2+i\epsilon}\,.$$

[29]Strictly, we should introduce separate γ, γ' and δ, δ' counterterms in (A20), but we anticipate the results following from the Ward identities, that $\gamma' = \gamma$ and $\delta' = \delta$.

[30]Bessis and Zinn-Justin (Ref. 7) only required that $\tilde{A}^{(2)}(s,t,u)$ should be finite in the limit $\mu \to 0$. They did not require it to reduce to (B50). They fixed the constant Γ [equivalently $I(-a^2)$ in their paper] by imposing a normalization condition on the real part of the σ propagator: $\mathrm{Re}D_\sigma(M^2)^{-1} = 0$.

[31]The formula is given in an appendix of Ref. 7. We have only checked a number of special cases which we have used in Appendix B. In so doing we have corrected one sign misprint in the formula in Ref. 7.

[32]S. Mandelstam, Phys. Rev. 115, 1741 (1959).

[33]R. C. Brunet, Phys. Rev. D 2, 3064 (1970).

[34]Unitarization of the Weinberg $\pi\pi$ scattering matrix element in the effective-range approximation was initiated by J. Sucher and C. H. Woo, Phys. Rev. Lett. 18, 723 (1967), and L. S. Brown and R. L. Goble, *ibid.* 20, 346 (1968).

[35]H. Lehmann has constructed the $\pi\pi$ scattering amplitude directly in the chiral-invariant effective-range limit, Phys. Lett. 41B, 529 (1972) and DESY Report No. DESY 72/54 (unpublished).

[36]G. Ecker and J. Honerkamp, Nucl. Phys. B52, 211 (1973).