PHYSICAL REVIEW D VOLUME 9, NUMBER 10 15 MAY 1974

Estimates of the eighth-order corrections to the anomalous magnetic moment of the muon

Mark A. Samuel
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
*and Quantum Theoretical Research Group, Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74074**
(Received 7 January 1974)

All the diagrams which contribute to the eighth-order muon anomaly are classified and their contributions are estimated. The Kinoshita method is used whenever possible. The total correction is estimated to be $230(\alpha/\pi)^4 = 6.7 \times 10^{-9}$.

I. INTRODUCTION

This is the current situation in the comparison between theory and experiment of the anomalous magnetic moment of the muon.[1] The theoretical value for the anomaly $a = \frac{1}{2}(g-2)$ is given by[2]

$$a_{\text{th}} = 0.5\left(\frac{\alpha}{\pi}\right) + 0.76578\left(\frac{\alpha}{\pi}\right)^2 + (21.4 \pm 1.1)\left(\frac{\alpha}{\pi}\right)^3 + a_H = (116588 \pm 2) \times 10^{-8},$$

where a_H is the hadronic contribution, which is estimated[3] to be $(73 \pm 9) \times 10^{-9}$. There is also a weak-interaction contribution which contributes[3] $+3 \times 10^{-9}$ to a_{th}. The current experimental value is

$$a_{\text{ex}} = (116616 \pm 31) \times 10^{-8}.$$

The next generation of experiments at CERN may have a precision as good as $\pm 1 \times 10^{-8}$ (10 ppm), and the question of whether or not the eighth-order contribution might soon be significant arises. With this in mind, Lautrup[4] recently estimated the contributions for certain classes of eighth-order diagrams. It seems desirable to have alternative estimates of these contributions, as well as estimates of the other eighth-order contributions. We will use, whenever possible, the Kinoshita method[5] to evaluate the coefficients of the terms $\ln^n(m_\mu/m_e)$ which typically occur in the contributions to a_{th}.

II. CLASSIFICATION OF THE DIAGRAMS

We have classified all the nonzero topologically distinct proper diagrams which contribute to the eighth-order muon anomaly. We find, in agreement with Lautrup,[4] that there are 891 mass-independent graphs (these of course also give $a_e^{(8)}$), of which 373 contain muon loops and 518 do not contain any loops. There are 469 mass-dependent graphs[6] containing electron loops, making a total of 1360 Feynman diagrams contributing to the muon anomalous magnetic moment in eighth order. As a check to be certain that no mass-dependent graphs have been omitted, in each of the 373 mass-independent graphs with muon loops, we have substituted electron loops for muon loops in all possible ways. This generates the 469 mass-dependent graphs.

It is expected that, as in lower orders, there will be large cancellations among the contributions of the 891 mass-independent graphs. We estimate the value to be[7] $-(\alpha/\pi)^4$, which is negligible.

We now separate the 469 mass-dependent graphs into 11 classes. This classification scheme is slightly different from that of Lautrup.

Class A. This class is obtained by inserting a single electron loop in all possible ways in the sixth-order diagrams without electron loops. Diagram (A) of Fig. 1 is a representative graph of this class. There are 216 diagrams in this class.

Class B. This class includes the diagrams obtained by inserting two electron loops in all possible ways in the fourth-order diagrams without electron loops. Diagram (B) of Fig. 1 is representative of the graphs of this class. There are 21 diagrams in this class.

Class C. This class includes the single diagram (C) of Fig. 1 with three electron loops.

Class D. This class is obtained by making a (proper) fourth-order vacuum-polarization insertion in all possible ways in the fourth-order diagrams without electron loops. Graph (D) of Fig. 1 is representative of the graphs of this class. There are 42 such diagrams.

Class E. This class includes the 6 diagrams which contain both an electron loop and a (proper) fourth-order vacuum-polarization insertion in a single photon line [graph (E) of Fig. 1].

Class F. This class contains the 18 diagrams obtained by making a (proper) sixth-order vacuum-polarization insertion in the second-order graph. It includes both graphs of type (F) and (F′)

of Fig. 1.

Class G. This class contains the 18 diagrams obtained by inserting a single electron loop in all possible ways in the photon-photon scattering (with electron loop) sixth-order graphs. Diagram (G) of Fig. 1 is representative of this class.

Class H. This class contains the 18 diagrams represented by graph (H) of Fig. 1. These are photon-photon scattering graphs in which all 4 vertices on the single electron loop involve a virtual photon.

Class I. This class contains the 18 diagrams represented by graph (I) of Fig. 1, which are obtained by inserting a single muon loop in all possible ways in the photon-photon scattering (with electron loop) sixth-order graphs.

Class J. This class includes 3 diagrams which contain a muon loop inside an electron loop [graph (J) of Fig. 1].

Class K. This class is obtained by attaching a single virtual photon in all possible ways to the photon-photon scattering (with electron loops) sixth-order graphs. It includes both graphs of type (K) and type (K′) of Fig. 1. There are 108 diagrams in this class.

III. CONTRIBUTIONS OF CLASSES A THROUGH F

We now use the Kinoshita method of partially renormalized amplitudes to estimate the contributions of classes A through F. We will obtain exactly the coefficients of all the $\ln^n(m_\mu/m_e)$ terms which occur in each case (for class C we will obtain the mass-independent term as well), giving us what should be a very accurate value for the first six classes. In Sec. IV we will estimate the contributions of the last five classes, but the values obtained are considerably less reliable.

Class A. The basic equation for the partially renormalized amplitude here is

$$\Delta M^A_{\rm pr}(m_e, m_\mu, \Lambda) = 3Z_3^{(2)}(m_e, \Lambda)a_e^{(6)} + \Delta M^A_r(m_e, m_\mu), \qquad (1)$$

with

$$Z_3^{(2)}(m_e, \Lambda) = -\left(\frac{\alpha}{\pi}\right)\left[\tfrac{2}{3}\ln(\Lambda/m_e) - \tfrac{5}{9}\right]. \qquad (2)$$

In the usual way,[5] the fact that

$$\lim_{m_e\to 0} \Delta M^A_{\rm pr}(m_e, m_\mu, \Lambda)$$

exists enables us to obtain for the contribution of class A

$$\Delta M^A_r(m_e, m_\mu) = 2a_e^{(6)}\left(\frac{\alpha}{\pi}\right)\ln\frac{m_\mu}{m_e}.$$

This becomes, using[2,3] $a_e^{(6)} = 1.1(\alpha/\pi)^3$,

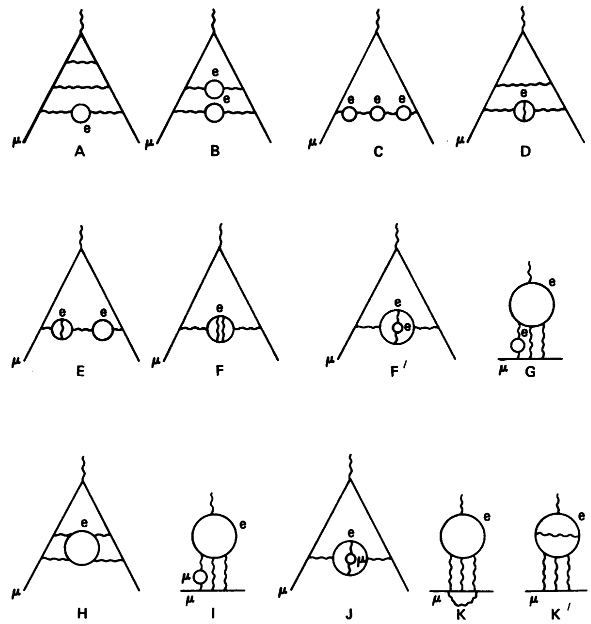

FIG. 1. Representative diagrams contributing to $a_\mu^{(8)} - a_e^{(8)}$.

$$\Delta M^A_r(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4\left[2.2\ln\frac{m_\mu}{m_e} + O(1)\right]$$

$$= 11.7\left(\frac{\alpha}{\pi}\right)^4; \qquad (3)$$

cf. Lautrup's estimate: $14(\alpha/\pi)^4$.

Class B. The relevant equation here is

$$\Delta M^B_{\rm pr}(m_e, m_\mu, \Lambda) = 3[Z_3^{(2)}(m_e, \Lambda)]^2 a_e^{(4)} + 3Z_3^{(2)}(m_e, \Lambda)a_{\mu 1}^{(6)}(m_e, m_\mu) + \Delta M^B_r(m_e, m_\mu), \qquad (4)$$

where $a_{\mu 1}^{(6)}$ is the contribution to $a_\mu^{(6)}$ due to second-order vacuum polarization inserted into the fourth-order vertex graphs. Using[8]

$$a_{\mu 1}^{(6)}(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)\left[\frac{4}{3}a_e^{(4)}\ln\frac{m_\mu}{m_e} + 0.032\left(\frac{\alpha}{\pi}\right)^2\right]$$

we obtain

$$\Delta M^B_r(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^2\left[\frac{4}{3}a_e^{(4)}\ln^2\frac{m_\mu}{m_e} + 0.064\left(\frac{\alpha}{\pi}\right)^2\ln\frac{m_\mu}{m_e}\right], \qquad (5)$$

and with

$$a_e^{(4)} = \left[\tfrac{197}{144} + \tfrac{1}{12}\pi^2 - \tfrac{1}{2}\pi^2\ln 2 + \tfrac{3}{4}\zeta(3)\right]\left(\frac{\alpha}{\pi}\right)^2$$

$$= -0.328\left(\frac{\alpha}{\pi}\right)^2$$

we obtain

$$\Delta M_r^B(m_e, m_\mu)$$
$$= -\left(\frac{\alpha}{\pi}\right)^4\left[0.44\ln^2\frac{m_\mu}{m_e} - 0.064\ln\frac{m_\mu}{m_e} + O(1)\right]$$
$$= -12.1\left(\frac{\alpha}{\pi}\right)^4; \tag{6}$$

cf. Lautrup's estimate: $-9(\alpha/\pi)^4$.

Class C. We begin with the equation

$$\Delta M_{\rm pr}^C(m_e, m_\mu, \Lambda) = [Z_3^{(2)}(m_e, \Lambda)]^3 a_e^{(2)} + 3[Z_3^{(2)}(m_e, \Lambda)]^2 a_{\mu 1}^{(4)}(m_e, m_\mu) + 3Z_3^{(2)}(m_e, \Lambda) a_{\mu 2}^{(6)}(m_e, m_\mu) + \Delta M_r^C(m_e, m_\mu), \tag{7}$$

where $a_{\mu 1}^{(4)}$ is the single mass-dependent contribution to $a_\mu^{(4)}$, and $a_{\mu 2}^{(6)}$ is the two-electron loop (double-bubble) contribution to $a_\mu^{(6)}$. Here[5]

$$a_{\mu 1}^{(4)} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\left(\ln\frac{m_\mu}{m_e} - \frac{25}{12}\right) \tag{8}$$

and

$$a_{\mu 2}^{(6)} = \frac{2}{9}\left(\frac{\alpha}{\pi}\right)^3\left(\ln^2\frac{m_\mu}{m_e} - \frac{25}{6}\ln\frac{m_\mu}{m_e} + \frac{317}{72} + \frac{\pi^2}{6}\right). \tag{9}$$

After some algebraic manipulations the contribution is found to be

$$\Delta M_r^C(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4\left[\frac{4}{27}\ln^3\frac{m_\mu}{m_e} - \frac{25}{27}\ln^2\frac{m_\mu}{m_e} + \left(\frac{2\pi^2}{27} + \frac{317}{162}\right)\ln\frac{m_\mu}{m_e} + O(1)\right]. \tag{10}$$

In this case the contribution can be calculated directly without a great deal of labor. As a check then we evaluate[5]

$$\Delta M_r^C(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4\int_0^1 du(1-u)\left[\int_0^1 \frac{dv\, u^2 v^2(1-\frac{1}{3}v^2)}{u^2(1-v^2) + 4(1-u)(m_e/m_\mu)^2}\right]^3, \tag{11}$$

obtaining for the "exact" contribution of class C

$$\Delta M_r^C(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4\left[\frac{4}{27}\ln^3\frac{m_\mu}{m_e} - \frac{25}{27}\ln^2\frac{m_\mu}{m_e} + \left(\frac{2\pi^2}{27} + \frac{317}{162}\right)\ln\frac{m_\mu}{m_e} - \frac{2\zeta(3)}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832}\right]. \tag{12}$$

It is seen that the logarithmic coefficients agree with those obtained in Eq. (10), and that the $O(1)$ term is obtained as well. This yields

$$\Delta M_r^C(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4(22.5 - 26.3 + 14.3 - 3.3)$$
$$= 7.2\left(\frac{\alpha}{\pi}\right)^4; \tag{13}$$

c.f. Lautrup's estimate: $14\,(\alpha/\pi)^4$. The contribution of each logarithmic term and the $O(1)$ term is shown separately. Note that, just as occurs in lower orders, there is tremendous cancellation among large terms. Interestingly, the $\ln^2$ term here is larger than the $\ln^3$ term.

The $\ln^3$ coefficient is consistent with Terazawa's infinite summation of graphs of this type,[9] but as can be seen from Eq. (13) and as Terazawa clearly points out, the so-called leading logarithmic terms cannot be taken to be approximately $a_{\rm th}$ under the physically realized situation with $(2\alpha/3\pi)\ln(m_\mu/m_e) \approx \frac{1}{120}$.

Class D. The relevant starting point here is

$$\Delta M_{\rm pr}^D(m_e, m_\mu, \Lambda) = 2Z_3^{(4)}(m_e, \Lambda)a_e^{(4)} + \Delta M_r^D(m_e, m_\mu), \tag{14}$$

with $Z_3^{(4)}$ given by[10]

$$Z_3^{(4)}(m_e, \Lambda) = -(\alpha/\pi)^2[\tfrac{1}{2}\ln(\Lambda/m_e) + \zeta(3) - \tfrac{5}{24}]. \tag{15}$$

Actually we need only the ln term here. The contribution obtained is

$$\Delta M_r^D(m_e, m_\mu) = a_e^{(4)}\left(\frac{\alpha}{\pi}\right)^2\ln\frac{m_\mu}{m_e}$$
$$= \left(\frac{\alpha}{\pi}\right)^4\left[-0.328\ln\frac{m_\mu}{m_e} + O(1)\right]$$
$$= -1.8\left(\frac{\alpha}{\pi}\right)^4; \tag{16}$$

cf. Lautrup's estimate: $-2.4(\alpha/\pi)^4$.

Class E. The equation for this class is

$$\Delta M_{\rm pr}^E(m_e, m_\mu, \Lambda) = 2Z_3^{(2)}(m_e, \Lambda)a_{\mu 3}^{(6)}(m_e, m_\mu) + 2Z_3^{(2)}(m_e, \Lambda)Z_3^{(4)}(m_e, \Lambda)a_e^{(2)} + 2Z_3^{(4)}(m_e, \Lambda)a_{\mu 1}^{(4)}(m_e, m_\mu) + \Delta M_r^E(m_e, m_\mu), \tag{17}$$

where $a_{\mu 3}^{(6)}$ is the fourth-order (proper) vacuum-polarization contribution to $a_\mu^{(6)}$ given by

$$a_{\mu 3}^{(6)} = (\alpha/\pi)^3[\tfrac{1}{4}\ln(m_\mu/m_e) + \tfrac{1}{2}\zeta(3) - \tfrac{5}{12}]. \tag{18}$$

This leads to the contribution

$$\Delta M_r^E(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4 \left\{\frac{1}{3}\ln^2\frac{m_\mu}{m_e} - \left[\frac{5}{4} - \frac{2}{3}\zeta(3)\right]\ln\frac{m_\mu}{m_e} + O(1)\right\}$$
$$= 7.1\left(\frac{\alpha}{\pi}\right)^4 . \tag{19}$$

Class F. The equation we use here is

$$\Delta M_{\rm pr}^F(m_e, m_\mu, \Lambda) = Z_3^{(6)}(m_e, \Lambda)a_e^{(2)} + Z_3^{(2)}(m_e, \Lambda)a_{\mu 3}^{(6)}(m_e, m_\mu) + Z_3^{(2)}(m_e, \Lambda)Z_3^{(4)}(m_e, \Lambda)a_e^{(2)} + \Delta M_r^F(m_e, m_\mu), \tag{20}$$

with $Z_3^{(6)}$ given by[11]

$$Z_3^{(6)}(m_e, \Lambda) = -\left(\frac{\alpha}{\pi}\right)^3 \left\{\frac{1}{6}\ln^2\frac{\Lambda}{m_e} + \left[\frac{2}{3}\zeta(3) - \frac{141}{144}\right]\ln\frac{\Lambda}{m_e} + C_0\right\} . \tag{21}$$

The contribution of this class is obtained in the usual way and is found to be

$$\Delta M_r^F(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4 \left\{\frac{1}{12}\ln^2\frac{m_\mu}{m_e} - \left[\frac{67}{96} - \frac{\zeta(3)}{3}\right]\ln\frac{m_\mu}{m_e} + O(1)\right\}$$
$$= 0.78\left(\frac{\alpha}{\pi}\right)^4 . \tag{22}$$

No direct comparison can be made with Lautrup for classes E and F. He has estimated the sum of the contributions of diagrams of type (E) and type (F′) of Fig. 1 to be $16(\alpha/\pi)^4$. He has no estimate, however, for the contribution of diagrams of type (F).

Adding Eqs. (3), (6), (13), (16), (19), and (22) we obtain the total contribution of the 304 diagrams in classes A through F,

$$\Delta M_r^{A-F} = 12.9(\alpha/\pi)^4 . \tag{23}$$

IV. CONTRIBUTIONS OF CLASSES G THROUGH K

In this section we will estimate the contributions of classes G through K, but, unlike the estimates in Sec. III, we will be unable to exactly obtain the coefficients of all the $\ln^n(m_\mu/m_e)$ terms which occur. (For class G, however, we will obtain the leading $\ln^2$ coefficient exactly.)

Class G. Here we will use a generalization of the Kinoshita method which involves partially renormalized amplitudes which are functions of two masses $m, m' \ll m_\mu$. This method can be used for any class which has at least two electron loops, and is discussed (and further generalized) in more detail, with examples, in the Appendix.

At the end of the analysis one puts $m = m' = m_e$. Hence, we write

$$\Delta M_{\rm pr}^G(m, m', m_\mu, \Lambda) = 3Z_3^{(2)}(m', \Lambda)a_{\mu 4}^{(6)}(m, m_\mu) + \Delta M_r^G(m, m', m_\mu) . \tag{24}$$

Here $a_{\mu 4}^{(6)}$ represents the photon-photon scattering (with electron loop) contribution to $a_\mu^{(6)}$, for which we use[12]

$$a_{\mu 4}^{(6)}(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^3 \left(6.4\ln\frac{m_\mu}{m_e} - 15.7\right) . \tag{25}$$

Using the fact that $\lim_{m'\to 0}\Delta M_{\rm pr}^G(m, m', m_\mu, \Lambda)$ (with m fixed) exists, we obtain

$$\Delta M_r^G(m, m', m_\mu) = \left(\frac{\alpha}{\pi}\right)^4 \left[\left(2\ln\frac{m_\mu}{m'} + a\right)\left(6.4\ln\frac{m_\mu}{m} - 15.7\right) + O(1)\right], \tag{26}$$

with a left undetermined. Now putting $m = m' = m_e$, the $\ln^2$ coefficient is determined " exactly, " but, unfortunately, the ln coefficient is undetermined.

$$\Delta M_r^G(m_e, m_e, m_\mu) = \Delta M_r^G(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4 \left[12.8\ln^2\frac{m_\mu}{m_e} + O\left(\ln\frac{m_\mu}{m_e}\right)\right]$$
$$= 364\left(\frac{\alpha}{\pi}\right)^4 ; \tag{27}$$

this should be considered as an upper bound to the contribution since the ln term most probably will be negative. As our estimate for this class, we will put $a = 0$ and obtain

$$\Delta M_r^G(m_e, m_\mu) = \left(\frac{\alpha}{\pi}\right)^4 \left(12.8\ln^2\frac{m_\mu}{m_e} - 31.4\ln\frac{m_\mu}{m_e}\right)$$
$$= 197\left(\frac{\alpha}{\pi}\right)^4 ; \tag{28}$$

cf. Lautrup's estimate which corresponds to $a = -\frac{5}{3}$: $166(\alpha/\pi)^4$.

Class H. In order to estimate the contribution of this class, we make use of the high-energy limit of the forward photon-photon scattering cross section[13]:

$$\frac{d\sigma(\omega, 0)}{d\Omega} \sim \left(\frac{\alpha}{\pi}\right)^2 r_0^{\,2}\left(\frac{m_e}{\omega}\right)^2 \ln^4\frac{\omega}{m_e}, \tag{30}$$

where $r_0 = \alpha/m$ and $\omega \gg m_e$ is the energy of each photon in the center-of-momentum system. Equation (30) can be rewritten in terms of the forward amplitude $\pi(\omega)$ as

$$\frac{d\sigma(\omega, 0)}{d\Omega} \sim \frac{\pi^2}{16\omega^2}|\pi(\omega)|^2, \tag{31}$$

with

$$\pi(\omega) \sim \pm 4\left(\frac{\alpha}{\pi}\right)^2 \ln^2\frac{\omega}{m_e}. \quad (32)$$

In our case, the characteristic value of all the photon momenta is $q \sim m_\mu \gg m_e$. For our estimate of this class we take

$$\Delta M_r^H \sim a_{\rm el}^{(4)}\pi(q), \quad (33)$$

where $a_{\rm el}^{(4)}$ represents the sum of the uncrossed and crossed ladder-diagram contributions to $a_e^{(4)}$, and[14]

$$a_{\rm el}^{(4)} = (0.778 - 0.467)\left(\frac{\alpha}{\pi}\right)^2 = 0.311\left(\frac{\alpha}{\pi}\right)^2.$$

Hence, the contribution is estimated to be

$$\Delta M_r^H(m_e, m_\mu) \sim \pm\left(1.2\ln^2\frac{m_\mu}{m_e}\right)\left(\frac{\alpha}{\pi}\right)^4 = \pm 35\left(\frac{\alpha}{\pi}\right)^4. \quad (34)$$

[Lautrup disagrees here. He believes there are no ln terms in the contribution of class H (see Ref. 15).] The sign of the contribution is undetermined.

In the case of the $a_{\mu 4}^{(6)}$ contribution, although three of the photons have $q \sim m_\mu$, the external photon has $q \sim 0$. Some terms in the partial calculations did contain $\ln^2(m_\mu/m_e)$, but this dependence eventually canceled out.[16] It should be noted that the nonforward γ-γ cross section is of the form

$$\frac{d\sigma}{d\Omega} \sim \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{\omega^2} f(\Theta),$$

with $f(\Theta) \sim \ln^4\Theta$ for small Θ. If, however, $\Theta \sim O(1)$ in class H, then there is no $m_e \to 0$ singularity.

Class I. Making a muon bubble insertion in all possible ways in a class of diagrams yields a contribution which is $-\pi_\mu^{(2)}(\bar{q})$ times the contribution of the original class. $\pi_\mu^{(2)}$ is the second-order vacuum polarization due to a single muon loop, and $\bar{q}$ is some average value of q which depends on the class. For the known contributions to a_μ, $-\pi_\mu^{(2)}(\bar{q}) \sim 0.1(\alpha/\pi)$, corresponding to $\bar{q} \sim 1.3 m_\mu$. Hence, we take as our estimate for this class

$$\Delta M_r^I \sim -\pi_\mu^{(2)}(\bar{q}) a_{\mu 4}^{(6)} \sim 1.8(\alpha/\pi)^4. \quad (35)$$

Class J. Using the same reasoning as in class I above, the contribution of this class is estimated to be very small:

$$\Delta M_r^J \sim 0.1(\alpha/\pi) a_{\mu 3}^{(6)} \sim 0.15(\alpha/\pi)^4. \quad (36)$$

Class K. The contribution of this class can be considered to be a radiative correction to $a_{\mu 4}^{(6)}$. It is expected that there will be large cancellations among the contribution of the 108 diagrams of this class.

We estimate its value to be[7]

$$\Delta M_r^K \sim -(\alpha/\pi) a_{\mu 4}^{(6)} \sim -18(\alpha/\pi)^4. \quad (37)$$

V. DISCUSSION

It is seen from Eqs. (23), (27), (34), (35), (36), and (37) that the eighth-order muon anomalous magnetic moment might be as large as $400(\alpha/\pi)^4$. Interestingly, this correction would be $11.6\times 10^{-9} \sim 0.93(\alpha/\pi)^3$. The largest contributions come from classes G, H, and K, in that order. Our best estimate for the eighth-order muon moment, using Eq. (28) instead of (27) as well as the positive sign in Eq. (34), is

$$a_\mu^{(8)} = 230(\alpha/\pi)^4 = 6.7\times 10^{-9}. \quad (38)$$

Table I summarizes the contributions of each class and the coefficients of $\ln^n(m_\mu/m_e)$ are given in each case. After submitting this article for publication a report by Lautrup and de Rafael[15] was received in which $a_\mu^{(8)}$ is considered from the point of view of the Callan-Symanzik equation. A detailed comparison with their results can be made for each entry in Table I where an analytical expression for the coefficient of $\ln^n(m_\mu/m_e)$ is given. This includes class C, $n=0, 1, 2,$ and 3; class D, $n=1$; class E, $n=1$ and 2; and class F, $n=1$ and 2. The results, in each case, are in complete agreement.

ACKNOWLEDGMENTS

The author would like to thank S. D. Drell for the kind hospitality extended at SLAC, where most of this work was done. Helpful discussions with S. Brodsky and others at SLAC are gratefully acknowledged, as well as useful correspondence with B. E. Lautrup, to whom thanks are also due for reading the manuscript.

APPENDIX

We now examine in more detail the generalized Kinoshita method referred to in Sec. IV. First consider the double-bubble contribution to $a_\mu^{(6)}$ (one diagram only), with unequal bubble masses $m, m' \ll m_\mu$. We write

$$\Delta M_{\rm pr}^{(6)}(m, m', m_\mu, \Lambda) = Z_3^{(2)}(m', \Lambda) a_{\mu 1}^{(4)}(m, m_\mu) + a_{\mu 2}^{(6)}(m, m', m_\mu). \quad (A1)$$

TABLE I. Contributions of each class to $a_\mu^{(8)}$ in terms of $(\alpha/\pi)^4$. The coefficients of $\ln^n(m_\mu/m_e)$ are given in each case. The underlined values indicate coefficients whose values are merely estimates.

Class	$\ln^3(m_\mu/m_e)$	$\ln^2(m_\mu/m_e)$	$\ln(m_\mu/m_e)$	1	Total
A	0	0	2.2		11.7
B	0	−0.44	+ 0.064		−12.1
C	$\frac{4}{27}$	$-\frac{25}{27}$	$\frac{2}{27}\pi^2+\frac{317}{162}$	$-\frac{2}{9}\zeta(3)-\frac{25}{162}\pi^2-\frac{8609}{5832}$	7.2
D	0	0	$\frac{197}{144}+\frac{1}{12}\pi^2-\frac{1}{2}\pi^2\ln 2+\frac{3}{4}\zeta(3)$		− 1.8
E	0	$\frac{1}{3}$	$\frac{2}{3}\zeta(3)-\frac{5}{4}$		7.1
F	0	$\frac{1}{12}$	$\frac{1}{3}\zeta(3)-\frac{67}{96}$		.78
G	0	12.8	$-\underline{31.4}$		197
H	0	$\pm\underline{1.2}$			$\pm\underline{35}$
I	0	0	$\underline{0.64}$	$-\underline{1.6}$	$\underline{1.8}$
J	0	0	$\underline{0.025}$	$\underline{0.018}$	$\underline{0.15}$
K	0	0	$-\underline{6.4}$	$+\underline{15.7}$	$-\underline{18}$

Now using the fact that

$$\lim_{m'\to 0} \Delta M_{\text{pr}}^{(6)}(m,m',m_\mu,\Lambda) \quad (\text{with } m \text{ fixed})$$

exists, one obtains

$$a_{\mu 2}^{(6)}(m,m',m_\mu)=\left(\frac{\alpha}{\pi}\right)^3\left[\frac{2}{9}\ln\frac{m_\mu}{m}\ln\frac{m_\mu}{m'}-\frac{25}{54}\ln\frac{m_\mu}{m'}+f\left(\frac{m_\mu}{m}\right)\right]. \tag{A2}$$

Since the contribution must satisfy the symmetry relation

$$a_{\mu 2}^{(6)}(m,m',m_\mu)=a_{\mu 2}^{(6)}(m',m,m_\mu), \tag{A3}$$

f is determined and we get

$$a_{\mu 2}^{(6)}(m,m',m_\mu)=\left(\frac{\alpha}{\pi}\right)^3\left[\frac{2}{9}\ln\frac{m_\mu}{m}\ln\frac{m_\mu}{m'}-\frac{25}{54}\ln\frac{m_\mu}{m}-\frac{25}{54}\ln\frac{m_\mu}{m'}+O(1)\right]. \tag{A4}$$

Putting $m=m'=m_e$ we find that we have easily obtained the well-known result

$$a_{\mu 2}^{(6)}(m_e,m_e,m_\mu)=a_{\mu 2}^{(6)}(m_e,m_\mu)=\left(\frac{\alpha}{\pi}\right)^3\left[\frac{2}{9}\ln^2\frac{m_\mu}{m}-\frac{25}{27}\ln\frac{m_\mu}{m_e}+O(1)\right]. \tag{A5}$$

As one more example of the power of this method we consider the triple-bubble diagram of class C, with unequal bubble masses $m, m', m'', \ll m_\mu$. Hence we write

$$\Delta M_{\text{pr}}^{(8)}(m,m',m'',m_\mu,\Lambda)=Z_3^{(2)}(m'',\Lambda)a_{\mu 2}^{(6)}(m,m',m_\mu)+\Delta M_r^C(m,m',m'',m_\mu). \tag{A6}$$

We will use Eq. (A4) with

$$O(1)=\tfrac{2}{9}\left(\tfrac{317}{72}+\tfrac{1}{6}\pi^2\right). \tag{A7}$$

This is required so that Eqs. (A5) and (9) agree. This gives us

$$\Delta M_r^C(m,m',m'',m_\mu)=\left(\frac{\alpha}{\pi}\right)^4\left[\frac{4}{27}\ln\frac{m_\mu}{m}\ln\frac{m_\mu}{m'}\ln\frac{m_\mu}{m''}-\frac{25}{81}\left(\ln\frac{m_\mu}{m}\ln\frac{m_\mu}{m''}+\ln\frac{m_\mu}{m'}\ln\frac{m_\mu}{m''}\right)+\frac{4}{27}\left(\frac{\pi^2}{6}+\frac{317}{72}\right)\ln\frac{m_\mu}{m''}+g(m,m',m_\mu)\right]. \tag{A8}$$

The function g is determined by symmetrizing as before, and so

$$\Delta M_r^C(m,m',m'',m_\mu)=\left(\frac{\alpha}{\pi}\right)^4\left[\frac{4}{27}\ln\frac{m_\mu}{m}\ln\frac{m_\mu}{m'}\ln\frac{m_\mu}{m''}-\frac{25}{81}\left(\ln\frac{m_\mu}{m}\ln\frac{m_\mu}{m'}+\ln\frac{m_\mu}{m}\ln\frac{m_\mu}{m''}+\frac{m_\mu}{m'}\ln\frac{m_\mu}{m''}\right)+\frac{4}{27}\left(\frac{\pi^2}{6}+\frac{317}{72}\right)\left(\ln\frac{m_\mu}{m}+\ln\frac{m_\mu}{m'}+\ln\frac{m_\mu}{m''}\right)+O(1)\right]. \tag{A9}$$

Now we put $m=m'=m''=m_e$ and easily obtain, in agreement with Eq. (10),

$$\Delta M_r^C(m_e,m_e,m_e,m_\mu)=\Delta M_r^C(m_e,m_\mu)$$
$$=\left(\frac{\alpha}{\pi}\right)^4\left[\frac{4}{27}\ln^3\frac{m_\mu}{m_e}-\frac{25}{27}\ln^2\frac{m_\mu}{m_e}+\left(\frac{2\pi^2}{27}+\frac{317}{162}\right)\ln\frac{m_\mu}{m_e}+O(1)\right]. \quad (A10)$$

In conclusion, this generalization of Kinoshita's method provides a neat and effective technique for evaluating, very easily, the contribution of a class with at least two electron loops. Unfortunately, for the case of the diagrams of class G, we have no relation, like a symmetry relation, which we can impose in Eq. (26). If we did, we could obtain the ln coefficient for ΔM_r^G.

---

*Permanent address.

[1]B. E. Lautrup, A. Peterman, and E. de Rafael, Phys. Rep. 3C, 193 (1972).

[2]We use for the electron anomaly $a_e^{(6)}=1.1(\alpha/\pi)^3$. See M. J. Levine and J. Wright, Phys. Rev. Lett. 26, 1351 (1971); Phys. Rev. D 8, 3171 (1973). See also T. Kinoshita and P. Cvitanovic, Phys. Rev. Lett. 29, 1534 (1972); Raymond Carroll and York-Peng Yao, Phys. Lett. 48B, 125 (1974).

[3]S. Brodsky, in proceedings of the SLAC Summer Institute on Particle Physics, 1973 (unpublished). This contribution is model-dependent.

[4]B. Lautrup, Phys. Lett. 38B, 408 (1972).

[5]T. Kinoshita, Nuovo Cimento 51B, 140 (1967).

[6]B. Lautrup, private communication.

[7]We speculate that in making radiative corrections to a class of graphs by inserting a single photon in all possible ways, one obtains a contribution which is roughly $-(\alpha/\pi)$ times the contribution of the class. This seems to be obeyed by the known contributions.

[8]B. Lautrup, Phys. Lett. 32B, 627 (1970).

[9]H. Terazawa, Prog. Theor. Phys. 40, 830 (1968).

[10]C. R. Hagen and M. A. Samuel, Phys. Rev. Lett. 20, 1405 (1968); B. Lautrup and E. de Rafael, Phys. Rev. 174, 1835 (1968).

[11]M. A. Samuel, Nucl. Phys. B47, 526 (1972); E. de Rafael and J. L. Rosner, Ann. Phys. (N.Y.) 82, 369 (1974).

[12]J. Aldins, S. Brodsky, A. Dufner, and T. Kinoshita, Phys. Rev. Lett. 23, 441 (1969); Phys. Rev. D 1, 2378 (1970).

[13]J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Cambridge, Mass., 1955).

[14]See, for example, A. Rich and J. C. Wesley, Rev. Mod. Phys. 44, 250 (1972).

[15]B. Lautrup and E. de Rafael, report 1973 (unpublished).

[16]S. Brodsky, private communication.