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Generalized recoil theorem for soft quantum emission including photon-pair correlations to infinite order in the coupling constant

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A generalized recoil theorem to arbitrary order in the coupling constant including photon-pair correlations is derived. The corresponding factorizable (because of pair effects not completely exponentiable) radiative transition rate is valid for an arbitrary process with arbitrarily many particles of arbitrary charge and mass. The theorem is first derived for spinless source particles and then generalized to particles with spin.

I. INTRODUCTION

First-order recoil effects to arbitrary order in the coupling constant including photon-pair correlations are evaluated. Recoil here means corrections to the soft (eikonal) approximation, which is known to lead to a perfectly exponentiable result. Due to the pair effects this exponentiation is not obtainable for the recoil process, although it is still factorizable and has a very convenient form. The technique is here developed within the framework of scalar electrodynamics and quantum electrodynamics (QED), but the application to other types of interactions is straightforward. This is because nowhere in the derivation have we made use of the smallness of the electromagnetic (em) coupling constant. The theorem is generalized to a process with arbitrarily many particles of arbitrary mass, charge, and spin. Before going into the detailed calculation, let us first review some earlier work on this topic.

The Low theorem was established for processes in which one soft photon is emitted.[1] It was further developed by Burnett and Kroll[2] and Bell and Van Royen.[3] By the use of charge conservation, and an assumption that the internal radiation amplitudes are nonsingular, the theorem gives the form of the recoil term σ_1 in the series expansion of the radiative differential cross section:

$$\frac{d\sigma}{dk_0} = \frac{\sigma_0}{k_0} + \sigma_1 + \sigma_2 k_0 + \cdots , \tag{1.1}$$

where k_0 is the photon energy. The term σ_1 comes from the interference between a term of order k^{-1}, i.e., the usual soft-photon current[4]

$$j^0_\mu(k) = \frac{ie}{(2\pi)^{3/2}}\,\theta(\epsilon - k_0)\left(\frac{p'_{1\mu}}{p'_1\cdot k} - \frac{p_{1\mu}}{p_1\cdot k}\right) \tag{1.2}$$

and a recoil term of order k^0. Here ϵ is the energy cutoff introduced to distinguish between hard and soft photons. It is introduced purely for calculational reasons, because the hard spectrum cannot be calculated to infinite order. The actual value to be used for ϵ is determined by the desired accuracy of the calculation. p_1 and k are the four-momenta of the accelerated charged particle and the emitted soft photon, respectively. Primed variables refer to outgoing massive particles. If T is the matrix element for an arbitrary nonradiative process, then from Feynman rules with a soft approximation, the radiative matrix element with one emitted soft photon is found to be

$$e^\mu j^0_\mu(k)\,T\,, \tag{1.3}$$

where e^μ is the photon polarization vector. It should be mentioned that the cutoff ϵ in (1.2) could equally well be introduced in the spatial photon momentum. The same contribution is obtained also from the virtual soft photon when the photon energy integration is performed, only that this contribution has the opposite sign compared with the cross section from the real photons.[5] Virtual plus real soft corrections then in fact give no contribution if it were not for the fact that $k_{0\,\max}$ for the hard photons, which depends on the energy E of the outgoing charged particle, sometimes becomes less than ϵ. One finds that when E tends to its maximum value, $k_{0\,\max}$ goes to zero (see, e.g., Ref. 6)

$$\lim_{E\to E_{\max}} k_{0\,\max}(E) = 0\,. \tag{1.4}$$

Integrating (1.1) one then finds

$$\sigma(E) = \sigma_0 \ln\left\{\frac{\min[\epsilon;\, k_{0\,\max}]}{\epsilon}\right\} + \text{recoil} \tag{1.5}$$

which appearently diverges in the limit (1.4). *This is the only infrared divergence which survives when virtual and real corrections are added and it is entirely caused by the kinematical condition (1.4).* (We investigated this in Ref. 6, which was later corrected by Ross.[6]) This is the point where the perturbation expansion breaks down and we must sum up diagrams to all orders in the coupling constant. The corresponding multiphoton states are most comfortably handled by the use of coherent states[7]

$$|j_0\rangle = U(j_0)|0\rangle , \qquad (1.6)$$

where U is the Weyl operator

$$U(j_0) = e^{(a^\dagger, j_0)-(j_0^*, a)} \qquad (1.7)$$

and $a^\dagger$ and a are the creation and annihilation operators for the photon field. Summing over all such diagrams to infinite order, where the nth order is essentially proportional to $e^n k^{-n}$, the soft radiative matrix element is then given by the factorized expression

$$M = |e^\mu j_\mu^0\rangle \; T . \qquad (1.8)$$

It is only in this soft corner of momentum space that we have got an exact radiative S matrix, in exact correspondence with the soft interacting Hamiltonian. The latter is obtained by a soft cutoff ϵ for the spatial photon momenta in

$$\mathcal{H}_I = e : \bar{\psi}\gamma^\mu\psi : A_\mu = j^\mu A_\mu . \qquad (1.9)$$

In passing we notice that the result (1.8) also obtains when the operator current in (1.9) is replaced by the classical particle trajectory,[4] the c-number current

$$j_\mu(x) = e \int u'_\mu \delta^4(x - x'(\tau))\, d\tau, \qquad u'_\mu = \frac{dx'_\mu}{d\tau} . \qquad (1.10)$$

The soft limit is therefore essentially a classical limit where the concept of a particle for the em field, i.e., the photon, is hardly a meaningful concept. To understand how quantum effects enter we start from this exponentiable limit and derive the recoil to infinite order in the coupling constant (nth order is proportional to $e^n k^{-n+1}$); *a combination of infinite summation methods for soft quanta*[4] *and Low's recoil theorem for one soft quantum.*[1]

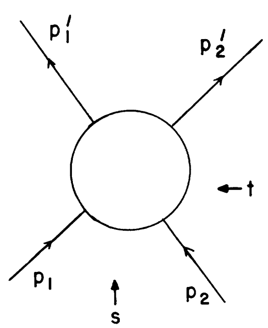

FIG. 1. The mutual scattering of two spin-zero particles by an arbitrary scattering mechanism. Particle one carries the charge e and particle two is neutral.

During the performance of this generalization, the full result of which is here going to be presented, we guessed part of it. In order to derive the energy loss spectrum we simply inserted $j^0 \to j^0 + J^{\mathrm{Recoil}}$ in (1.8) according to Low's result.[8] Neglecting pair effects that simple recipe holds when charged particles are spinless, whereas for particles with spin magnetic terms appear in the amplitude. In Ref. 8 it was assumed that all magnetic terms vanish in the spin-averaged transition rate. However, this is only true for the anomalous terms. *The regular magnetic terms are precisely needed to obtain the same transition rate as in the spinless case.* For the sake of simplicity, let us here assume that the virtual recoil effects are included in the core matrix. Once having obtained the technical clue, this can then be used for virtual corrections. It can also be used for improvements on the relativistic eikonal model.[9] Clearly, in strong interactions, where the coupling constant is large, pair effects will become more important. The situation will of course be much more complicated because of isospin and unitary spin, but on the other hand we cannot avoid an investigation of recoil terms if we want to understand the difference between a field theory with infraparticles and one with a mass gap.

II. RADIATIVE MATRIX ELEMENT FOR TWO SPIN-ZERO PARTICLES

The scattering of two spin-zero particles (Fig. 1) when particle one carries the charge e and mo-

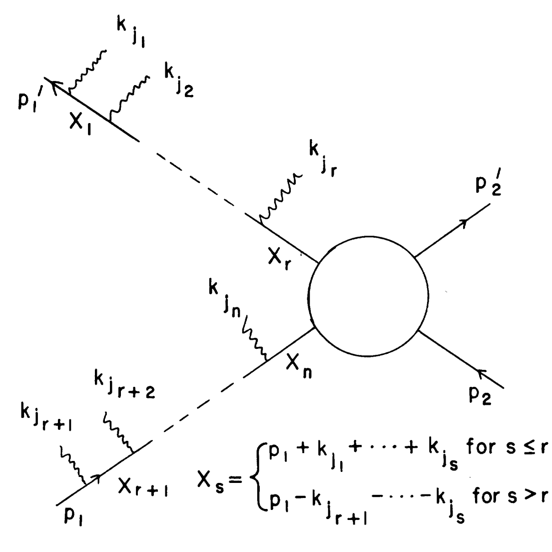

FIG. 2. The same process as in Fig. 1 but now with n emitted photons.

mentum p_1, and particle two is neutral and has momentum p_2, is first considered. The masses of the two particles are m_1 and m_2, respectively. For an arbitrary scattering mechanism, if no photons are emitted, the scattering matrix element T depends on two variables, essentially the square of their total energy, s, and the scattering angle t:

$$s = p_1 \cdot p_2 + p_1' \cdot p_2' , \tag{2.1}$$

$$t = (p_2' - p_2)^2 . \tag{2.2}$$

We then let n soft photons with momenta $k_1, k_2, \ldots, k_n$ and polarizations $e_1, e_2, \ldots, e_n$ be emitted in this process. Let r photons be attached to the outgoing external charged particle leg and $n-r$ photons to the corresponding ingoing leg (Figs. 2 and 3), with rising photon index towards the vertex. For the radiative matrix element we then get

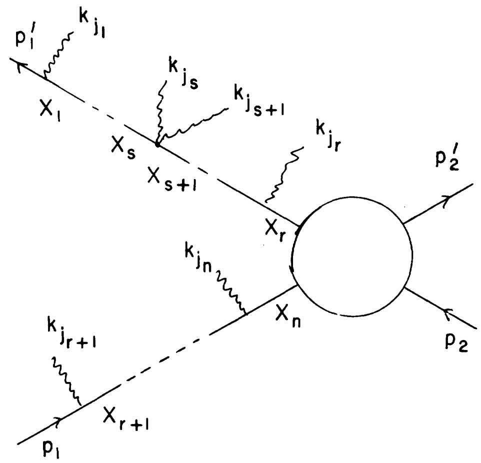

FIG. 3. A typical "seagull" diagram with n emitted photons.

$$\begin{aligned} M_r^{(1)} = (ie)^n \Bigg\{ & \sum_{\Pi(1\cdots n)} \left(\frac{p_1'\mu_{j(1)}}{p_1' \cdot k_{j_1}} \cdots \frac{2(p_1' + k_{j_1} + \cdots + k_{j_{r-1}})_{\mu_{j(r)}}}{2p_1' \cdot (k_{j_1} + \cdots + k_{j_r}) + (k_{j_1} + \cdots + k_{j_r})^2} \right) \\ & \times \left(\frac{p_1\mu_{j(r+1)}}{-p_1 \cdot k_{j_{r+1}}} \cdots \frac{2(p_1 - k_{j_{r+1}} - \cdots - k_{j_{n-1}})_{\mu_{j(n)}}}{-2p_1 \cdot (k_{j_{r+1}} + \cdots + k_{j_n}) + (k_{j_{r+1}} + \cdots + k_{j_n})^2} \right) \\ & - (-1)^{n-r} \Bigg[\sum_{\substack{s=1 \\ \Pi^s}}^{r-1} \left(\frac{p_1'\mu_{j(1)}}{p_1' \cdot k_{j_1}} \cdots \frac{p_1'\mu_{j(s-1)}}{p_1' \cdot (k_{j_1} + \cdots + k_{j_{s-1}})} \frac{g_{\mu_{j(s)}\mu_{j(s+1)}}}{p_1' \cdot (k_{j_1} + \cdots + k_{j_{s+1}})} \cdots \frac{p_1'\mu_{j(r)}}{p_1' \cdot (k_{j_1} + \cdots + k_{j_r})} \right) \\ & \qquad \times \left(\frac{p_1\mu_{j(r+1)}}{p_1 \cdot k_{j_{r+1}}} \cdots \frac{p_1\mu_{j(n)}}{p_1 \cdot (k_{j_{r+1}} + \cdots + k_{j_n})} \right) \\ & \qquad - \sum_{\substack{s=r+1 \\ \Pi^s}}^{n-1} \left(\frac{p_1'\mu_{j(1)}}{p_1' \cdot k_{j_1}} \cdots \frac{p_1'\mu_{j(r)}}{p_1' \cdot (k_{j_1} + \cdots + k_{j_r})} \right) \\ & \qquad \times \left(\frac{p_1\mu_{j(r+1)}}{p_1 \cdot k_{j_{r+1}}} \cdots \frac{p_1\mu_{j(s-1)}}{p_1 \cdot (k_{j_{r+1}} + \cdots + k_{j_{s-1}})} \frac{g_{\mu_{j(s)}\mu_{j(s+1)}}}{p_1 \cdot (k_{j_{r+1}} + \cdots + k_{j_{s+1}})} \cdots \frac{p_1\mu_{j(n)}}{p_1 \cdot (k_{j_{r+1}} + \cdots + k_{j_n})} \right) \Bigg] \Bigg\} \\ & \times \langle p_1' + k_{j_1} + \cdots + k_{j_r} ; p_2' | T | p_1 - k_{j_{r+1}} - \cdots - k_{j_n} ; p_2 \rangle e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n} . \end{aligned} \tag{2.3}$$

We let $D(p;q)$ denote the set of all permutations $\Pi(j_p \cdots j_q)$. Then clearly $\Pi(j_1 \cdots j_n)$ should run through the set $D(1;n)$ of all $n!$ permutations among the n photons. The seagull terms, corresponding to Fig. 3, shall be summed over all permutations Π^s of the factor set D^s.

$$D^s(1;n) = D(1;n)/D(s;s+1) . \tag{2.4}$$

As is well known from[4] the soft approximation is simply to count terms essentially of order $e^n k^{-n}$. After some straightforward algebraical work, omitting polarization vectors, (2.3) then factorizes to

$$M^{*(1)}_{\mu_1 \cdots \mu_n} = (ie)^n \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \prod_{l=1}^{r} \frac{p_1'\mu_l}{p_1' \cdot k_l} \prod_{l=r+1}^{n} \frac{p_1\mu_l}{p_1 \cdot k_l} \left\langle p_1' + \sum_{s=1}^{r} k_s ; p_2' \middle| T \middle| p_1 - \sum_{s=r+1}^{n} k_s ; p_2 \right\rangle , \tag{2.5}$$

where as well the k dependence in the T matrix could be dropped. The factor $\binom{n}{r}$ comes from the number of permutations in the factor set

$$D_1 = [D(1;n)/D(1;r)]/D(r+1;n), \tag{2.6}$$

where we have already summed over the $r!$ permutations on the outgoing side and the $(n-r)!$ permutations on the ingoing side. Summing over r in (2.5) and neglecting recoil in the T matrix we obtain the formula (3.6).

The main purpose of this work is to evaluate the recoil effects in (2.3), which is the collection of all terms $M^{**(1)}$ of essential order $e^n k^{-n+1}$. Thus from (2.3) we get

$$M^{(1)}=M^{*(1)}+M^{**(1)}\ , \tag{2.7}$$

where $M^{**(1)}$ partially factorizes to

$$M^{**(1)}_{\mu_1\cdots\mu_n}=(ie)^{n-2}\sum_{r=0}^{n}\sum_{s<t}^{n}\chi_{\mu_s\mu_t}(-1)^{n-r}\sum_{\Pi\in D_1^{st}}\prod_{\substack{l=1\\ l\neq s,t}}^{r}\frac{p'_{1\mu_{j(l)}}}{p'_1\cdot k_{j_l}}\prod_{\substack{l=r+1\\ l\neq s,t}}^{n}\frac{p_{1\mu_{j(l)}}}{p_1\cdot k_{j_l}}\langle p'_1;p'_2|T|p_1;p_2\rangle \tag{2.8}$$

and the tensor is the pair-correlation current

$$\chi_{\mu_s\mu_t}=(ie)^2[X_{\mu_s\mu_t}(p'_1;k_s;k_t)-X_{\mu_s\mu_t}(p_1;k_s;k_t)]\ , \tag{2.9}$$

with

$$X_{\mu_s\mu_t}=\frac{1}{p_1\cdot(k_s+k_t)}\left(\frac{p_{1\mu_s}k_{s\mu_t}}{p_1\cdot k_s}+\frac{p_{1\mu_t}k_{t\mu_s}}{p_1\cdot k_t}-\frac{p_{1\mu_s}p_{1\mu_t}}{p_1\cdot k_s p_1\cdot k_t}k_s\cdot k_t-g_{\mu_s\mu_t}\right)\ . \tag{2.10}$$

D_1^{st} is the factor set D_1/Π_{st} where the two s and t quanta are already permuted. The form of (2.9) follows from a first-order expansion of (2.3), and is proved by induction. The binomial theorem then gives

$$M^{**(1)}_{\mu_1\cdots\mu_n}=(ie)^{n-2}\sum_{s<t}^{n}\chi_{\mu_s\mu_t}\prod_{\substack{l=1\\ l\neq s,t}}^{n}\left(\frac{p'_{1\mu_l}}{p'_1\cdot k_l}-\frac{p_{1\mu_l}}{p_1\cdot k_l}\right)T(s;t) \tag{2.11}$$

which is gauge-invariant since

$$k_i^{\mu_i}\chi_{\mu_s\mu_t}=0\quad\text{for}\quad i=s,t\ . \tag{2.12}$$

We now turn to the k dependence in the T matrices (2.5), which also give rise to recoil effects.

III. EXPANSION OF THE T MATRICES

The T matrices in (2.5) conserve energy and momentum but not mass for particle one. They will therefore depend on the squares of the initial and final masses M_1^2 and $M_1'^2$, and of course also on s and t. In the case of (2.5) they are

$$y_1\equiv M_1'^2=M_1^2+2p'_1\cdot\sum_{s=1}^{r}k_{j_s}\ , \tag{3.1}$$

$$y_2\equiv M_1^2=m_1^2-2p_1\cdot\sum_{s=r+1}^{n}k_{j_s}\ , \tag{3.2}$$

$$y_3\equiv s^*=p_1\cdot p_2+p'_1\cdot p'_2+p'_2\cdot\sum_{s=1}^{r}k_{j_s}-p_2\cdot\sum_{s=r+1}^{n}k_{j_s}\ , \tag{3.3}$$

$$y_4\equiv t=(p'_2-p_2)^2\ . \tag{3.4}$$

We assume that the T matrices are analytic in all these four variables, and can then expand them into power series around $k_j=0$. To the order e^nk^{-n+1} we then have

$$M^{*(1)}_{\mu_1\cdots\mu_n}=N^{(1)}_{\mu_1\cdots\mu_n}+R^{(1)}_{\mu_1\cdots\mu_n}\ , \tag{3.5}$$

where $N^{(1)}$ is of essential order e^nk^{-n} and given by

$$N^{(1)}_{\mu_1\cdots\mu_n}=(ie)^n\prod_{l=1}^{n}\left(\frac{p'_{1\mu_l}}{p'_1\cdot k_l}-\frac{p_{1\mu_l}}{p_1\cdot k_l}\right)T\ . \tag{3.6}$$

This is just (2.5) after summation over r, if recoil is neglected in the T matrix. The correction term $R^{(1)}$ is of the order e^nk^{-n+1} and given by

$$R^{(1)}_{\mu_1\cdots\mu_n}=(ie)^n\sum_{\substack{r=0\\ \Pi\in D_1}}^{n}\prod_{l=1}^{r}\frac{p'_{1\mu_{j(l)}}}{p'_1\cdot k_{j_l}}\prod_{l=r+1}^{n}\frac{p_{1\mu_{j(l)}}}{p_1\cdot k_{j_l}}(-1)^{n-r}$$
$$\times\left[T_12p'_1\cdot\sum_{l=1}^{r}k_{j_l}-T_22p_1\cdot\sum_{l=r+1}^{n}k_{j_l}+T_3\left(p'_2\cdot\sum_{l=1}^{r}k_{j_l}-p_2\cdot\sum_{l=r+1}^{n}k_{j_l}\right)\right]\ . \tag{3.7}$$

T_1, T_2, and T_3 are the first-order partial derivatives with respect to y_1, y_2, and y_3.

There are $\binom{n}{r}$ elements in D_1 for a given r. Let us fix one arbitrary out of these $\Pi(j_1\cdots j_n)=\Pi'$ and for the corresponding term in (3.7) we get

$$(ie)^n \prod_{\substack{l=1\\ l\neq s}}^{r} \frac{p'_{1\mu_{j(l)}}}{p'_1\cdot k_{j_l}} \frac{p'_{1\mu_{j(s)}}}{p'_1\cdot k_{j_s}} \prod_{\substack{l=r+1\\ l\neq s}}^{n} \frac{p_{1\mu_{j(l)}}}{p_1\cdot k_{j_l}} (-1)^{n-r} \times\left[T_1 2p'_1\cdot\sum_{l=1}^{r} k_{j_l} - T_2 2p_1\cdot\sum_{l=r+1}^{n} k_{j_l} + T_3\left(p'_2\cdot\sum_{l=1}^{r}k_{j_l} - p_2\cdot\sum_{l=r+1}^{n}k_{j_l}\right)\right]. \qquad (3.8)$$

We then pick out that permutation Π'' which corresponds to that configuration, where the arbitrarily chosen j_sth photon in (3.8) is moved from the outgoing side to the ingoing

$$(ie)^n \prod_{\substack{l=1\\ l\neq s}}^{r} \frac{p'_{1\mu_{j(l)}}}{p'_1\cdot k_{j_l}} \frac{p_{1\mu_{j(s)}}}{p_1\cdot k_{j_s}} \prod_{\substack{l=r+1\\ l\neq s}}^{n} \frac{p_{1\mu_{j(l)}}}{p_1\cdot k_{j_l}} (-1)^{n-r+1} \times\left[T_1 2p'_1\cdot\sum_{l=1}^{r} k_{j_l} - T_2\left(2p_1\cdot\sum_{\substack{l=r+1\\ l\neq s}}^{n} k_{j_l} + 2p_1\cdot k_{j_s}\right) + T_3\left(p'_2\cdot\sum_{\substack{l=1\\ l\neq s}}^{r}k_{j_l} - p_2\cdot\sum_{\substack{l=r+1\\ l\neq s}}^{n}k_{j_l} - p_2\cdot k_{j_s}\right)\right]. \qquad (3.9)$$

If we multiply (3.8) and (3.9) with k_{j_s} and add these expressions, we get

$$(ie)^n \prod_{\substack{l=1\\ l\neq s}}^{r} \frac{p'_{1\mu_{j(l)}}}{p'_1\cdot k_{j_l}} \prod_{\substack{l=r+1\\ l\neq s}}^{n} \frac{p_{1\mu_{j(l)}}}{p_1\cdot k_{j_l}} (-1)^{n-r}\left[2p'_1\cdot k_{j_s} T_1 + 2p_1\cdot k_{j_s} T_2 + (p'_2\cdot k_{j_s} + p_2\cdot k_{j_s})T_3\right]. \qquad (3.10)$$

This is the part of $M^{(1)}$, which is projected out by k_{j_s} for a given r and given permutations Π' and Π''. We then again break out k_{j_s} and sum over r, s, and Π, where Π now runs through the set D_1^s. D_1^s is the same as D_1 except that the j_sth photon is missing. $R^{(1)}$ is then obtained to be

$$R^{(1)}_{\mu_1\ldots\mu_n} = (ie)^n \sum_{s=1}^{n} \prod_{\substack{l=1\\ l\neq s}}^{n}\left(\frac{p'_{1\mu_l}}{p'_1\cdot k_l} - \frac{p_{1\mu_l}}{p_1\cdot k_l}\right)\left[2p'_{1\mu_s}T_1 + 2p_{1\mu_s}T_2 + \left(\frac{p'_{1\mu_s}}{p'_1\cdot k_s}p'_2\cdot k_s + \frac{p_{1\mu_s}}{p_1\cdot k_s}p_2\cdot k_s\right)T_3\right], \qquad (3.11)$$

where we have made use of the binomial theorem. Components proportional to $p'_{2\mu_s}$ and $p_{2\mu_s}$ are absent in (3.7) and therefore also in (3.11).

Equation (3.10) is nonvanishing and accordingly $M^{(1)}$ given by (2.7) is not gauge-invariant. We therefore must add a correction $M^{(2)}$, which is the amplitude for radiative processes when not all photons are emitted from external particles (Fig. 4). This could be understood as follows:

The current given by (3.11) is not conserved. Because of charge conservation it must therefore exist a certain leakage somewhere. All processes, where every photon is emitted or absorbed before or after the scattering, are counted. The leakage must therefore take place during the time of interaction. In the "interaction-bubble" (Fig. 4), symbolically denoted by T, we do not know how matter and charge distribute or propagate etc. Thus radiation emitted from this state of excitation cannot directly be expressed in propagators and vertices. The corresponding leakage current is denoted by $M^{(2)}$.

The total matrix element is

$$M_{\mu_1\ldots\mu_n} = M^{(1)}_{\mu_1\ldots\mu_n} + M^{(2)}_{\mu_1\ldots\mu_n}. \qquad (3.12)$$

Since the over-all current is conserved, for an arbitrary s such that $1\leqslant s\leqslant n$, we have

$$k_{j_s}^{\mu_{j(s)}} M_{\mu_1\ldots\mu_n} = 0 \qquad (3.13)$$

from which we get that $M^{(1)}$ and $M^{(2)}$ are related by

$$\begin{aligned} k_{j_s}^{\mu_{j(s)}} M^{(2)}_{\mu_1\ldots\mu_n} &= -k_{j_s}^{\mu_{j(s)}} M^{(1)}_{\mu_1\ldots\mu_n} \\ &= -k_{j_s}^{\mu_{j(s)}} M^{*(1)}_{\mu_1\ldots\mu_n} \\ &= -k_{j_s}^{\mu_{j(s)}} R^{(1)}_{\mu_1\ldots\mu_n}. \end{aligned} \qquad (3.14)$$

However, in order to derive the j_sth component of $M^{(2)}$, which appears when we know how to break k_{j_s} out of (3.14), we must further conjecture

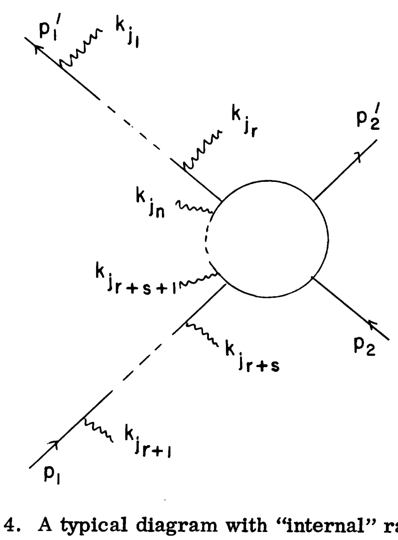

FIG. 4. A typical diagram with "internal" radiation.

(i) $M^{(2)}$ contains no charged one-particle propagator due to the j_sth photon, and is accordingly nonsingular for vanishing k_{j_s}. This has proved to be true for a large number of diagrams, where internal closed-loop integrations remove the j_sth photon singularity.

Conjecture (i) directly excludes terms like (2.11) and (3.6) in $M^{(2)}$ and thus

$$N^{(2)}_{\mu_1\ldots\mu_n}=0\ , \tag{3.15}$$

$$M^{**(2)}_{\mu_1\ldots\mu_n}=0\ . \tag{3.16}$$

It further gives the j_sth component of $M^{(2)}$ out of (3.14) and (3.10)

$$M^{(2)}_{\mu_1\ldots\mu_n}=-(ie)^n\sum_{s=1}^{n}\prod_{\substack{l=1\\ l\neq s}}^{n}\left(\frac{p'_{1\mu_l}}{p'_1\cdot k_l}-\frac{p_{1\mu_l}}{p_1\cdot k_l}\right)[2p'_{1\mu_s}T_1+2p_{1\mu_s}T_2+(p'_{2\mu_s}+p_{2\mu_s})T_3]\ . \tag{3.17}$$

The T_3 term in this is different from that in (3.11). Namely, the T_3 term projected out from the sum of (3.8) and (3.9) by k_{j_s} has the form

$$\left(\frac{p'_1\cdot k_{j_s}}{p'_1\cdot k_{j_s}}p'_2\cdot k_{j_s}+\frac{p_1\cdot k_{j_s}}{p_1\cdot k_{j_s}}p_2\cdot k_{j_s}\right)T_3 \tag{3.18}$$

and provides two possible ways to break out k_{j_s}: (3.11) and (3.17). The latter has no j_s propagator because of (i).

From (2.5), (2.7), (2.11), (3.5), (3.11), (3.12), and (3.17) the soft radiative matrix element, to the order $e^n k^{-n+1}$ is obtained to be

$$\begin{aligned}M_{\mu_1\ldots\mu_n}=\Bigg[&(ie)^n\prod_{l=1}^{n}\left(\frac{p'_{1\mu_l}}{p'_1\cdot k_l}-\frac{p_{1\mu_l}}{p_1\cdot k_l}\right)\\&+ie\sum_{s=1}^{n}\left(\frac{p'_{1\mu_s}}{p'_1\cdot k_s}p'_2\cdot k_s+\frac{p_{1\mu_s}}{p_1\cdot k_s}p_2\cdot k_s-p'_{2\mu_s}-p_{2\mu_s}\right)\frac{\partial}{\partial s_T}(ie)^{n-1}\prod_{\substack{l=1\\ l\neq s}}^{n}\left(\frac{p'_{1\mu_l}}{p'_1\cdot k_l}-\frac{p_{1\mu_l}}{p_1\cdot k_l}\right)\\&+\sum_{\substack{s,t=1\\ s<t}}^{n}\chi_{\mu_s\mu_t}(k_s;k_t)(ie)^{n-2}\prod_{\substack{l=1\\ l\neq s,t}}^{n}\left(\frac{p'_{1\mu_l}}{p'_1\cdot k_l}-\frac{p_{1\mu_l}}{p_1\cdot k_l}\right)\Bigg]T(s;t)\ .\end{aligned} \tag{3.19}$$

Denoting the soft current (1.2) without normalization and cutoff by f^0 and introducing an f^R for the operative recoil current, we get

$$M_{\mu_1\ldots\mu_n}=\left[\prod_{l=1}^{n}f^0_{\mu_l}(k_l)+\sum_{s=1}^{n}f^R_{\mu_s}(k_s)\prod_{\substack{l=1\\ l\neq s}}^{n}f^0_{\mu_l}(k_l)+\sum_{\substack{s,t=1\\ s<t}}^{n}\chi_{\mu_s\mu_t}(k_s;k_t)\prod_{\substack{l=1\\ l\neq s,t}}^{n}f^0_{\mu_l}(k_l)\right]T(s;t)\ . \tag{3.20}$$

The generalization to arbitrary many particles of arbitrary charge and mass is straightforward.

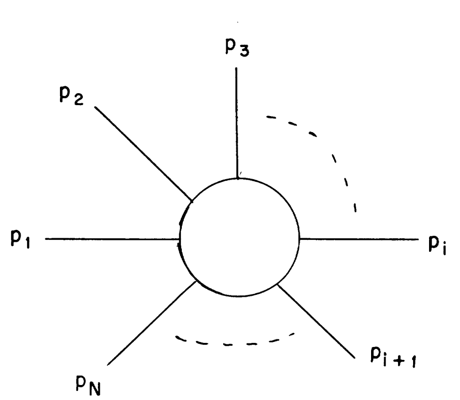

FIG. 5. A process with N particle legs.

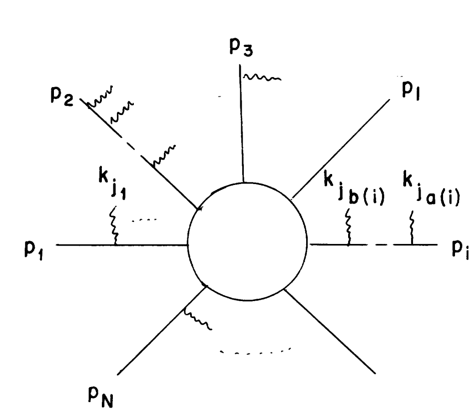

FIG. 6. A process with N particle legs and an arbitrary number of photons.

IV. GENERALIZATION TO ARBITRARILY MANY PARTICLES OF ARBITRARY MASS AND CHARGE

We first consider a process in which $\frac{1}{2}N$ charged spin-zero particles are scattered by an arbitrary mechanism (Fig. 5). The corresponding nonradiative matrix element we denote by T, which as before we assume to be an analytic function of the external momenta. The particles have masses $m_1, m_2, \ldots, m_N$ and charges $Q_1, Q_2, \ldots, Q_N$. Thus, attaching n soft photons with momenta $k_1, k_2, \ldots, k_n$ and polarizations $e_1, e_2, \ldots, e_n$, we get for the radiative matrix element (Fig. 6)

$$
\begin{aligned}
M^{(1)}_{\mu_1 \cdots \mu_n} = e^n \Bigg\{ \sum_{\substack{b(\nu)=0 \\ (\nu=1,\ldots,N)}}^{n} \Bigg[\sum_{\Pi \in D(1;n)} \prod_{i=1}^{n} (Q_i \epsilon_i)^{b(i)-a(i)+1} \\
\times \left(\frac{p_{i\mu_{j(a(i))}} \cdots 2[p_i + \epsilon_i (k_{j(a(i))} + \cdots + k_{j(b(i)-1)})_{\mu_{j(b(i))}}]}{p_i \cdot k_{j(a(i))} \cdots [2p_1 \cdot (k_{j(a(i))} + \cdots + k_{j(b(i))}) + \epsilon_1 \cdot (k_{j(a(i))} + \cdots + k_{j(b(i))})^2]} \right) \\
- \sum_{r=1}^{N} \sum_{s=a(r)}^{b(r)-1} \sum_{\Pi \in D^s(1;n)} \prod_{\substack{i=1 \\ i \neq r}}^{N} (Q_i \epsilon_i)^{b(i)-a(i)+1} \epsilon_r \left(\frac{p_{i\mu_{j(a(i))}} \cdots p_{i\mu_{j(b(i))}}}{p_i \cdot k_{j(a(i))} \cdots p_i \cdot (k_{j(a(i))} + \cdots + k_{j(b(i))})} \right) \\
\times \left(\frac{p_{r\mu_{j(a(r))}} \cdots p_{r\mu_{j(s-1)}} g_{\mu_{j(s)}\mu_{j(s+1)}} \cdots p_{r\mu_{j(b(r))}}}{p_r \cdot k_{j(a(r))} \cdots p_r \cdot (k_{j(a(r))} + \cdots + k_{j_{s-1}}) p_r \cdot (k_{j_{a(r)}} + \cdots + k_{j_{s+1}}) \cdots p_r \cdot (k_{j_{a(r)}} + \cdots + k_{j_{b(r)}})} \right) \Bigg] \Bigg\} .
\end{aligned}
\tag{4.1}
$$

Here, the sets of permutations $D(1;n)$ and $D^s(1;n)$ are given in Sec. II. The ϵ_i is defined by

$\epsilon_i = +1$ (-1) if the ith particle is outgoing (ingoing) .

For the factorizable part corresponding to (2.5) we obtain

$$
M^{*(1)}_{\mu_1 \cdots \mu_n} = (ie)^n \sum_{\substack{b(\nu)=0 \\ \nu=1,\ldots,N}}^{n} \sum_{\Pi \in D_N} \prod_{i=1}^{N} (Q_i \epsilon_i)^{b(i)-a(i)+1} \prod_{l=a(i)}^{b(i)} \frac{p_{i\mu_{j(l)}}}{p_i \cdot k_{j(l)}} T\left(p_r + \epsilon_r \sum_{t=a(r)}^{b(r)} k_{j_t} ;\ r=1,\ldots,N \right), \tag{4.2}
$$

where D_N is the factor set

$$
D_N = D(1;n) \Big/ \prod_{i=1}^{N} D(a(i);b(i)) .
$$

Then collecting all terms of order $e^n k^{-n+1}$ and proceeding exactly like in the previous case, the result is

$$
\begin{aligned}
M_{\mu_1 \cdots \mu_n} = (ie)^n \prod_{l=1}^{n} \sum_{i=1}^{N} \frac{Q_i \epsilon_i p_{i\mu_l}}{p_i \cdot k_l} T \\
+ ie \sum_{s=1}^{n} \sum_{r=1}^{N} Q_r \epsilon_r \left(\frac{p_{r\mu_s}}{p_r \cdot k_s} \epsilon_r k_s \cdot \frac{\partial}{\partial p_r} - \epsilon_r \frac{\partial}{\partial p_{r\mu_s}} \right)_T (ie)^{n-1} \prod_{\substack{l=1 \\ l \neq s}}^{n} \left(\sum_{i=1}^{N} \frac{Q_i \epsilon_i p_{i\mu_l}}{p_i \cdot k_l} \right) T \\
+ \sum_{\substack{s,t=1 \\ s<t}}^{n} \chi_{\mu_s \mu_t}(k_s;k_t) (ie)^{n-2} \prod_{\substack{l=1 \\ l \neq s;t}}^{n} \left(\sum_{i=1}^{N} \frac{Q_i \epsilon_i p_{i\mu_l}}{p_i \cdot k_l} \right) T .
\end{aligned}
\tag{4.3}
$$

In a first-order momentum approximation this gives

$$
M_{\mu_1 \cdots \mu_n} = \left\{ \prod_{l=1}^{n} [f^0_{\mu_l}(k_l) + f^R_{\mu_l}(k_l)] + \sum_{\substack{s,t=1 \\ s<t}}^{n} \chi_{\mu_s \mu_t} \prod_{\substack{l=1 \\ l \neq s,t}}^{n} f^0_{\mu_l}(k_l) \right\} T , \tag{4.4}
$$

where the pair-correlation current is now defined by

$$
\chi_{\mu_s \mu_t} = (ie)^2 \sum_{r=1}^{N} Q_r^{\,2} \epsilon_r X^r_{\mu_s \mu_t} \tag{4.5}
$$

and r denote the rth particle channel. Then like before we define

$$
f^0_\mu(k) = ie \sum_{i=1}^{N} Q_i \epsilon_i \frac{p_{i\mu}}{p_i \cdot k} \tag{4.6}
$$

and for the recoil current

$$f_\mu^R(k) = ie\sum_{i=1}^{N} Q_i\epsilon_i{}^2 D_{i\mu}(k)\ , \tag{4.7}$$

where the D's are the differential operators

$$D_{i\mu}(k) = \frac{p_{i\mu}}{p_i\cdot k}\, k\cdot\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_{i\mu}}\ . \tag{4.8}$$

The first term in (4.8) corresponds to the translation in momentum, $p_i \to p_i + \epsilon_i k$, and the second is the leakage current. In a first-order momentum approximation (3.20) equals (4.4), which is gauge-invariant due to charge conservation $\sum_{i=1}^{N} Q_i\epsilon_i = 0$. Derivatives of T with respect to masses,

$$y_r = M_r{}^2 = m_r{}^2 + 2\epsilon_r p_r\cdot\sum_{t=a(r)}^{b(r)} k_{j_t}\ , \tag{4.9}$$

do not contribute. This is shown similar to Sec. III.

V. GENERALIZATION TO PARTICLES WITH SPIN

We here choose the case when the source particle has spin one-half and anomalous magnetic moment λ and will everywhere indicate when some formulas are valid for arbitrary spin. According to the foregoing it is no simplification to let all photons be emitted from one of the particle legs. The nth-order matrix element corresponding to (2.3) is then defined by

$$\begin{aligned} M^{(1)}_{\mu_1\dots\mu_n} = (ie)^n \sum_{\Pi(1\cdots n)} \bar u(p_1')\, & \frac{(\gamma_{\mu_{j(1)}} - \lambda[\gamma_{\mu_{j(1)}}, \not k_{j_1}])(\not p_1' + \not k_{j_1} + m)}{(p_1' + k_{j_1})^2 - m^2} \\ \times & \frac{(\gamma_{\mu_{j(2)}} - \lambda[\gamma_{\mu_{j(2)}}, \not k_{j_2}])(\not p_1' + \not k_{j_1} + \not k_{j_2} + m)}{(p_1' + k_{j_1} + k_{j_2})^2 - m^2}\cdots \\ \times & \frac{(\gamma_{\mu_{j(n)}} - \lambda[\gamma_{\mu_{j(n)}}, \not k_{j_n}])(\not p_1' + \not k_{j_1} + \cdots + \not k_{j_n} + m)}{(p_1' + k_{j_1} + \cdots + k_{j_n})^2 - m^2}\ \mathcal{T}(p_1' + k_{j_1} + \cdots + k_{j_n}; p_2'; p_1; p_2) u(p_1). \end{aligned} \tag{5.1}$$

Straightforward algebraical calculations and iterated use of the spinor relations

$$\bar u(p_1')(\not p_1' - m) = 0$$

and

$$\bar u(p_1')\gamma_{\mu_s}(\not p_1' + m) = u(p_1')2p_{1\mu_s}'$$

leaves the following formula

$$\begin{aligned} M_{\mu_1\dots\mu_n} = \bar u(p_1')\Bigg[& \prod_{l=1}^{n} f^0_{\mu_l}(k_l) + \sum_{s=1}^{n} f^R_{\mu_s}(k_s) \prod_{\substack{l=1\\ l\neq s}}^{n} f^0_{\mu_l}(k_l) + ie\sum_{s=1}^{n}\frac{\gamma_{\mu_s}\not k_s}{p_1'\cdot k_s}[\tfrac12 - \lambda(\not p_1' + m)] \prod_{\substack{l=1\\ l\neq s}}^{n} f^0_{\mu_l}(k_l) \\ & + \sum_{\substack{s,t=1\\ s<t}}^{n} \chi_{\mu_s\mu_t}(k_s; k_t) \prod_{\substack{l=1\\ l\neq s,t}}^{n} f^0_{\mu_l}(k_l)\Bigg] \mathcal{T}(p_1'; p_2'; p_1; p_2) u(p_1)\ , \end{aligned} \tag{5.2}$$

where the currents are given in (4.5) to (4.8) with $N = 1$. The operative part of the recoil current f^R shall here act only on $\mathcal{T}$ and *not on the spinors*. The form (5.2) is made gauge-invariant as before and is easily proved by induction. Except for the magnetic terms

$$\gamma_{\mu_s}\not k_s = \tfrac12[\gamma_{\mu_s}, \not k_s] = -i\sigma_{\mu_s\nu}k^\nu\ , \tag{5.3}$$

it is identical to the result in the spinless case (3.20). Covariance and gauge invariance implies that the nonmagnetic part is valid for arbitrary spin. The transition rate we derive in two steps and we first demonstrate that anomalous terms, proportional to λ, in (5.1) do not contribute. It will do to study the sth photon contribution, since the rest may for the moment be included in $\mathcal{T}$.

$$M = ie\bar u(p_1')\,\frac{2p_{1\mu_s}' - \lambda[\gamma_{\mu_s}, \not k_s](\not p_1' + m)}{2p_1'\cdot k_s}\,\mathcal{T}u(p_1)e_s^{\mu_s}\ . \tag{5.4}$$

From this we get a recoil (by use of the notation $\bar\Gamma = \gamma^0\Gamma^\dagger\gamma^0$)

$$|M_{\text{recoil}}|^2=-\frac{e^2\lambda}{4m^2}\frac{p_1'\cdot e_s}{2(p_1'\cdot k_s)^2}\,\text{Tr}\,\{(\not{p}_1'+m)\mathcal{T}(\not{p}_1+m)\gamma^0\{[\not{e}_s,\not{k}_s](\not{p}_1'+m)\mathcal{T}\}^\dagger\gamma^0$$
$$+(\not{p}_1'+m)[\not{e}_s,\not{k}_s](\not{p}_1'+m)\mathcal{T}(\not{p}_1+m)\overline{\mathcal{T}}\}$$
$$=-\frac{e^2\lambda}{4m^2}\frac{p_1'\cdot e_s}{2(p_1'\cdot k_s)^2}\cdot\text{Tr}\{\gamma^0(\not{p}_1'+m)^\dagger\gamma^0[\not{k}_s,\not{e}_s](\not{p}_1'+m)\mathcal{T}(\not{p}_1+m)\overline{\mathcal{T}}$$
$$+(\not{p}_1'+m)[\not{e}_s,\not{k}_s](\not{p}_1'+m)\mathcal{T}(\not{p}_1+m)\overline{\mathcal{T}}\}=0\ . \tag{5.5}$$

This is because from $p_1'=(p_{10}',0)$ we have $\gamma^0(\not{p}_1'+m)^\dagger\gamma^0=(\not{p}_1'+m)$, and then (5.5) is always identically zero, which follows from covariance. It now remains to be demonstrated, the role of the non anomalous magnetic terms in (5.2) when it is squared and spin is summed over. For the moment we drop pair effects and anomalous terms and the matrix element is then given by

$$M_{\text{rad}}=(ie)^n\bar{u}(p_1')\left[\prod_{l=1}^n\frac{p_1'\cdot e_l}{p_1'\cdot k_l}+\sum_{s=1}^n\frac{\not{e}_s\not{k}_s}{2p_1'\cdot k_s}\prod_{\substack{l=1\\l\neq s}}^n\frac{p_1'\cdot e_l}{p_1'\cdot k_l}+\sum_{s=1}^n e_s\cdot D_1^{\mathcal{T}}(k_s)\prod_{\substack{l=1\\l\neq s}}^n\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right]\mathcal{T}(p_1')u(p_1) \tag{5.6}$$

and the corresponding transition rate is proportional to

$$|M_{\text{rad}}|^2=\frac{e^{2n}}{4m^2}\prod_{l=1}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)^2\text{Tr}[(\not{p}_1'+m)\mathcal{T}(\not{p}_1+m)\overline{\mathcal{T}}]$$
$$+\frac{e^{2n}}{4m^2}\prod_{l=1}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)\sum_{s=1}^n\prod_{\substack{l=1\\l\neq s}}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)\text{Tr}\left[\frac{\not{k}_s\not{e}_s(\not{p}_1'+m)\mathcal{T}(\not{p}_1+m)\overline{\mathcal{T}}}{2p_1'\cdot k_s}+\frac{(\not{p}_1'+m)\not{e}_s\not{k}_s\mathcal{T}(\not{p}_1+m)\overline{\mathcal{T}}}{2p_1'\cdot k_s}\right]$$
$$+\frac{e^{2n}}{4m^2}\prod_{l=1}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)\sum_{s=1}^n\prod_{\substack{l=1\\l\neq s}}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)\text{Tr}\{(\not{p}_1'+m)[e_s\cdot D_1(k_s)\mathcal{T}](\not{p}_1+m)\overline{\mathcal{T}}$$
$$+(\not{p}_1'+m)\mathcal{T}(\not{p}_1+m)[e_s\cdot D_1(k_s)\overline{\mathcal{T}}]\}\ . \tag{5.7}$$

We then insert

$$\frac{(\not{p}_1'+m)\not{e}_s\not{k}_s+\not{k}_s\not{e}_s(\not{p}_1'+m)}{2p_1'\cdot k_s}=\frac{2p_1'\cdot e_s\not{k}_s-2p_1'\cdot k_s\not{e}_s}{2p_1'\cdot k_s}$$
$$=e_s\cdot D_1(k_s)(\not{p}_1'+m) \tag{5.8}$$

which gives

$$|M_{\text{rad}}|^2=e^{2n}\prod_{l=1}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)^2|T|^2+e^{2n}\prod_{l=1}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)\sum_{s=1}^n\prod_{\substack{l=1\\l\neq s}}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)e_s\cdot D_1(k_s)|T|^2$$
$$=e^{2n}\prod_{l=1}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)\prod_{t=1}^n\left(\frac{p_1'\cdot e_t}{p_1'\cdot k_t}+e_t\cdot D_1(k_t)\right)|T|^2\ , \tag{5.9}$$

where T is defined by

$$|T|^2=\sum_{\text{spin}}|\bar{u}(p_1')\mathcal{T}u(p_1)|^2$$
$$=\frac{1}{4m^2}\,\text{Tr}[(\not{p}_1'+m)\mathcal{T}(\not{p}_1+m)\overline{\mathcal{T}}]\ . \tag{5.10}$$

Including also pair correlations we obtain

$$|M_{\text{rad}}|^2=e^{2n}\prod_{l=1}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)\left[\prod_{l=1}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}+e_l\cdot D_1^{\mathcal{T}}(k_l)\right)+2\sum_{\substack{s,t=1\\s<t}}^n\prod_{\substack{l=1\\l\neq s,t}}^n\left(\frac{p_1'\cdot e_l}{p_1'\cdot k_l}\right)e_s\cdot X_1(k_s;k_t)\cdot e_t\right]|T|^2 \tag{5.11}$$

independent of spin. In the one-photon case this agrees with the result of Ref. 2. For simplicity we have here dropped the cutoff functions and the normalization factors of the currents. The generalization to arbitrarily many particles of arbitrary mass and charge now follows trivially from the preceding section, Eqs. (4.1)–(4.4). As mentioned before the validity of (5.11) for arbitrary spin follows from covariance and gauge invariance. (See also Ref. 3.)

VI. SUMMARY

We have thus derived a generalized first-order recoil theorem for real soft quantum emission to infinite order in the coupling constant, including pair-correlation effects. It is valid for any process with an arbitrary scattering mechanism and where the source particles have *arbitrary mass, charge, and spin,* when spin is averaged over. When polarized particles are studied the nonmagnetic part of the amplitude is always the same, whereas the spin part naturally varies with spin. Nowhere in our derivation have we used the assumption that the coupling constant shall be small, and we therefore can apply this result to theories with an arbitrarily large coupling constant. Straightforward generalization of (5.2) to a process with arbitrarily many (N) spin-$\frac{1}{2}$ particles with arbitrary mass and charge Q gives the soft radiative transition amplitude

$$M^n_{\beta\alpha}=\bar{u}(p_\beta)\Bigg\{\prod_{l=1}^{n} f^0(k_l)\cdot e_l+\sum_{s=1}^{n} f^R(k_s)\cdot e_s\prod_{\substack{l=1\\ p\neq s}}^{n} f^0(k_l)\cdot e_l+\sum_{s=1}^{n}\Bigg[\sum_{i=1}^{N} Q_i\frac{\not{e}_s\not{k}_s}{p_i\cdot k_s}\left(\frac{1}{2}\frac{\lambda_i}{Q_i}(\not{p}_i+m)\right)\Bigg]\prod_{\substack{l=1\\ l\neq s}}^{n} f^0(k_l)\cdot e_l$$

$$+\sum_{\substack{s,t=1\\ s<t}}^{n} e_s\cdot\chi(k_s\cdot k_t)\cdot e_t\prod_{\substack{l=1\\ l\neq s,t}}^{n} f^0(k_l)\cdot e_l\Bigg\}\,\mathcal{T}u(p_\alpha)\,. \tag{6.1}$$

Here the spinors stand for the direct spinor products of ingoing and outgoing particles, respectively. For spinless particles the magnetic term disappears and we obtain (3.20). In a process where the initial particles are unpolarized and where final spin is unobserved the transition rate follows from (5.11) and (6.1). Defining $f=f^0+f^R$ and summing over photon polarizations we obtain

$$\sum_{\text{pol}}|M^n_{\text{rad}}|^2=(-1)^n\prod_{l=1}^{n} f^{0*}(k_l)\cdot\Bigg[\prod_{l=1}^{n} f(k_l)+2\sum_{\substack{s,t=1\\ s<t}}^{n}\chi(k_s;k_t)\prod_{\substack{l=1\\ l\neq s,t}}^{n} f^0(k_l)\Bigg]\,|T|^2$$

$$=(-1)^n\Bigg[\prod_{l=1}^{n}|f^0(k_l)|^2+\prod_{l=1}^{n} f^{0*}(k_l)\cdot f^R(k_l)+2\sum_{\substack{s,t=1\\ s<t}}^{n} f^{0*}(k_s)\cdot\chi(k_s;k_t)f^{0*}(k_t)\prod_{\substack{l=1\\ l\neq s,t}}^{n}|f^0(k_l)|^2\Bigg]|T|^2\,. \tag{6.2}$$

With the densities

$$f^{0*}(k_1)\cdot f(k_1)=\delta^0(k_1)+\delta^R(k_1)$$

Eq. (6.2) is

$$\sum_{\text{pol}}|M^n_{\text{rad}}|^2=(-1)^n\Bigg[\prod_{l=1}^{n}\delta^0(k_l)+\prod_{l=1}^{n}\delta^R(k_l)+2\sum_{\substack{s,t=1\\ s<t}}^{n} f^{0*}(k_s)\cdot\chi(k_s k_t)f^{0*}(k_t)\prod_{\substack{l=1\\ l\neq s,t}}^{n}\delta^0(k_l)\Bigg]\,|T|^2\,, \tag{6.3}$$

where the first term is the well-known exponentiable part.

The obtained result shows how correlation effects modify the exponentiation. Still, however, it is partially factorizable and explicitly obtainable to infinite order in the coupling constant, and is therefore in exact correspondence with the Lagrangian model in the soft corner of momentum space. In a second paper we shall treat the complete answer by including also virtual recoil effects in an elastic nearly forward scattering process. Technically we have got the clue to this since the approximation in a certain infinite-momentum limit is exactly the same. We are then able to study how eikonalization[9] should be modified in an exact treatment and not by selecting certain graphs. It is also nice to notice the form invariance of the result with respect to change of mass, charge, and spin, which enables us to sum also over intermediate resonances of the charged particles.

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Gauge-invariant signal for gauge-symmetry breaking*

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The effective potential is computed to order $\hbar$ in an Abelian gauge theory—scalar electrodynamics. The calculation is performed first in the ghost-requiring R_ξ gauges. The corresponding expression is also derived from the unitary Lagrangian. We discuss the gauge dependence of the effective potential and its minima in connection with spontaneous symmetry breakdown; and we interpret the unitary computation to be the physically relevant one.

I. INTRODUCTION

Minima in the field-theoretic effective potential V indicate symmetry properties of solutions for the theory.[1] Unfortunately, an exact calculation of V is rarely possible; often the best answer to be had is the first few terms in a loop expansion.[2-6] In general, one goes beyond the lowest tree approximation, so that effects due to accidental symmetries,[4,5] finite temperature,[7] or radiative corrections[3] can be examined. However, any approximate calculation may be unreliable; it may exhibit unphysical minima. (For example, we show in the Appendix that in ordinary quantum mechanics one frequently commits errors when the exact V is approximated by a finite series since the series does not converge in the region of the true minimum.)

In gauge theories the effective potential is gauge-dependent.[3,4,6] This presents difficulty in assessing the validity of any approximation to the complete V, since the gauge dependence may create false minima. Also, a direct physical interpretation cannot be given to a gauge-dependent quantity. Furthermore, it has been alleged that in some gauges (the R_ξ gauges[8]) V cannot be defined.[4]

In this paper we compute the effective potential to order $\hbar$ for an Abelian gauge theory—scalar electrodynamics. We show how even in the R_ξ gauges a potential *can* be defined. The calculation in this gauge is of additional interest as it involves a treatment of ghost loops. The problems with gauge dependence are vividly portrayed in our calculation. In the R_ξ gauge, V is already gauge-dependent in the tree approximation and possesses stationary points which do not correspond to physical solutions of the theory.

We suggest that the difficulty of the gauge dependence may be resolved by considering the unitary Lagrangian $\mathcal{L}_U$ (frequently called the Lagrangian in the unitary gauge). This unitary Lagrangian can be obtained as the limit of the corresponding object in the R_ξ gauge. However, we shall argue that $\mathcal{L}_U$ may be viewed not merely as a Lagrangian in a special gauge, but also as the Lagrangian for the theory when all gauge degrees of freedom have been removed.[9] The unitary Lagrangian reflects the physical spectrum for its fields, and the effective potential V_U associated with it merits the physical interpretation given by Symanzik.[10]

The danger with computations based on the unitary Lagrangian is that they may not be renor-