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Method for proof of asymptotic theorems in presence of oscillations

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A simple method is derived which gives the asymptotic behavior of the forward scattering amplitude directly on the physical cut without any restriction on the nature of oscillations. Applications are given to high-energy theorems such as the Pomeranchuk theorems and consequences deduced from unitarity. Using the phase representation and the method developed, a number of results due to Khuri and Kinoshita, Jin and MacDowell, and Bessis and Kinoshita are derived and improved.

I. INTRODUCTION

Within the framework of axiomatic field theory, very little is known on the high-energy behavior of the scattering amplitudes. As the energy tends to infinity, because of the accumulation of branch points corresponding to normal thresholds of production processes, the presence of oscillations is generally expected. This phenomenon makes it difficult to study the asymptotic behavior of dispersion integrals which is needed in the proof of the Pomeranchuk theorems for the difference of particle-antiparticle total cross sections, the differential cross sections, and also in the proof of the asymptotic relation between the phase and the modulus of the scattering amplitude.

There have been many treatments of this problem in the literature and widely different methods have been used. The purpose of this paper is to give a simple and elementary but nevertheless rigorous method to handle the problem of oscillations; in many cases it leads to more precise and general results than those obtained previously.

Let us briefly review the situation of the proof of the Pomeranchuk theorem, where the problem of oscillations was first encountered. Since the original publication of Pomeranchuk,[1] there have been many attempts to give a precise proof of this theorem regardless of the nature of the oscillations of the difference $\Delta\sigma$ of the particle-antiparticle total cross sections.[2]

The first rigorous proof is due to Weinberg[3] who used the properties of Herglotz functions to show that the integral

$$\int_{\mu}^{\infty} \frac{dE'}{E'} \Delta\sigma(E') < +\infty ,$$

under the assumption that $\Delta\sigma$ does not change sign for large enough energies (together with the usual physical assumption that the ratio $\mathrm{Re}F/\mathrm{Im}F$ is bounded). The case excluded by Weinberg's proof is when $\Delta\sigma$ changes sign indefinitely; in this case, because of the continuity of $\Delta\sigma$, it is clear that $\Delta\sigma$ must go through zero an infinite number of times. Later Meiman[4] used a generalization of the Phragmén-Lindelöf theorem to prove that the set of limiting values of $\Delta\sigma$ contains zero. Finally a simple proof of the Pomeranchuk theorem under slightly more general physical assumptions was given by Martin.[5] Martin's proof has the interesting feature of implying some average constraint on $\Delta\sigma$. This was recently made more precise by Truong and Lam,[6] who showed from Martin's result that

$$\lim_{E\to\infty} \frac{1}{\ln E} \int_{\mu}^{E} \frac{dE'}{E'} \Delta\sigma(E') = 0 ;$$

hence they could make a statement about the asymptotic density of zeros and generalized Weinberg's result when $\Delta\sigma$ changes sign indefinitely. This method cannot however be used to study the restriction due to unitarity in general. This led Truong and Lam to introduce a method of averaging to handle the oscillations directly on the physical cut. The drawback of this method is that further assumptions must be made in order to extract physical consequences. This weak point will be eliminated in this article.

It turns out that the average method of Truong and Lam after some modifications is especially suitable for the phase representation. We shall show that it gives asymptotic Regge-like relations between the phase and the modulus of the forward scattering amplitude, in the spirit of the work of Sugawara and Tubis[7] and Jin and MacDowell[8] (see also Gervais[9]). It can also be used to derive in a simple manner and to sharpen some of the results given by Khuri and Kinoshita[10,11] on the forward crossing-even amplitude. The discussion of the phase representation presented in this paper thus serves the useful purpose of bridging the gap between the work of Jin and MacDowell[8] and that of Khuri and Kinoshita.[10,11] The Pomeranchuk theorem together with the restriction due to unitarity will be studied by a direct method which circumvents some drawback of the method of Truong and Lam.

We would like to stress that the asymptotic theorems stated in the integral form given above contain the maximum amount of information. To make this clear, let us discuss this point further.

(i) If $\Delta\sigma$ does not change sign, this condition implies that $\lim_{E\to\infty}\Delta\sigma=0$, except on a set of points of zero asymptotic density.

(ii) If $\Delta\sigma$ changes sign indefinitely, this condition shows to what extent the cancellation of oscillations exists. As an example let us take $\Delta\sigma=C+\sin(\ln x)$, with $|C|<1$. This integral condition shows that $C=0$, which cannot be obtained by the usual method (since the set of limiting values by construction contains zeros).

The plan of this paper is organized as follows. In Sec. II the problem of oscillations is reviewed. Our analysis leads naturally to the average procedure of Sec. III, where the method of Truong and Lam[6] is developed into a form which is useful for the phase representation. In Sec. IV, the physical consequences of the phase representation are given. We give a simple demonstration of the Jin-Martin lower bound[12] which also gives the density of the set of points where this bound can possibly be violated. We also derive the most general condition on the phase to get a generalized "Regge" behavior of the Bessis-Kinoshita type[13] (which is in our case a behavior in the physical region). As a special case, our results lead naturally to those of Gervais and Yndurain.[9,14] The connection with the work of Khuri and Kinoshita and its generalization are given in Sec. V. Section VI deals with the direct proof of the Pomeranchuk theorem and the improved upper bound for $|\Delta\sigma|$, as well as with general consequences deduced from unitarity. In Sec. VII the case of slowly decreasing function is discussed.

Because of the wide variety of subjects dealt with in this paper, we would suggest the reader start with the section which is of interest to him. For example, if he is interested only in the proof of the Pomeranchuk theorem, he should start first with Sec. VI and read the other sections later.

II. PROBLEMS RELATED TO THE PRINCIPAL-PART INTEGRATION

Let us take the proof of the Pomeranchuk theorem as an example to discuss the problem of oscillations. The following discussion applies also to the phase representation; in this case the imaginary part in the dispersion integral is replaced by the phase of the amplitude instead of the difference of the total cross sections.

Consider the odd-crossing forward amplitude $f_a(E)=f_P(E)-f_A(E)$, where f_P and f_A are the particle and antiparticle forward scattering amplitudes and E is the laboratory energy. Let us now define

$$f(E)=\frac{f_a(E)}{E}-f_a'(0).$$

It is obvious that $f(E)$ is even under crossing and thus satisfies the following dispersion relation:

$$\begin{aligned}\mathrm{Re}f(E)&=\frac{E}{\pi}\mathrm{P}\int_{-\infty}^{+\infty}\frac{dE'}{E'}\frac{\mathrm{Im}f(E')}{E'-E}\\&=\frac{2E^2}{\pi}\mathrm{P}\int_{E_0}^{+\infty}\frac{dE'}{E'}\frac{\mathrm{Im}f(E')}{E'^2-E^2},\end{aligned}\qquad(1)$$

where $\mathrm{Im}f(E)=(1/4\pi)(q/E)\Delta\sigma(E)$ by the optical theorem, q is the laboratory momentum, and $\Delta\sigma$ is the difference of particle and antiparticle total cross sections. We assume in the following $\Delta\sigma$, and hence $\mathrm{Im}f$, is bounded.

Equation (1) can be rewritten in terms of $x=E^2$ variable as

$$\mathrm{Re}f(x)=\frac{x}{\pi}\mathrm{P}\int_{x_0}^{+\infty}\frac{dx'}{x'}\frac{\mathrm{Im}f(x')}{x'-x}.\qquad(2)$$

Let us first assume that $\lim_{x\to+\infty}\mathrm{Im}f(x)=C$. A crude calculation, with $\mathrm{Im}f(x')$ on the right-hand side of (2) replaced by its limit C for sufficiently large x', yields the following result:

$$\lim_{x\to+\infty}\frac{\mathrm{Re}f(x)}{\ln x}=-\frac{C}{\pi},\qquad(3)$$

which is incompatible with the physical assumption $|f(x)|/\ln x\to 0$ unless $C=0$. The derivation of Eq. (3) cannot be regarded as strictly correct, since it does not take into account the dependence of the principal-part integration (2) on the derivative of the integrand. To see this, let us suppose for the time being that the derivative of $\mathrm{Im}f(x)$ exists (we will not have to make this assumption later). Subsequent analysis shows that Eq. (3) is correct only if an assumption is made on the asymptotic behavior of the derivative of $\mathrm{Im}f(x)$. It is precisely to avoid this difficulty and to do away with any assumption about the derivative that Martin[5] uses the Phragmén-Lindelöf theorem to show that the physical assumption $\lim_{x\to+\infty}f(x)/\ln x=0$ is valid also outside the physical cut, in particular when $x\to-\infty$. In this region we no longer have to deal with the principal-part integration and the physical assumption leads immediately to

$$\lim_{x\to+\infty}\frac{x}{\ln x}\int_{x_0}^{+\infty}\frac{dx'}{x'}\frac{\mathrm{Im}f(x')}{x'+x}=0,\qquad(4)$$

which is compatible only with $C=0$. Equation (4) already yields some average constraint on $\mathrm{Im}f$. This point was made more precise recently by Truong and Lam,[6] who have shown that (see also Appendix C)

$$f(-x) = -\frac{x}{\pi}\int_{x_0}^{+\infty}\frac{dx'}{x'}\frac{\mathrm{Im}f(x')}{x'+x} \qquad x>0$$

$$= -\frac{1}{\pi}\int_{x_0}^{x}\frac{dx'}{x'}\mathrm{Im}f(x') + O(1)\,, \tag{5}$$

where $O(1)$ involves only terms which stay bounded as $x\to+\infty$. Using this result in Eq. (4) we have

$$\lim_{x\to+\infty}\frac{1}{\ln x}\int_{x_0}^{x}\frac{dx'}{x'}\mathrm{Im}f(x') = 0\,, \tag{6}$$

which is the result given by Truong and Lam.[6]

It is now natural to find the sufficient condition to get the behavior expressed by Eq. (6), but working directly on the physical cut. For this purpose, let us split the right-hand side of Eq. (2) as follows:

$$\mathrm{Re}f(x) = -\frac{1}{\pi}\int_{x_0}^{x}\frac{dx'}{x'}\mathrm{Im}f(x') + \frac{1}{\pi}\mathrm{P}\int_{x_0}^{2x}dx'\frac{\mathrm{Im}f(x')}{x'-x}$$

$$-\frac{1}{\pi}\int_{x}^{2x}\frac{dx'}{x'}\mathrm{Im}f(x')$$

$$+\frac{1}{\pi}\int_{2x}^{+\infty}\frac{dx'}{x'}\frac{x}{x'-x}\mathrm{Im}f(x')\,. \tag{7}$$

The last two integrals on the right-hand side of (7) are $O(1)$ as can be seen by using the change of variable $y' = x'/x$ and by noticing that $\mathrm{Im}f(xy')$ is bounded by assumption. Let us now introduce the average[6]

$$\int_{x_0}^{x}\frac{dx'}{x'}\mathrm{Im}f(x') = \langle\mathrm{Im}f\rangle_x \ln x \tag{8}$$

into Eq. (7):

$$\mathrm{Re}f(x) = -\frac{1}{\pi}\langle\mathrm{Im}f\rangle_x\ln x + I(x) + R(x)\,, \tag{9}$$

where

$$R(x) = -\frac{1}{\pi}\int_{x}^{2x}\frac{dx'}{x'}\mathrm{Im}f(x')$$

$$+\frac{1}{\pi}\int_{2x}^{+\infty}\frac{dx'}{x'}\frac{x}{x'-x}\mathrm{Im}f(x') = O(1)$$

and

$$I(x) = \frac{1}{\pi}\mathrm{P}\int_{x_0}^{2x}dx'\frac{\mathrm{Im}f(x')}{x'-x}.$$

It is easy to see that, by changing the variable as done previously, one can write

$$I(x) = J(x) + O(1)\,,$$

with

$$J(x) = \frac{1}{\pi}\mathrm{P}\int_{x(1-\lambda)}^{x(1+\lambda)}dx'\frac{\mathrm{Im}f(x')}{x'-x}\,, \tag{10}$$

where $0<\lambda<1$. This principal-part integral was previously considered by Sugawara and Kanazawa[15] and by Gervais and Yndurain.[14] It is only when $J(x)$ is bounded that one gets the same asymptotic behavior for $f(x)$ on the physical cut as in other directions in the complex-x plane. From the definition of the principal-part integral, it is simple to show that $J(x)$ depends on the derivative of $\mathrm{Im}f(x)$ which is not necessarily bounded for a bounded $\mathrm{Im}f(x)$. If one is willing to make the assumption that $d\,\mathrm{Im}f(x)/dx$ is bounded, it will be shown in Appendix A that $J(x) = O(\ln x)$, i.e., $J(x)/\ln x$ is bounded (this is a particular case of a more general situation considered by Lanz and Prosperi[16]). In fact examples where $\mathrm{Im}f(x)$ stays bounded and its derivative approaches zero, but where $J(x)$ diverges like $\ln x$ on a set of points extending to infinity are given in Appendix B. In these cases, although it is possible to bound $\mathrm{Re}f(x)$ by $\ln x$, the unbounded contribution to $\mathrm{Re}f(x)$ is not entirely given by the term $-(1/\pi)\langle\mathrm{Im}f\rangle_x\ln x$.

Let us note parenthetically that if we consider the "nonoscillating" case, where

$$\lim_{x\to+\infty}\mathrm{Im}f(x) = C\,,$$
$$\frac{d}{dx}\mathrm{Im}f(x) = O(1)\,, \tag{11a}$$

it is shown in Appendix A that

$$\lim_{x\to\infty}\frac{J(x)}{\ln x} = 0\,. \tag{11b}$$

It is interesting to note that if $\mathrm{Im}f(x)$ belongs to the class of nonoscillating functions introduced by Gervais and Yndurain,[14] $J(x)$ may still give trouble; in fact these authors met basically the same difficulty as in the general case of oscillations considered here, and had to use an average procedure similar to the one to be introduced in Sec. III.

The problem of bounding $J(x)$ appears then to be quite complicated. It is shown in Appendix A that a sufficient condition for $J(x)$ to be bounded as $x\to+\infty$ is

$$x\frac{d}{dx}\mathrm{Im}f(x) = O(1)\,. \tag{12}$$

One should remark that condition (12) follows automatically from the boundedness of $\mathrm{Im}f(x)$ for some *simple* nonoscillating functions. We thus see that with enough smoothness as required by condition (12), it is possible to bound $J(x)$, and hence the asymptotic behavior of $\mathrm{Re}f(x)$ on the physical cut is also given by the right-hand side of Eq. (5), i.e., the same as outside the cut when

$x\to -\infty$.

It appears that we are faced with the possibility of having to make some assumption on the derivative of $\mathrm{Im}f(x)$ to carry out our program. Fortunately this is not needed as suggested by Martin's proof of the Pomeranchuk theorem and its generalization by Truong and Lam. Instead of dealing directly with $f(x)$, we shall work with an average function which is also analytic in the complex-x plane and is sufficiently smooth such that (12) is automatically satisfied by construction. This is done in Sec. III.

III. AVERAGE ASYMPTOTIC BEHAVIOR OF $\mathrm{Re}f(x)$ ON THE REAL AXIS

Truong and Lam, in their proof of the Pomeranchuk theorem,[6] introduced the following function (apart from a factor of $\ln x$):

$$g(x)=\int_0^x \frac{dx'}{x'}f(x'). \tag{13}$$

(A similar equation for the even-crossing amplitude was introduced earlier by Khuri and Kinoshita[11] to construct univalent functions.) This integral converges at $x=0$, since $\mathrm{Im}f(x)=0$ for $x<x_0$ and $[\mathrm{Re}f(x)/x]_{x=0}$ is finite as can be seen from Eq. (2). The average function $g(x)$ when divided by $\ln x$ has the property of preserving the magnitude of $f(x)$, if $f(x)$ belongs to the class of slowly varying functions like $\ln x$, $\ln\ln x$, etc. It is analytic in the complex-x plane with the same cut as $f(x)$ and has the same number of subtractions. Furthermore its derivative exists and satisfies the smoothness condition (12) since

$$\frac{d}{dx}\mathrm{Im}g=\frac{\mathrm{Im}f(x)}{x}$$

and $\mathrm{Im}f(x)=O(1)$. We may expect that by working with the function $g(x)$ it will be possible to get rid of the difficulties associated with $J(x)$. This indeed happens. Since $g(x)$ is analytic, a dispersion relation similar to Eq. (2) can be written, from which the analysis of Sec. II shows

$$\begin{aligned}\mathrm{Re}g(x)&=-\frac{1}{\pi}\int_{x_0}^x\frac{dx'}{x'}\mathrm{Im}g(x')+O(\ln x)\\&=-\frac{1}{\pi}\int_{x_0}^x\frac{dx'}{x'}\int_{x_0}^{x'}\frac{dx''}{x''}\mathrm{Im}f(x'')+O(\ln x)\,;\end{aligned} \tag{14}$$

the difference between Eq. (14) and Eq. (9) is that $I(x)$ and $R(x)$ are not bounded by $O(1)$ but $O(\ln x)$ since $\mathrm{Im}g(x)$ is $O(\ln x)$. Equation (14) was previously obtained by Truong and Lam by the average method. It turns out that this method is more useful and leads naturally to other interesting consequences. We shall now give a more compact and slightly improved derivation of $\mathrm{Re}g(x)$.

We have from Eqs. (13) and (2), after interchanging order of integration which is legitimate,[17]

$$\begin{aligned}\mathrm{Re}g(x)&=\int_{x_0}^x\frac{dx'}{x'}\mathrm{Re}f(x')\\&=-\frac{1}{\pi}\int_{x_0}^{+\infty}\frac{dx'}{x'}\ln\left|1-\frac{x}{x'}\right|\mathrm{Im}f(x').\end{aligned} \tag{15}$$

The integral in Eq. (15) can be rewritten as follows:

$$\begin{aligned}-\mathrm{Re}g(x)&=\frac{1}{\pi}\int_{x_0}^x\ln\frac{x}{x'}\mathrm{Im}f(x')\frac{dx'}{x'}\\&+\frac{1}{\pi}\int_{x_0}^x\ln\left|\frac{x'}{x}-1\right|\mathrm{Im}f(x')\frac{dx'}{x'}\\&+\frac{1}{\pi}\int_x^{+\infty}\ln\left|1-\frac{x}{x'}\right|\mathrm{Im}f(x')\frac{dx'}{x'}.\end{aligned}$$

Again the last two integrals are seen to be bounded, if $\mathrm{Im}f$ is bounded, by the change of variable $y'=x'/x$, hence

$$\mathrm{Re}g(x)=-\frac{1}{\pi}\int_{x_0}^x\left(\ln\frac{x}{x'}\right)\mathrm{Im}f(x')\frac{dx'}{x'}+O(1). \tag{16}$$

Integrating the right-hand side of Eq. (16) by parts, we have

$$\mathrm{Re}g(x)=-\frac{1}{\pi}\int_{x_0}^x\frac{dx'}{x'}\int_{x_0}^{x'}\frac{dx''}{x''}\mathrm{Im}f(x'')+O(1). \tag{17}$$

Equations (16) and (17) show that the asymptotic behavior of $\mathrm{Re}g(x)$ is, in general, given by a convolution integral. They have the interesting features that the remaining terms are $O(1)$ and not $O(\ln x)$ as given by Eq. (14). Comparing Eqs. (16) and (17) with (13) and (9), we get

$$\int_0^x[I(x')+R(x')]\frac{dx'}{x'}=O(1). \tag{18}$$

[One can show that the integrals of $I(x)$ and $R(x)$ are individually bounded by constants for the case considered here.] This is a remarkable result which illustrates the usefulness of the averaging technique.

Equation (18) gives a strong condition on $I(x)+R(x)$ [assuming $I(x)+R(x)$ is continuous, which is the case if $f(x)$ is continuous]:

(i) If $I(x)+R(x)$ has a limit as $x\to+\infty$, this limit is zero.

(ii) If it does not have a limit, but does not change sign for x large enough, $I(x)+R(x)$ still tends to zero except on a set of points of zero asymptotic density.

(iii) If $I(x)+R(x)$ change sign indefinitely, there must be an infinite sequence $\{x_n\}$ ($x_n \to +\infty$) of points where $I(x)+R(x)=0$, and (18) shows to what extent the cancellation of oscillations of $I(x)+R(x)$ around zero occurs.

From now on we work directly with Eq. (16).

As it stands, the right-hand side of (16) does not lead to an easy physical interpretation. We shall modify it by defining the following average:

$$g_b(x)=\frac{1}{\Delta(b)}\int_{x(1-b)}^{x(1+b)}\frac{dx'}{x'}f(x')=\frac{1}{\Delta(b)}[g(x(1+b))-g(x(1-b))], \qquad (19)$$

where $0<b<1$ and $\Delta(b)=\ln[(1+b)/(1-b)]$ is a normalization factor.

$g_b(x)$ is an analytic function with the same properties as $g(x)$, but its advantage over g is that the average of $f(x)$ is taken on an interval which moves to infinity together with x.

Using Eq. (16), one gets by direct calculation

$$\begin{aligned}-\pi\Delta(b)\mathrm{Re}\,g_b(x)&=\int_{x_0}^{x(1+b)}\frac{dx'}{x'}\ln\left(\frac{x(1+b)}{x'}\right)\mathrm{Im}f(x')+\int_{x_0}^{x(1-b)}\frac{dx'}{x'}\ln\left(\frac{x(1-b)}{x'}\right)\mathrm{Im}f(x')+O(1)\\&=\Delta(b)\int_{x_0}^{x}\frac{dx'}{x'}\mathrm{Im}f(x')+\int_{x(1-b)}^{x(1+b)}\frac{dx'}{x'}\ln\frac{x}{x'}\mathrm{Im}f(x')\\&\quad+\ln|1+b|\int_{x}^{x(1+b)}\frac{dx'}{x'}\mathrm{Im}f(x')+\ln|1-b|\int_{x(1-b)}^{x}\frac{dx'}{x'}\mathrm{Im}f(x')+O(1).\end{aligned}$$

The last three integrals are seen to be bounded by the usual change of variable as above, hence we end with the result

$$\mathrm{Re}\,g_b(x)=-\frac{1}{\pi}\langle\mathrm{Im}f\rangle_x\ln x+O(1), \qquad (20)$$

where we have used the definition of the average as given by Eq. (8). This simple result is somewhat surprising, since in general one would expect $\mathrm{Re}\,g_b(x)$ to be a convolution-type integral.

Comparing Eqs. (5) and (20), we arrive at the following important result that for $|x|$ sufficiently large, the high-energy behavior of $f(x)$ as $x\to-\infty$ is the same as that of the average of $f(x)$ on the physical cut. What we have accomplished is, for the case considered here, the extension of Montel's theorem[18] onto the boundary of an analytic function and its generalization to oscillating functions (see Appendix C).

As an application of this result, Eq. (20), let us derive the Pomeranchuk theorem in its most general form, assuming $\mathrm{Re}f(x)/\ln x\to 0$ or $\mathrm{Re}f(x)=o(\ln x)$. It is easy to verify that this condition also implies $\mathrm{Re}\,g_b(x)=o(\ln x)$; hence Eq. (20) implies

$$\lim_{x\to+\infty}\langle\mathrm{Im}f\rangle_x=0 \qquad (21a)$$

or

$$\lim_{E\to+\infty}\frac{1}{\ln E}\int_{E_0}^{E}\frac{\Delta\sigma(E')}{E'}dE'=0. \qquad (21b)$$

Conversely if Eq. (21) is valid, it implies only $\mathrm{Re}\,g_b(x)=o(\ln x)$, i.e., it controls the growth of $\mathrm{Re}f(x)$ on the physical cut on the *average* only. In order to have $\mathrm{Re}f(x)=o(\ln x)$ everywhere, we must assume smoothness in the sense $J(x)=o(\ln x)$. The method presented in this section can be extended to study the restrictions due to unitarity and leads to improvements of results previously obtained.[6] It turns out that the method to be discussed in Sec. VI is more direct and more general; we hence limit ourself to the application of the results obtained in this section to the phase representation, where we derive a number of important consequences which will be needed for the proof of the Pomeranchuk theorem for unbounded total cross sections and of general theorems for the real part of the even-crossing forward amplitude.

IV. APPLICATIONS TO THE PHASE REPRESENTATION

Let us denote $f_s=f_P+f_A$ the forward even-crossing amplitude and define $x=E^2$ as above. We shall assume that the amplitude f_s is properly averaged,[19] so that $\mathrm{Im}f_s(x)$ has no zero on the physical cut $x\geq 1$, where for convenience the cut starts at $x=1$. Since $\mathrm{Im}f_s$ is positive definite, it is known that f_s has only a finite number of zeros z_i and that the following phase representation holds[8]:

$$f_s(x)=\frac{A}{x-x_P}\prod_{i=1}^{N}(x-z_i)\exp\left[\frac{x}{\pi}\int_1^{+\infty}\frac{dx'}{x'}\frac{\delta(x')}{x'-x-i\epsilon}\right], \qquad (22)$$

where x_p is the location of the pole and A is a real constant. If z_i are complex they must occur as complex conjugate pairs. The phase $\delta(x)$ is assumed to be continuous and we use the convention $\delta(1)=0$. We shall assume for convenience that $f_s(1)>0$, so that using positivity one gets the following bounds for the phase:

$$0 \leq \delta(x) \leq \pi .$$

The case $f_s(1)<0$ can similarly be treated. The exponential factor on the right-hand side of Eq. (22) is exactly the same type of integral studied previously.

Introducing

$$\mathrm{Re}h(x)=\frac{x}{\pi}\mathrm{P}\int_1^{+\infty}\frac{dx'}{x'}\frac{\delta(x')}{x'-x},\quad \mathrm{Im}\,h(x)=\delta(x), \tag{23}$$

we can write (22) as follows:

$$\ln|f_s(x)|=\nu\ln x+\mathrm{Re}h(x)+\ln|A|+o(1),$$

where ν = (number of zeros − number of poles) and x is taken positive. It is simple now to apply the results of Secs. II and III concerning $\mathrm{Re}f(x)$. Owing to the Froissart bound, ν can take only the values of zero or one.

First of all, using the consequences of (18), there exists at least an infinite sequence of neighborhoods $\{x_n\}$ $(x_n\to+\infty)$, where

$$|f_s(x)|=|A(x)|\,x^{\nu-(1/\pi)\langle\delta\rangle_x}, \tag{24}$$

with

$$\lim_{\substack{x\to+\infty\\ x\in\{x_n\}}}|A(x)|=|A|.$$

Moreover using the result of Sec. III, Eq. (20), we obtain immediately

$$[\ln|f_s|]_x=\left(\nu-\frac{1}{\pi}\langle\delta\rangle_x\right)\ln x+O(1), \tag{25}$$

where we have defined

$$[\ln|f_s|]_x=\frac{1}{\Delta(b)}\int_{x(1-b)}^{x(1+b)}\ln|f_s(x')|\frac{dx'}{x'} \tag{26a}$$

and

$$\langle\delta\rangle_x=\frac{1}{\ln x}\int_1^x\delta(x')\frac{dx'}{x'}. \tag{26b}$$

Equation (25) gives the most general result relating the average modulus on the cut and its average phase. [It should be noted that the averages on the left-hand side and the right-hand side of Eq. (25) are different.] No assumption has been made on the ratio $\mathrm{Re}f_s/\mathrm{Im}f_s$, which may be unbounded. Equation (25) gives an explicit average relation between the modulus and the phase of f_s. It shows the average behavior of $\ln|f_s|$ is controlled directly by the average $\langle\delta\rangle_x$ of the phase rather than the phase $\delta(x)$ itself. Let us consider some special cases.

(i) $\langle\delta\rangle_x$ does not have a limit, but is bounded below and above by δ_1 and δ_2:

$$0<\delta_1\leq\langle\delta\rangle_x\leq\delta_2;$$

then one gets from Eq. (24)

$$C_2x^{\nu-(1/\pi)\delta_2}<|f_s(x)|<C_1x^{\nu-(1/\pi)\delta_1} \tag{27}$$

for $x\in\{x_n\}$. This is especially useful when $\delta(x)$ has narrow high peaks which do not show up in its average $\langle\delta\rangle_x$, but will nevertheless weaken considerably the bounds given by Jin and MacDowell.[8] [See Eqs. (22) and (23) of this reference, where δ_1 and δ_2 are bounds of $\delta(x)$ instead of $\langle\delta\rangle_x$.] We thus obtain a result which represents a definite improvement over their work. If we do not want to have a result which is valid on a set of points then $|f_s(x)|$ in Eq. (27) should be replaced by its average value $\exp\{[\ln|f_s|]_x\}$.

The meaning of the nonexistence of the limit of $\langle\delta\rangle_x$ has been examined by Bessis and Kinoshita.[13] They pointed out that the short-range oscillations of $\delta(x)$ do not affect the existence of the limit. This limit does not exist only when $\delta(x)$ undergoes an extremely slow change as $x\to+\infty$ with rapidly increasing period of oscillation.

(ii) Let us now examine a more special case where $\lim_{x\to+\infty}\langle\delta\rangle_x$ exists. Bessis and Kinoshita showed that

$$\frac{\ln f_s(z)}{\ln z}\to\alpha\quad\text{for }|z|\to+\infty,\ \epsilon<\arg z\leq\pi,\ \epsilon>0 \tag{28a}$$

if and only if

$$\lim_{x\to+\infty}\langle\delta\rangle_x=\beta, \tag{28b}$$

where $\mathrm{Re}z=x$. We would like to point out that their result can be improved to show that $\beta=\pi(\nu-\alpha)$. This can be done simply by considering the analytic function $\ln f_s(z)/\ln z$, using Montel's theorem and the method of Truong and Lam in their proof of the Pomeranchuk theorem for $\Delta\sigma$. (See also Appendix C.)

The result given by Eq. (25) enables us to generalize this result on to the cut. In this case existence of the limit $\langle\delta\rangle_x$ implies the existence of the limit of

$$\frac{[\ln|f_s|]_x}{\ln x}$$

and vice versa. In general we cannot infer the behavior of $|f_s|$ for all x from the behavior of $|f_s|$ in complex directions in the z plane. It is suspected that it might be related to some sort of average behavior on the real axis but this point was never proved. We have proved here this result, Eq. (25); the average involved is defined by Eq. (26a).

If one wants to get the behavior of (28a) at all points on the physical cut, then one must add to (28b), as discussed in Sec. II, the following condition:

$$\lim_{x\to+\infty}\frac{J(x)}{\ln x}=0\,, \tag{29a}$$

where

$$J(x)=\frac{1}{\pi}\mathrm{P}\int_{x(1-\lambda)}^{x(1+\lambda)}dx'\,\frac{\delta(x')}{x'-x}. \tag{29b}$$

A sufficient condition for (29a) to be valid is given by Eq. (12), namely,

$$x\frac{d}{dx}\delta(x)=O(1)\,. \tag{29c}$$

In the more special case, i.e., $\lim_{x\to+\infty}\delta(x)$ exists, and with condition $J(x)\to 0$ we recover the results of Sugawara and Tubis,[7] Jin and MacDowell,[8] and Gervais.[9] The meaning of the existence of a limit for $\delta(x)$ was previously discussed by Gervais who showed that it implies that the amplitude has a nonoscillating behavior in the sense defined by Gervais and Yndurain[14]; one still needs the condition $J(x)\to 0$ to get this behavior at all points on the real axis.

Condition (28b) yields the following result on the real axis:

$$|f_s(x)|=x^\alpha\varphi(x)\,, \tag{30}$$

where $\varphi(x)$ is a "slowly varying function" in the sense that

$$\lim_{x\to+\infty}\frac{[\ln\varphi]_x}{\ln x}=0\,. \tag{31}$$

(It should be noted that a slowly varying function in our definition can be a rapidly oscillating function.)

(iii) Regarding some properties of slowly varying functions, we see that the nature of the slowly varying function is related to the behavior of the integral

$$\int_1^x\frac{dx'}{x'}\Delta(x')\,, \tag{32a}$$

where

$$\begin{aligned}\Delta(x)&=\delta(x)-\pi(\nu-\alpha)\\&=\delta(x)-\beta\,.\end{aligned} \tag{32b}$$

It follows from Eq. (25) that the average of $\ln\varphi$ satisfies

$$[\ln\varphi]_x=-\frac{1}{\pi}\int_1^x\frac{dx'}{x'}\Delta(x')+O(1)\,. \tag{33}$$

One immediately sees from (33) that

$$\begin{aligned}&\lim_{x\to+\infty}\varphi(x)=+\infty \text{ implies } \lim_{x\to+\infty}\int_1^x\frac{dx'}{x'}\Delta(x')=-\infty\,,\\&\lim_{x\to+\infty}\varphi(x)=0 \text{ implies } \lim_{x\to+\infty}\int_1^x\frac{dx'}{x'}\Delta(x')=+\infty\,.\end{aligned} \tag{34}$$

If, on the other hand, φ is bounded from above and below by constants then $\int_1^x(dx'/x')\Delta(x')$ must be bounded.

As an example, let us consider the case $\alpha=\frac{1}{2}$, then $\varphi=|f_s/E|$ is essentially the total cross section and $\beta=\frac{1}{2}\pi$. In this case the total cross section will be bounded from above and below if $\int_1^x[\delta(x')-\frac{1}{2}\pi]dx'/x'$ is bounded. If we assume moreover that $\mathrm{Re}f_s$ does not change sign above some energy, then this implies essentially that $|\mathrm{Re}f_s/\mathrm{Im}f_s|$ goes to zero faster than $1/\ln x$. This conclusion is no longer valid if $\mathrm{Re}f_s$ is allowed to change sign indefinitely. As an example, we can let $|f_s/E|$ be bounded and

$$\delta(x)=\tfrac{1}{2}\pi+\sin x$$

or

$$\delta(x)=\tfrac{1}{2}\pi+\sin\ln x\,.$$

Similarly if one assumes that $C_1\ln^\alpha x\leq\varphi(x)\leq C_2\ln^\beta x$ with $0<\alpha<\beta$, then

$$\alpha\ln\ln x<-\frac{1}{\pi}\int_1^x\frac{dx'}{x'}\Delta(x')<\beta\ln\ln x\,.$$

Again if $\mathrm{Re}f_s$ does not change sign, this implies essentially that $(\ln x)|\mathrm{Re}f_s/\mathrm{Im}f_s|$ is bounded above and below by constants. Similar results can be obtained when α and β are negative.

We note finally that if $\int_1^x(dx'/x')\Delta(x')$ is negative, it cannot diverge faster than $\ln\ln x$ because of the Froissart bound. It can diverge faster than $\ln\ln x$ with positive sign corresponding to a total cross section decreasing faster than any negative power of logarithm of energy. These results are in agreement with the analyses of Jin and MacDowell[8] and of Khuri and Kinoshita.[10]

We now return to the further consequences of Eq. (25).

(i) Regarding the Jin-Martin lower bound, we see that the most negative value of the right-hand side of (25) can be obtained by taking $\nu=0$ and $\langle\delta\rangle=\pi$, where one gets

$$[\ln|f_s|]_x\geq\ln\frac{\mathrm{const}}{x}. \tag{35}$$

From (35), it follows that there must exist a sequence of intervals $\{x_n\}$, $x_{n\to+\infty}$, on which

$$|f_s(x)| \geq \frac{\text{const}}{x},$$

which is the Jin-Martin lower bound.[12] Moreover, if $|f_s|$ oscillates down to arbitrary small values, (35) gives the density of the set of points where this bound can be violated. This lower bound was obtained previously by Sugawara[20] using the phase representation under the stronger assumption that f_s has a definite phase at infinity.

Further consequences of (25) can be obtained according to the number of zeros of f_s and the sign of $\text{Re}f_s$.

(ii) If f_s has only one zero then $\nu = 0$ (we have assumed there is only one pole), and (25) shows that the total cross section σ_s is bounded from above in average by $1/E$ (the bound being saturated when $\langle\delta\rangle = 0$).

(iii) If $\nu = 1$, (25) can be written as

$$\left[\ln\left|\frac{f_s}{E}\right|\right]_x = \frac{1}{\pi}(\tfrac{1}{2}\pi - \langle\delta\rangle_x)\ln x + O(1) \tag{36a}$$

$$= \frac{1}{\pi}\int_1^x \frac{dx'}{x'}[\tfrac{1}{2}\pi - \delta(x')] + O(1). \tag{36b}$$

It is clear from (36) that $|f_s/E|$ is bounded from below in average by $1/E$. One sees also that $|f_s/E|$ (i.e., essentially σ_s) will be bounded from above and below in average by finite constants if $\int_1^x (dx'/x')[(\frac{1}{2}\pi) - \delta(x')]$ is bounded.

The possibility of $|f_s/E|$ becoming arbitrarily small or large depends on the sign of $\frac{1}{2}\pi - \langle\delta\rangle_x$.

(a) If $\langle\delta\rangle_x \geq \frac{1}{2}\pi$, the coefficient of $\ln x$ on the right-hand side of (36) cannot become positive, hence $|f_s/E|$ is bounded from above in average by a finite constant. This case is realized in particular when $\text{Re}f_s$ stays negative for large x.

(b) If, on the other hand, $\langle\delta\rangle_x \leq \frac{1}{2}\pi$, $|f_s/E|$ is bounded from below in average by a finite constant, hence it cannot go smoothly to zero. It is interesting to note that consistency with the Froissart bound requires in case (b) that $\lim_{x\to+\infty}\langle\delta\rangle_x = \frac{1}{2}\pi$ as shown below, and therefore that $|f_s/E|$ is a "slowly varying function."

To see this, let us use the Froissart bound in the form

$$\ln\left|\frac{f_s}{E}\right| \leq 2\ln\ln x. \tag{37}$$

It is easy to check that the bound (37) must also hold for the average on the left-hand side of (36). This bound is not compatible with the right-hand side of (36) when $\langle\delta\rangle_x \leq \frac{1}{2}\pi$ unless $\lim_{x\to+\infty}\langle\delta\rangle_x = \frac{1}{2}\pi$. More precisely, one must have in this case

$$0 \leq \frac{1}{\pi}\int_1^x \frac{dx'}{x'}[\tfrac{1}{2}\pi - \delta(x')] \leq 2\ln\ln x. \tag{38}$$

Equation (38) gives a strong condition on $\delta(x)$ if one assumes moreover that $\text{Re}f_s$ stays positive for large x. Then (38) implies

$$0 \leq \tfrac{1}{2}\pi - \delta(x) \leq \frac{2\pi}{\ln x}$$

or (39)

$$0 \leq \frac{\text{Re}f_s}{\text{Im}f_s} \leq \frac{2\pi}{\ln x},$$

except on a set of points of asymptotic zero density.

Let us note finally that even in the most general case when $\langle\delta\rangle_x$ is allowed to oscillate around $\frac{1}{2}\pi$, the amplitude of the oscillations must become arbitrarily small below $\frac{1}{2}\pi$ to be consistent with the Froissart bound.

This analysis generalizes a similar one by Jin and MacDowell[8] who treated the case where $\text{Re}f_s$ does not change sign for large x. The connection between the sign of $\text{Re}f_s$ and the behavior of the total cross section will be considered in more detail by a more direct method in Sec. V.

Finally we mention that the methods of this section can be applied to the unsymmetric forward amplitude,[19] for which a phase representation also holds, as well as to the antisymmetric amplitude f_a, if one assumes for instance that $\Delta\sigma$ does not change sign for E large enough. In this case the phase δ_a of f_a is bounded and the number of zeros of f_a is finite; one can write a phase representation for $f = f_a/E$ which is identical to that of f_s:

$$[\ln|f_a|]_x = \left(\nu_a + \tfrac{1}{2} - \frac{1}{\pi}\langle\delta_a\rangle_x\right)\ln x + O(1), \tag{40}$$

where the $\frac{1}{2}$ factor on the right-hand side provides the correct signatured phase for an antisymmetric amplitude and ν_a is the number of zeros of f_a/E.

V. APPLICATIONS TO THE UNIVALENT FUNCTIONS $G(E)$ OF KHURI AND KINOSHITA

Information on the high-energy behavior of the forward scattering amplitude can also be obtained by studying suitable averages of the amplitude such as those introduced by Khuri and Kinoshita.[11] These averages were constructed to be univalent functions which allow the deduction of various inequalities between their phase and their modulus. We would like to show that some of their results which are useful to us later can be obtained very simply and made more precise by the methods developed in Secs. II–IV.

A. Properties of the functions $G(E)$

Following Khuri and Kinoshita we shall denote throughout this section $f_s(E)$ as

$f_s(E)=f_P(E)+f_A(E)-$ (pole terms),

where f_P, f_A are, respectively, particle and antiparticle forward amplitudes. Because of the Froissart bound we have

$$f_s(E)-f_s(0)=\frac{2E^2}{\pi}\int_\mu^{+\infty}\frac{dE'}{E}\frac{\text{Im}f_s(E')}{E'^2-E^2-i\epsilon}, \qquad (41)$$

where $\text{Im}f_s(E)=(q/4\pi)(\sigma_P+\sigma_A)$ by the optical theorem. Let us consider the following average:

$$G(E)=\int_0^E\frac{dE'}{E'}\frac{f_s(E')-f_s(0)}{E'}, \qquad (42)$$

which is an analytic function in the upper half plane, odd under crossing, and univalent. A direct calculation leads to the following formulas:

$$\text{Re}G(E)=\frac{1}{\pi}\int_\mu^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|\frac{\text{Im}f_s(E')}{E'}, \qquad (43a)$$

$$\text{Im}G(E)=\int_0^E\frac{dE'}{E'}\frac{\text{Im}f_s(E')}{E'}, \qquad (43b)$$

where

$$\text{Im}f_s(E)=0 \text{ for } |E|<\mu .$$

The essential property which allows us to establish the univalence of $G(E)$ and which is also basic in the method discussed below is that $\text{Im}G(E)$ is a positive and monotonically increasing function of E on the positive real axis; the results of Sec. IV will be applied to the function $G(E)$, then the monotonicity of $\text{Im}G(E)$ will be used to get bounds on $G(E)$ at every point.

It is easy to verify from Eq. (42) that $G(E)$ has no zero on the real axis except at $E=0$; in particular $\text{Re}G(E)$ is positive for $E>0$ since $\ln|(E'+E)/(E'-E)|>0$. From Eqs. (41) and (42) it is also clear that $\text{Im}G(E)$ is positive and definite in the upper half E plane so that $G(E)$ has no complex zero. These properties are sufficient to establish the validity of the phase representation for the function $G(E)/E$, which has no zero and is even under crossing so that one can write

$$\frac{1}{\Delta(b)}\int_{x(1-b)}^{x(1+b)}\frac{dx'}{x'}\ln|G|=\left(\tfrac{1}{2}-\frac{1}{\pi}\langle\delta_G\rangle_x\right)\ln x+O(1), \qquad (44)$$

where δ_G is the phase of G. Moreover since $\text{Re}G(E)\geq 0$ and $\text{Im}G(E)>0$ on the positive E axis, one gets the following bounds for the phase:

$$0\leq\delta_G(E)\leq\tfrac{1}{2}\pi . \qquad (45)$$

Let us now examine the following consequences.

1. *Froissart bound*

The upper bound of $|f_s(E)|$ as $E\to+\infty$ is given by

$$|f_s(E)|<CE\ln^2E . \qquad (46)$$

This leads to the following bound for $|G(E)|$:

$$|G(E)|<C\ln^3E ,$$

and hence in terms of the variable $x=E^2$,

$$\ln|G(x)|<3\ln\ln x , \qquad (47)$$

where we have neglected a constant term on the right-hand side of (47). We now wish to explore the consequences of the upper bound (47) on the ratio of $\text{Re}G/\text{Im}G$. The proof can proceed the same way as in Sec. IV. First of all let us note that due to the bounds (45) and (47) together with Eq. (44) we must have

$$\lim_{x\to+\infty}\langle\delta_G\rangle_x=\tfrac{1}{2}\pi . \qquad (48)$$

More precisely Eq. (44) enables us to get some information on the way $\delta_G(x)$ approaches $\frac{1}{2}\pi$ and hence on the ratio $\text{Re}G/\text{Im}G$. For this purpose let us put $\Delta_G(x)=\frac{1}{2}\pi-\delta_G(x)$. We can rewrite Eq. (44) as

$$\frac{1}{\Delta(b)}\int_{x(1-b)}^{x(1+b)}\frac{dx'}{x'}\ln|G|=\frac{1}{\pi}\int_\mu^x\frac{dx'}{x'}\Delta_G(x')+O(1). \qquad (49)$$

Hence inequality (47) implies

$$0\leq\frac{1}{\pi}\int_\mu^x\frac{dx'}{x'}\Delta_G(x')\leq 3\ln\ln x . \qquad (50)$$

Since $\Delta_G(x)\geq 0$, there must exist at least an infinite sequence $\{x_n\}$, $x_n\to+\infty$ for which

$$0\leq\frac{\text{Re}G(x)}{\text{Im}G(x)}\leq\frac{3\pi}{\ln x} . \qquad (51)$$

The density of the set of points where (51) may be violated is given by (50) and hence this set has an asymptotic zero density. This result improves that given by Khuri and Kinoshita whose method could lead neither to an evaluation of the constant involved in the upper bound of $\text{Re}G/\text{Im}G$ nor to Eq. (50). Let us examine some more special cases.

2. *Unbounded total cross section*

We define unbounded total cross section as

$$\lim_{E\to+\infty}\langle\sigma_s\rangle_E=+\infty ,$$

where

$$\langle\sigma_s\rangle_E\equiv\frac{1}{\ln E}\int_\mu^E\frac{dE'}{E'}\sigma_s(E') \qquad (52)$$

(we define σ_s as the sum of particle-antiparticle total cross sections). $\langle\sigma_s\rangle_E$ is proportional to $\text{Im}G(E)/\ln E$. It is straightforward to show in this case $\ln|G|>\ln\ln x$. Using the same method as above we have the following bounds:

$$\frac{\pi}{\ln x} < \frac{\operatorname{Re}G(x)}{\operatorname{Im}G(x)} \leq \frac{3\pi}{\ln x} \tag{53a}$$

or in terms of variable $E = \sqrt{x}$,

$$\frac{\pi}{2\ln E} < \frac{\operatorname{Re}G(E)}{\operatorname{Im}G(E)} \leq \frac{3\pi}{2\ln E}, \tag{53b}$$

which is valid everywhere except on a set of points of asymptotic zero density.

3. Bounded total cross section

If $\langle \sigma_s \rangle_E$ is bounded from below and above by constants, $\operatorname{Im}G(E)$ is bounded from below and above by $\ln E$. Using Eq. (51) we have $\ln|G(E)| \simeq \ln|\operatorname{Im}G(E)|$, hence

$$\frac{\operatorname{Re}G(E)}{\operatorname{Im}G(E)} \simeq \frac{\pi}{2\ln E}. \tag{54}$$

This result is an improvement over that given by Khuri and Kinoshita who showed that $\operatorname{Re}G(E)/\operatorname{Im}G(E)$ must tend to zero not much faster or much slower than $1/\ln E$.

4. Decreasing total cross sections

It is interesting to examine the case where

$$\lim_{E\to+\infty}\int_\mu^E \frac{dE'}{E'}\,\sigma_s(E') < +\infty,$$

i.e., $\sigma_s(E)$ tends to zero faster than $1/\ln E$. In this case $\operatorname{Im}G(E)$ is bounded from below and above by constants. Using Eq. (51), we have $\ln|G(E)| \simeq \ln|\operatorname{Im}G(E)|$. It follows from Eq. (49) that

$$\lim_{E\to+\infty}\int_\mu^E \frac{dE'}{E'}\Delta_G(E') < +\infty, \tag{55}$$

which implies essentially that $\operatorname{Re}G/\operatorname{Im}G$ tends to zero faster than $1/\ln E$.

5. Converse of 3.

It is useful to examine the converse of case examined in Sec. V A 3, i.e., we wish to explore the consequences of

$$C \leq \operatorname{Re}G(E) \leq C' \text{ for } E > E_1, \tag{56}$$

where C and C' are positive constants.

Since $G(E)$ is odd under crossing, it satisfies the following "inverse" dispersion relation:

$$\operatorname{Im}G(E) = -\frac{2E^2}{\pi}\,\mathrm{P}\int_0^{+\infty}\frac{dE'}{E'}\frac{\operatorname{Re}G(E')}{E'^2 - E^2}. \tag{57}$$

Moreover since $\operatorname{Re}G(E)$ is assumed to be bounded, the methods of Sec. III can be applied to $G(E)$ to give us

$$\frac{1}{\Delta(b)}\int_{x(1-b)}^{x(1+b)}\frac{dx'}{x'}\operatorname{Im}G(x') = \frac{1}{\pi}\int_{x_1}^{x}\frac{dx'}{x'}\operatorname{Re}G(x') + O(1). \tag{58}$$

From Eqs. (56) and (58) it follows that

$$\frac{C}{\pi}\ln\frac{x}{x_1} + \text{const} \leq \frac{1}{\Delta(b)}\int_{x(1-b)}^{x(1+b)}\frac{dx'}{x'}\operatorname{Im}G(x') \leq \frac{C'}{\pi}\ln\frac{x}{x_1} + \text{const}. \tag{59}$$

Finally using the monotonicity of $\operatorname{Im}G$ we get

$$\frac{C}{\pi}\ln\frac{x}{x_1} + \text{const} \leq \operatorname{Im}G(x) \leq \frac{C'}{\pi}\ln\frac{x}{x_1} + \text{const}, \tag{60}$$

which implies that $\langle \sigma_s \rangle_E$ is bounded below and above by constants.

B. Physical applications

The previous results allow us to relate the behavior of the integral $\int^E (dE'/E')[\operatorname{Re}f_s(E')/E']$ to that of the average $\langle \sigma_s \rangle$ defined in (52). They are therefore particularly useful to discuss the connection between the sign and rate of growth of $\operatorname{Re}f_s$ and the behavior of the total cross section at high energies. We shall consider successively the cases of unbounded and bounded cross sections.

1. Unbounded total cross section

Let us first assume that

$$\lim_{E\to+\infty}\int^E \frac{dE'}{E'}\frac{\operatorname{Re}f_s}{E'} = +\infty. \tag{61}$$

Condition (61) is equivalent to $\lim_{E\to+\infty}\operatorname{Re}G(E) = +\infty$. It then follows from (51) that $\lim_{E\to+\infty}\langle \sigma_s \rangle_E = +\infty$ except on a set of asymptotic zero density.

Conversely if $\lim_{E\to+\infty}\langle \sigma_s \rangle_E = +\infty$, (53) shows that $\lim_{E\to+\infty}\operatorname{Re}G(E) = +\infty$ on almost all sequences, so that (61) follows.

Note that condition (61) implies that $\operatorname{Re}f_s$ takes large positive values; if one assumes that $\operatorname{Re}f_s$ does not change sign for E large enough, it means essentially that $\operatorname{Re}f_s$ is positive and larger than $E/\ln E$. On the other hand, it can be shown as in Sec. VI A that the Froissart bound gives the upper bound

$$\int^E \frac{dE'}{E'}\frac{\operatorname{Re}f_s}{E'} \leq \frac{\pi}{4\mu^2}\ln^2 E.$$

Hence $\operatorname{Re}f_s$ is essentially bounded from above by $E\ln E$ if it stays positive at high energy.

2. Bounded total cross section

Assume now that

$$\left|\int^E \frac{dE'}{E'}\frac{\operatorname{Re}f_s}{E'}\right| = O(1), \tag{62}$$

so that $\mathrm{Re}G(E)$ is bounded from above. Equation (60) then shows that $\langle\sigma_s\rangle_E$ is also bounded.

Conversely if $\langle\sigma_s\rangle_E = O(1)$, (51) shows that condition (62) holds on almost all sequences.

It is interesting to note that if one makes the slightly stronger assumption $\sigma_s <$ const, one can prove condition (62) directly from (43a), without using the previous theorem. By replacing $\mathrm{Im}f_s(E')/E'$ by its upper bound C, one gets

$$\left|\int_0^E \frac{dE'}{E'}\,\frac{\mathrm{Re}f_s(E')-f_s(0)}{E'}\right| < \frac{C}{\pi}\int_0^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right| = \tfrac{1}{2}C\pi\,,$$

where in the last step we have used the identity

$$\int_0^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right| = \int_0^{+\infty}\frac{dy'}{y'}\ln\left|\frac{y'+1}{y'-1}\right| = \tfrac{1}{2}\pi^2 \qquad (63)$$

(the change of variable $y'=E'/E$ has been used).

This example shows that from a bound on σ_s one can very easily get a bound on an integral of $\mathrm{Re}f_s$; a similar method will be applied more extensively in Sec. VI to the antisymmetric amplitude to derive Pomeranchuk-like theorems. Assuming $\lim_{E\to+\infty}\sigma_s(E)=+\infty$, one could have shown in the same way that $\lim_{E\to+\infty}\int^E(dE'/E')[\mathrm{Re}f_s(E')/E'] = +\infty$ for case 1 above.

3. *Condition for the existence of* $\lim_{E\to+\infty}\langle\sigma_s\rangle_E$

If one assumes that the integral $\int^E(dE'/E') \times[\mathrm{Re}f_s(E')/E']$ converges to a finite limit so that $\lim_{E\to+\infty}\mathrm{Re}G(E)=C$ (C may take the value zero), then (60) shows that $\lim_{E\to+\infty}\langle\sigma_s\rangle_E = (2C/\pi)4\pi = 8C$.

Conversely it is shown below from (43a) that $\lim_{E\to+\infty}\sigma_s(E)=\sigma_\infty$ implies

$$\lim_{E\to+\infty}\mathrm{Re}G(E) = (\tfrac{1}{2}\pi)\sigma_\infty\frac{1}{4\pi} = \tfrac{1}{8}\,\sigma_\infty\,.$$

If $\mathrm{Re}f_s$ does not change sign for E large enough, the convergence of $\int^E(dE'/E')[\mathrm{Re}f_s(E')/E']$ essentially means that $|\mathrm{Re}f_s| \ll E/\ln E$.

5. *Converse of 3 and 4*

The case where $\mathrm{Re}f_s<0$ for sufficiently large E is of particular interest, since then one can prove the convergence of $\int^E(dE'/E')[\mathrm{Re}f_s(E')/E']$ using positivity. This follows from the fact that this integral is necessarily bounded from below if $\mathrm{Re}f_s<0$ for E large enough, otherwise the condition $\mathrm{Re}G(E)\geq 0$ is violated; since it is also monotonically decreasing in this case, it must converge to a finite limit; as we saw in Sec. VB3, this implies that $\lim_{E\to+\infty}\langle\sigma_s\rangle_E$ exists. We can thus state the following result, which sharpens a previous one of Khuri and Kinoshita[11]: If $\mathrm{Re}f_s$ stays negative for E large enough, $\lim_{E\to+\infty}\langle\sigma_s\rangle_E$ exists and is a finite number (zero is not excluded).

Finally a partial converse to the previous results is provided by the following theorem, which is a slight generalization of a result of Jin and MacDowell[8] [we do not assume that $\sigma_s(E)$ is monotonic].

4. *Consequence of a negative* $\mathrm{Re}f_s$

If $\sigma_s(E)$ tends from above (or from below) to σ_∞ but not as fast as $1/E$ (in the sense that $|\sigma_s(E)-\sigma_\infty| >$ const$/E$ for $E>E_1$) then the integral $\int^E(dE'/E') \times[\mathrm{Re}f_s(E')/E']$ converges as $E\to\infty$ and $|\mathrm{Re}f_s(E)|$ is unbounded. Moreover $\int_E^{+\infty}(dE'/E')[\mathrm{Re}f_s(E')/E']$ is negative if $\sigma_s(E)$ reaches its limit from above, positive in the opposite case.

Proof. Let us show $\lim_{E\to+\infty}\sigma_s(E)=\sigma_\infty$ implies that the integral $\int^E(dE'/E')[\mathrm{Re}f_s(E')/E']$ converges and that

$$\int_E^{+\infty}\frac{dE'}{E'}\,\frac{\mathrm{Re}f_s(E')-f_s(0)}{E'} = -\frac{1}{4\pi^2}\int_0^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|[\sigma_s(E')-\sigma_\infty]\,. \qquad (64)$$

We first prove that the right-hand side of (64) tends to zero when $E\to+\infty$. One can split this integral into a low-energy and a high-energy part as follows:

$$\int_0^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|[\sigma_s(E')-\sigma_\infty] = \int_0^{E_1}\frac{dE'}{E'}\ln\left|\frac{E'+E'}{E'-E}\right|[\sigma_s(E')-\sigma_\infty] + \int_{E_1}^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|[\sigma_s(E')-\sigma_\infty]\,. \qquad (65)$$

The low-energy part is $O(1/E)$ since it has a compact support. The high-energy part is bounded by

$$\sup_{E>E_1}|\sigma_s(E)-\sigma_\infty|\int_0^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right| = \tfrac{1}{2}\pi^2\sup_{E>E_1}|\sigma_s(E)-\sigma_\infty|$$

and can be made arbitrarily small for sufficiently large E_1, which proves our statement about the right-hand side of Eq. (64).

Now one can write

$$\frac{1}{4\pi^2}\int_0^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|[\sigma_s(E')-\sigma_\infty]=\frac{1}{4\pi^2}\int_0^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|\sigma_s(E')-\frac{1}{4\pi^2}\sigma_\infty\int_0^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|$$

$$=\int_0^E\frac{dE'}{E'}\frac{\mathrm{Re}f_s(E')-f_s(0)}{E'}-\tfrac{1}{8}\sigma_\infty\,, \tag{66}$$

where in the last step we have used (43a) and (63). We have thus proved

$$\lim_{E\to+\infty}\int_0^E\frac{dE'}{E'}\frac{\mathrm{Re}f_s(E')-f_s(0)}{E'}\equiv\int_0^{+\infty}\frac{dE'}{E'}\frac{\mathrm{Re}f_s(E')-f_s(0)}{E'}=\tfrac{1}{8}\sigma_\infty\,. \tag{67}$$

Equation (64) follows from (66) and (67). Let us now consider the case where $\sigma_s(E)-\sigma_\infty$ stays positive and larger than $(1/E^{1-\epsilon})$ $(0<\epsilon<1)$ for $E>E_1$. We have already seen that the low-energy part on the right-hand side of (65) is $O(1/E)$. The high-energy part tends to zero slower than $1/E$, since one can write

$$\int_{E_1}^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|[\sigma_s(E')-\sigma_\infty]$$

$$>\int_{E_1}^{+\infty}\frac{dE'}{E'}\ln\left|\frac{E'+E}{E'-E}\right|\frac{1}{E'^{1-\epsilon}}$$

$$=\frac{1}{E^{1-\epsilon}}\int_{E_1/E}^{+\infty}\frac{dy'}{y'}\ln\left|\frac{y'+1}{y'-1}\right|\frac{1}{y'^{1-\epsilon}}$$

$$\sim\frac{\mathrm{const}}{E^{1-\epsilon}}$$

(the change of variable $y'=E'/E$ has been used). Note that the last integral converges at $y'=0$. (If $\epsilon=0$ the high-energy part behaves as $\ln E/E$, hence the theorem is also true for $\epsilon=0$.)

It follows that in the case considered here the asymptotic behavior of the right-hand side of (65) is controlled by the high-energy part, which has the same sign as $\sigma_s(E)-\sigma_\infty$. In particular using (64) one sees that $\int_E^{+\infty}(dE'/E')[\mathrm{Re}f_s(E')-f_s(0)]/E'$ is negative and tends to zero slower than $1/E$. Since the subtraction constant $f_s(0)$ gives a contribution of order $1/E$, it can be neglected. We therefore end with the result that $\int_E^{+\infty}(dE'/E')\times[\mathrm{Re}f_s(E')/E']$ becomes negative for E sufficiently large, and tends to zero slower than $1/E$ (which implies that $\mathrm{Re}f_s(E)$ is unbounded).

The case where $\sigma_s(E)$ reaches its limit from below can be similarly treated; in this case $\int_E^{+\infty}(dE'/E')[\mathrm{Re}f_s(E')/E']$ becomes positive for sufficiently large E.

VI. THEOREMS FOR THE DIFFERENCE OF PARTICLE-ANTIPARTICLE TOTAL CROSS SECTION

We have shown in Secs. II–V that the integral $\int^E(dE'/E')\mathrm{Im}f(E')$ is a useful quantity to express the asymptotic behavior of the scattering amplitude in the presence of oscillations. In this section we deal with asymptotic theorems for $\Delta\sigma$. Although the methods developed in Secs. II–V in particular Sec. III can also be used, they do not lead to the most general result when $\mathrm{Im}f(E)$ becomes unbounded. We give in this section a different method to prove directly theorems related to $\Delta\sigma$ (and similarly theorems related to $\mathrm{Re}f_s$, the real part of the even-crossing forward amplitude). The method presented below is rigorously valid even when $\mathrm{Im}f$ is unbounded.[21]

Let us denote $f_a=f_P-f_A$, the difference of the forward particle-antiparticle amplitudes. It is odd under crossing. The optical theorem is $\mathrm{Im}f_a=(q/4\pi)\Delta\sigma$. Instead of writing an ordinary dispersion relation which relates the real part of the forward amplitude to its imaginary part, we write the inverse dispersion relation relating the imaginary part to its real part. Using the odd-crossing property of $f_a(E)$, we have

$$\mathrm{Im}f_a(E)=-\frac{2E^2}{\pi}\,\mathrm{P}\int_0^{+\infty}\frac{dE'}{E'}\frac{\mathrm{Re}f_a(E')-bE'}{E'^2-E^2}\,, \tag{68}$$

where b is a subtraction constant at $E=0$. This is to be compared with the ordinary dispersion relation

$$\mathrm{Re}f_a(E)=bE+\frac{2E^3}{\pi}\,\mathrm{P}\int_0^{+\infty}\frac{dE'}{E'^2}\frac{\mathrm{Im}f_a(E')}{E'^2-E^2}\,.$$

Let us integrate Eq. (68) between 0 and E:

$$\int_0^E\mathrm{Im}f_a(E')\frac{dE'}{E'^2}=-\frac{1}{\pi}\int_0^{+\infty}\ln\left|\frac{E'+E}{E'-E}\right|\mathrm{Re}f_a(E')\frac{dE'}{E'^2}\,, \tag{69}$$

where we have interchanged order of integrations which is allowed,[17] and neglected a constant $b\pi/2$ on the right-hand side of Eq. (69). Since $\mathrm{Im}f_a(E)=0$ for $|E|<\mu$, the threshold, the left-hand side of (69) converges at $E'=0$. This equation is basic to derive asymptotic theorems below. Although we have assumed the existence of the dispersion relation, the technique developed here is applicable to the situation where there is a finite region of nonanalyticity.

A. Upper bound for $\Delta\sigma$

From a simple example $f_a(E) \sim E(\ln E - \frac{1}{2}i\pi)^\beta$, we have $\beta \leq 2$ because of the Froissart-Martin bound and hence $\Delta\sigma$ is bounded by $\ln E$. The problem is then to generalize this result. Roy and Singh[22] showed that if $\Delta\sigma$ is a monotonic function of E for sufficiently large E, then $\Delta\sigma$ is bounded by $\ln E$. A weaker result but without assumption on the high-energy behavior of $\Delta\sigma$ was discussed by Cornille.[23] We give here the most general result for the upper bound of the average of $\Delta\sigma$ without making any assumption.

It is known that $|f_{P,A}|$ obey the following upper bound:

$$|f_{P,A}| \leq CE\ln^2 E\,, \tag{70a}$$

where $C = 1/(4\mu^2)$ and μ is the pion mass. It is obvious that

$$|\mathrm{Re}\, f_a(E)| \leq |f_P| + |f_A| \leq 2CE\ln^2 E\,, \tag{70b}$$

which holds for sufficiently large $E > E_1$. Using this in Eq. (69) we get

$$\left|\int_0^E \mathrm{Im} f_a(E')\frac{dE'}{E'^2}\right| \leq \frac{1}{\pi}\int_0^{E_1}\ln\left|\frac{E'+E}{E'-E}\right| |\mathrm{Re}\, f_a(E')|\frac{dE'}{E'^2} + \frac{2C}{\pi}\int_{E_1}^{+\infty}\ln\left|\frac{E'+E}{E'-E}\right|\ln^2 E'\frac{dE'}{E'}\,. \tag{70c}$$

The first integral on the right-hand side has a compact support and behaves as $1/E$ as $E \to +\infty$. After changing the variable of integration to $x = E'/E$, we see that the second integral I_2 yields

$$I_2 = \frac{2C}{\pi}\int_{E_1/E}^{+\infty}\ln\left|\frac{1+x}{1-x}\right|[\ln(Ex)]^2\frac{dx}{x}\,,$$

which yields the leading term in I_2 as (the other terms behave as $\ln E$ and constant)

$$I_2 \simeq 2C\ln^2 E\,\frac{1}{\pi}\int_{E_1/E}^{+\infty}\ln\left|\frac{1+x}{1-x}\right|\frac{dx}{x}$$
$$\leq 2C\ln^2 E\,\frac{1}{\pi}\int_0^{+\infty}\left|\frac{1+x}{1-x}\right|\frac{dx}{x}$$

and hence finally

$$I_2 \leq \pi C\ln^2 E\,.$$

Let us now define

$$\langle\Delta\sigma\rangle_E = \frac{1}{\ln E}\int_0^E\frac{\Delta\sigma(E)}{E'}\,dE'\,. \tag{71}$$

Putting the above results together into Eq. (70c) and using (71) we have finally

$$|\langle\Delta\sigma\rangle_E| \leq \frac{\pi^2}{\mu^2}\ln E\,. \tag{72}$$

This result holds for any high-energy behavior of $\Delta\sigma$.

B. Pomeranchuk theorem for bounded total cross sections

It is known that the Pomeranchuk theorem for bounded total cross sections cannot be proved from unitarity. We give here the proof of this theorem for $\Delta\sigma$ under the usual assumption that $|f_{P,A}|$ are bounded and hence $|\mathrm{Re}\, f_a|$ is bounded. Restrictions due to unitarity for bounded total cross sections will be given later. Let us impose the condition $|\mathrm{Re}\, f_a(E)| < M$ for $E > E_1$ in Eq. (69) and use again the change of variable. We get immediately

$$\left|\int_0^E \Delta\sigma(E')\frac{dE'}{E'}\right| < \frac{M'}{\pi}\int_{E_1/E}^{+\infty}\ln\left|\frac{1+x}{1-x}\right|\frac{dx}{x}\,, \tag{73}$$

where we have neglected an integral from 0 to E as above. From Eq. (73) it follows that the set of limiting values of the left-hand side of (73) is bounded, or

$$\lim_{E\to+\infty}\left|\int_0^E \Delta\sigma(E')\frac{dE'}{E'}\right| < \tfrac{1}{2}\pi M'\,, \tag{74a}$$

and if $\Delta\sigma$ does not change sign, then the integral converges:

$$\left|\int_0^{+\infty}\Delta\sigma(E')\frac{dE'}{E'}\right| < \tfrac{1}{2}\pi M' \tag{74b}$$

Equation (74b) shows that if $\Delta\sigma$ does not change sign, it must go to zero faster than $1/\ln E$. If we make the weaker assumption $(\mathrm{Re}\, f_{P,A})/(\mathrm{Im}\, f_{P,A}\ln E) \to 0$ and hence $(\mathrm{Re}\, f_a)/(E\ln E) \to 0$, it is straightforward to show one has in this case

$$\lim_{E\to+\infty}\langle\Delta\sigma\rangle_E = 0\,. \tag{75}$$

C. Consequences of unitarity on $\Delta\sigma$ and proof of the Pomeranchuk theorem for unbounded total cross sections

We shall now deal with consequences of unitarity on $\Delta\sigma$ in general and prove in particular the Pomeranchuk theorem for unbounded total cross sections.

It is known that unitarity restricts the growth of the modulus of $|f_P|$ and $|f_A|$ for sufficiently large E:[22, 24-26]

$$|f_{P,A}| \leq \frac{1}{4\sqrt{\pi}\,\mu}E\ln E(\sigma_{\mathrm{el}\,P,A})^{1/2}\,, \tag{76a}$$

where $\sigma_{\mathrm{el}\,P,A}$ refer, respectively, to the particle and antiparticle elastic cross sections. Using σ_{tot}

$\geqslant \sigma_{el}$, we also have

$$|f_{P,A}| \leqslant \frac{1}{4\sqrt{\pi}\,\mu} E \ln E (\sigma_{\text{tot}\,P,A})^{1/2}. \tag{76b}$$

Since the following analysis applies equally well for σ_{tot} or σ_{el}, we shall drop the subscript total and elastic.

Using inequality (76b) together with the assumption $\sigma_P \sim C_P(\ln E)^m$, $\sigma_A \sim C_A(\ln E)^m$ with $m > 0$, Eden and Kinoshita[25,26] showed that $C_P = C_A$. Our purpose is to generalize this result without making any assumption on the functional dependence of σ_P and σ_A.

For this purpose, let us note that for sufficiently large E, inequalitites (76a) and (76b) also imply

$$|f_a(E)| \leqslant \frac{1}{4\sqrt{\pi}\,\mu} E \ln E[(\sigma_P(E))^{1/2} + (\sigma_P(E))^{1/2}]. \tag{77}$$

Using this inequality in (69) and following the above analysis, we arrive at

$$\left| \int_0^E \operatorname{Im} f_a(E') \frac{dE'}{E'^2} \right| < \frac{1}{4\mu\pi^{3/2}} \int_0^{+\infty} \ln\left|\frac{E'+E}{E'-E}\right| \ln E' [(\sigma_P(E'))^{1/2} + (\sigma_A(E'))^{1/2}] \frac{dE'}{E'}. \tag{78}$$

We want to express the right-hand side of (78) in terms of quantities which are easier to handle. For this purpose let us construct the following function $H(E)$ which is the analytic in the upper half plane:

$$H(E) = \frac{2E}{\pi} \int_0^E \frac{\ln E'[(\sigma_P(E'))^{1/2} + (\sigma_A(E'))^{1/2}]}{E'^2 - E^2 - i\epsilon} dE'. \tag{79}$$

Let us construct the following univalent function $\tilde{G}(E)$ which is also analytic in the upper half plane:

$$\tilde{G}(E) = \int_0^E H(E') \frac{dE'}{E'}, \tag{80a}$$

$$\operatorname{Im}\tilde{G}(E) = \int_0^E (\ln E')[(\sigma_P(E'))^{1/2} + (\sigma_A(E'))^{1/2}] \frac{dE'}{E'}, \tag{80b}$$

$$\operatorname{Re}\tilde{G}(E) = \frac{1}{\pi} \int_0^{+\infty} (\ln E') \ln\left|\frac{E'+E}{E'-E}\right| \times [(\sigma_P(E'))^{1/2} + (\sigma_A(E'))^{1/2}] \frac{dE'}{E'}. \tag{80c}$$

We have discussed in Sec. V A the properties of the function $\tilde{G}(E)$. Because of the Froissart bound, using Eq. (51) we have

$$\operatorname{Re}\tilde{G}(E) \leqslant \frac{3\pi}{2\ln E} \operatorname{Im}\tilde{G}(E).$$

Using this result, the optical theorem, and Eq. (71) in (78), it is simple to show

$$|\langle \Delta\sigma \rangle_E| \leqslant \frac{3\pi^{3/2}}{2\mu \ln^2 E} \int_0^E [(\sigma_P(E')^{1/2} + (\sigma_A(E'))^{1/2}] \times \ln E' \frac{dE'}{E'}. \tag{81}$$

Using the Schwarz inequality for the right-hand side of (81), we finally arrive at

$$|\langle \Delta\sigma \rangle_E| \leqslant \frac{\sqrt{3}\,\pi^{3/2}}{2\mu} [(\langle \sigma_P \rangle_E)^{1/2} + (\langle \sigma_A \rangle_E)^{1/2}]. \tag{82}$$

This is the most general restriction on $\Delta\sigma$ due to unitarity and analyticity in terms of the average elastic or total cross sections (to get meaningful results on the asymptotic behavior of $\Delta\sigma$ and $\sigma_{P,A}$, it is necessary that $\ln E \langle \Delta\sigma \rangle_E$, $\ln E \langle \sigma_{P,A} \rangle_E$ diverge as $E \to +\infty$).

Let us now examine the consequences of (82) for unbounded total cross sections. For this purpose, let us define the unboundness of the total cross sections as

$$\lim_{E\to+\infty} \langle \sigma_{\text{tot}P,A} \rangle_E = +\infty. \tag{83}$$

Dividing both sides of (82) by $\langle \sigma_P + \sigma_A \rangle_E$, where σ_P, σ_A refer now to the total cross sections, it is straightforward to show

$$\frac{|\langle \Delta\sigma \rangle_E|}{\langle \sigma_P + \sigma_A \rangle_E} \leqslant \frac{\sqrt{3}\,\pi^{3/2}}{2\mu} \left(\frac{1}{(\langle \sigma_P \rangle_E)^{1/2}} + \frac{1}{(\langle \sigma_A \rangle_E)^{1/2}} \right). \tag{84}$$

Since $\langle \sigma_P \rangle_E$ and $\langle \sigma_A \rangle_E$ are unbounded as defined by (83) the right-hand side of (84) tends to zero as $E \to +\infty$, hence

$$\lim_{E\to+\infty} \frac{|\langle \Delta\sigma \rangle|_E}{\langle \sigma_P + \sigma_A \rangle_E} = 0. \tag{85}$$

This equation is valid for any behavior or any type of oscillation of σ_P and σ_A. This version of the Pomeranchuk theorem for unbounded total cross sections as given by Eq. (85) can be regarded as strictly a consequence of axiomatic field theory.

Equation (82) shows the same result holds if only one of the total cross sections is unbounded; it then follows that the other total cross section must be unbounded too. From Eq. (85), it is clear that there exists an infinite sequence of points $\{E_i\}$, $E_i \to +\infty$ on which $(\Delta\sigma)/(\sigma_P+\sigma_A)\to 0$; in particular if $(\Delta\sigma)/(\sigma_P+\sigma_A)$ has a limit, this limit is zero. Equation (84) also gives the rate of $|(\Delta\sigma)|/(\sigma_P+\sigma_A)$ tending to zero depending how fast $\langle\sigma_{P,A}\rangle_E$ diverge.

As it was remarked above, the Pomeranchuk theorem cannot be proved for bounded total cross sections using unitarity. In this case, it is still possible to get a restriction on the magnitude of $\langle\Delta\sigma\rangle_E$ in terms of the particle-antiparticle total cross sections or elastic cross sections. It is straightforward to show $\mathrm{Re}\tilde{G}/\mathrm{Im}\tilde{G} \le \pi/\ln E$ and hence

$$|\langle\Delta\sigma\rangle_E| \le \frac{\pi^{3/2}}{\sqrt{3}\,\mu}\left[(\langle\sigma_P\rangle_E)^{1/2} + (\langle\sigma_A\rangle_E)^{1/2}\right], \tag{86}$$

where $\langle\sigma_P\rangle_E$ and $\langle\sigma_A\rangle_E$ are again either elastic or total cross sections. Equation (86) is a generalization of the results of Roy and Singh,[22,27] who discussed the special case where $\Delta\sigma$ has a limit. For pion-nucleon scattering, if isospin invariance is assumed, a restriction on $\langle\Delta\sigma\rangle_E$ in terms of the charge-exchange cross section $\pi^- p\to\pi^0 n$ can be similarly obtained.[27] The same remark holds for the $K,\bar{K}$ system.

VII. DECREASING TOTAL CROSS SECTIONS

For total cross sections which decrease in such a way that

$$\int_0^E \sigma_s(E')\frac{dE'}{E'} < +\infty \quad (\sigma_s \equiv \sigma_P+\sigma_A), \tag{87}$$

i.e., essentially faster than $1/\ln E$, the previous method has to be modified, since the average $\langle\sigma_s\rangle_E$ does not reflect the asymptotic behavior of the cross section. In this case, since $|\Delta\sigma|<\sigma_s$, the integral $\int_0^E[\Delta\sigma(E')](dE'|E')$ converges also, and it is shown below that it is convenient to state the Pomeranchuk theorem in the form of a restriction on the rate of convergence of this integral.

More precisely, we want to show that the physical condition

$$\left|\frac{\mathrm{Re}f_{P,A}}{\mathrm{Im}f_{P,A}}\right| < \mathrm{const} \tag{88}$$

implies that

$$\lim_{E\to+\infty}\frac{\int_E^{+\infty}\Delta\sigma(E')(dE'/E')}{\int_E^{+\infty}\sigma_s(E')(dE'/E')} = 0, \tag{89}$$

provided that the integral $\int_0^E\sigma_s(E')(dE'/E')$ converges slower than any negative power of E [it is known that the Pomeranchuk theorem in the form $\lim_{E\to+\infty}(\Delta\sigma/\sigma_s)=0$ does not hold for rapidly decreasing cross sections]. [A similar proof can be given with the more general condition $\mathrm{Re}f_{P,A}/\mathrm{Im}f_{P,A}=o(\ln E)$.]

The meaning of (89) is that $\int_0^E(dE'/E')\Delta\sigma(E')$ converges faster than $\int_0^E(dE'/E')\sigma_s(E')$; in the simple case where $\sigma_{P,A}\sim C_{P,A}(\ln E)^{-(1+\epsilon)}$ (with $\epsilon>0$), condition (89) says that $C_P=C_A$, i.e., that $\lim_{E\to\infty}(\Delta\sigma/\sigma_s)=0$. The form (89) of the Pomeranchuk theorem is more general than those proofs[4,6] which have to introduce slowly varying functions which are analytic in the upper half plane.

In the course of the proof, it will be useful to give the necessary and sufficient condition which allows us to write an unsubtracted dispersion relation for f_a/E, assuming that f_a/E requires at most one subtraction. We therefore begin with the discussion of this point.

It is clear that this condition is given by

$$b = \frac{2}{\pi}\int_0^{+\infty}\mathrm{Im}f_a(E')\frac{dE'}{E'^2} \tag{90}$$

[b is nothing but $f_a'(0)$].

Now Eq. (69) can be written as

$$\int_0^E \mathrm{Im}f_a(E')\frac{dE'}{E'^2} = -\frac{1}{\pi}\int_0^{+\infty}\ln\left|\frac{E'+E}{E'-E}\right|\mathrm{Re}f_a(E')\frac{dE'}{E'^2} + \tfrac{1}{2}\pi f_a'(0), \tag{91}$$

where we have included the constant $\frac{1}{2}b\pi=\frac{1}{2}\pi f_a'(0)$.

It follows that (90) is equivalent to

$$\lim_{E\to+\infty}\int_0^{+\infty}\ln\left|\frac{E'+E}{E'-E}\right|\mathrm{Re}f_a(E')\frac{dE'}{E'^2} = 0, \tag{92}$$

which is the condition we wanted to derive.

Using (90), we see that Eqs. (91) and (92) can be written in the form

$$\int_E^{+\infty}\mathrm{Im}f_a(E')\frac{dE'}{E'^2} = \frac{1}{\pi}\int_0^{+\infty}\ln\left|\frac{E'+E}{E'-E}\right|\mathrm{Re}f_a(E')\frac{dE'}{E'^2}. \tag{93}$$

Note that a sufficient condition for (92) to be valid is $\lim_{E\to+\infty}\mathrm{Re}f_a(E)/E=0$ [the proof is identical to that given at the beginning of Sec. V B 5 replacing $\sigma_s(E)-\sigma_\infty$ by $\mathrm{Re}f_a(E)/E$].

This condition is weaker than the condition $\lim_{E\to+\infty}f_a(E)/E=0$ generally assumed when one writes an unsubtracted dispersion relation for f_a/E. When $\mathrm{Re}f_a/E$ tends to zero slower than $1/E$ (in the sense of Sec. V B 5), (93) gives a useful correlation between the asymptotic behavior of $\mathrm{Im}f_a$ and $\mathrm{Re}f_a$; in particular $\mathrm{Im}f_a$ and $\mathrm{Re}f_a$ must have the same sign asymptotically (if they keep a constant sign for sufficiently large E).

In order to prove the Pomeranchuk theorem in the form of Eq. (89), we want first to show that conditions (87) and (88) imply that the dispersion relation for f_a/E needs no subtraction. This is expected, since from (87) one knows essentially that $\lim_{E\to+\infty}\sigma_s(E)=0$, which in turn implies

$\lim_{E\to+\infty} \mathrm{Re} f_a/E = 0$ using (88). To give a rigorous argument one has to prove that condition (92) is satisfied. But (88) implies

$$\left|\frac{\mathrm{Re} f_a(E)}{E}\right| < C\sigma_s(E), \tag{94}$$

so that we have the following bound:

$$\left|\int_0^{+\infty} \ln\left|\frac{E'+E}{E'-E}\right| \mathrm{Re} f_a(E') \frac{dE'}{E'^2}\right| < C\int_0^{+\infty} \ln\left|\frac{E'+E}{E'-E}\right| \sigma_s(E')\frac{dE'}{E'} . \tag{95}$$

The right-hand side of (95) is just $4\pi C[\mathrm{Re}G(E)]$ [where $\mathrm{Re}G(E)$ has been defined in Sec. V], and we get, using (51)

$$\left|\int_0^{+\infty} \ln\left|\frac{E'+E}{E'-E}\right| \mathrm{Re} f_a(E') \frac{dE'}{E'^2}\right| < \frac{\mathrm{const}}{\ln E}\int_0^{E} \sigma_s(E')\frac{dE'}{E'} . \tag{96}$$

Equation (92) follows from (96) and (87). [The bounds (51) and (96) may be violated on some exceptional sequences. However, from (91) and the convergence of

$$\int_0^{E} \mathrm{Im} f_a(E') \frac{dE'}{E'^2} ,$$

which follows from (87) using the optical theorem, we know that the limit of the left-hand side of (92) exists; since it is zero on a set of points, it must be zero on all sequences.]

Since (92) has been checked we can use Eq. (93) and write (95) in the form

$$\left|\int_E^{+\infty} \mathrm{Im} f_a(E') \frac{dE'}{E'^2}\right| < \frac{C}{\pi}\int_0^{+\infty} \ln\left|\frac{E'+E}{E'-E}\right| \sigma_s(E')\frac{dE'}{E'} . \tag{97}$$

We now follow a procedure quite similar to that used in Sec. VI. We would like to simplify the right-hand side of (97) and compare it to $\int_E^{+\infty}\sigma_s(E') \times (dE'/E')$. For this purpose, consider the function

$$K(E) = G(E) - i\int_0^{+\infty} \sigma_s(E')\frac{dE'}{E'} ,$$

which is the boundary value of a function analytic in the upper half E plane and odd under crossing [since $K(-E+i\epsilon) = -K^*(E+i\epsilon)$]. We have

$$\mathrm{Re}K(E) = \mathrm{Re}G(E) = \frac{1}{4\pi^2}\int_0^{+\infty} \ln\left|\frac{E'+E}{E'-E}\right| \sigma_s(E')\frac{dE'}{E'} , \tag{98}$$

$$\mathrm{Im}K(E) = -\frac{1}{4\pi}\int_E^{+\infty} \sigma_s(E')\frac{dE'}{E'} .$$

We see from (98) that $\mathrm{Re}K(E) \geq 0$, $\mathrm{Im}K(E) < 0$ for $E \geq 0$. Using the convention $\delta_K(0) = -\frac{1}{2}\pi$ [$\delta_K(E)$ is the phase of K], we have the following bounds for the phase:

$$-\tfrac{1}{2}\pi \leq \delta_K(E) \leq 0 \quad (E \geq 0) . \tag{99}$$

These conditions prove the validity of the phase representation for $K(E)$; applying Eq. (25) to $\tilde{K}(E) \equiv iK(E)$ [$\tilde{K}(E)$ is symmetric under crossing], we get

$$\frac{1}{\Delta(b)}\int_{x(1-b)}^{x(1+b)} \ln|K|\frac{dx'}{x'} = -\frac{1}{\pi}\int_0^x [\tfrac{1}{2}\pi + \delta_K(x')]\frac{dx'}{x'} + O(1) \qquad (x \equiv E^2) \tag{100}$$

[it can be shown from the Froissart bound and the bounds (99) on the phase that one must take $\nu = 0$ in (25)].

Let us now assume that $|K|$ is a slowly varying function in the sense that

$$\lim_{x\to+\infty} \frac{\ln|K|}{\ln x} = 0 . \tag{101}$$

It follows from (100) that

$$\int_0^x [\tfrac{1}{2}\pi + \delta_K(x')]\frac{dx'}{x'} = o(\ln x) . \tag{102}$$

Since $\frac{1}{2}\pi + \delta_K(x') \geq 0$, (102) means that

$$\lim_{E\to+\infty} \frac{\mathrm{Re}K(E)}{\mathrm{Im}K(E)} = 0 , \tag{103}$$

except on a set of points of asymptotic zero density. We thus conclude using (97), (98), and the optical theorem, that

$$\lim_{E\to+\infty} \frac{\int_E^{+\infty} \Delta\sigma(E')(dE'/E')}{\int_E^{+\infty} \sigma_s(E')(dE'/E')} = 0 ,$$

which is the desired result. The condition (101) is crucial; its physical meaning becomes clear if one notes that a sufficient condition for (101) is that

$$\lim_{x\to+\infty} \frac{\ln|\mathrm{Im}K|}{\ln x} = 0 . \tag{104}$$

Condition (104) means that $\int_0^E [\sigma_s(E')](dE'/E')$ converges slower than any negative power of E; it provides a convenient average definition of "slowly decreasing cross section." For instance, a behavior like $\sigma_s(E) \sim (\ln E)^{-(1+\epsilon)}$ ($\epsilon > 0$) is allowed by (104), but $\sigma_s(E) \sim E^{-\epsilon}$ is forbidden.

VIII. CONCLUDING REMARKS

We have presented in this paper a comprehensive treatment of oscillations for a class of slowly varying functions (but which can be rapidly oscillating) which are directly connected with the Pomeranchuk theorem, related theorems, and the phase representation. We are hopeful that the method discussed here can be extended to other problems in high-energy physics.

An obvious application of the technique developed here is the proof of the Pomeranchuk theorem for the forward particle-antiparticle differential cross sections,[19] where no assumption other than continuity is needed (after a smoothing procedure):

$$\int_0^E \ln\left|\frac{f_P(E',t=0)}{f_A(E',t=0)}\right|\frac{dE'}{E'} \leq C\,.$$

The constant C can explicitly be evaluated and depends on only the fact that the phases of f_P and f_A are bounded because their imaginary part is positive. (This improves previous considerations[28] as explained in the Introduction.) This result can be extended to the case $t\neq 0$; in this case one must assume that the phases of f_P and f_A are bounded or more generally $\sim o(\ln E)$.

APPENDIX A: BOUNDS ON $J(x)$

(1) Let us first show that $J(x)$ is $O(\ln x)$ if $\mathrm{Im}f(x)$ is $O(1)$ and satisfies a Hölder condition uniformly in x. One can split $J(x)$ as follows:

$$J(x)=J_1(x)+J_2(x)+J_3(x)\,,$$

$$J_1(x)=\frac{1}{\pi}\int_{x(1-\lambda)}^{x-\lambda}\frac{\mathrm{Im}f(x')}{x'-x}\,dx'\,,$$

$$J_2(x)=\frac{1}{\pi}\,\mathrm{P}\int_{x-\lambda}^{x+\lambda}\frac{\mathrm{Im}f(x')}{x'-x}\,dx'\,,$$

$$J_3(x)=\frac{1}{\pi}\int_{x+\lambda}^{x(1+\lambda)}\frac{\mathrm{Im}f(x')}{x'-x}\,dx'\,.$$

Assuming $|\mathrm{Im}f(x)|\leq C$, an explicit integration shows that J_1 and J_3 are bounded by $\ln x$:

$$|J_1(x)|\leq\frac{C}{\pi}\int_{x(1-\lambda)}^{x-\lambda}\frac{dx'}{x-x'}=\frac{C}{\pi}\ln x\,,$$

$$|J_3(x)|\leq\frac{C}{\pi}\int_{x+\lambda}^{x(1+\lambda)}\frac{dx'}{x'-x}=\frac{C}{\pi}\ln x\,.$$

J_2 can be written by a simple change of variable in the form

$$J_2(x)=\frac{1}{\pi}\int_0^{\lambda}\frac{\mathrm{Im}f(x+y')-\mathrm{Im}f(x-y')}{y'}\,dy'\,. \qquad \text{(A1)}$$

If one assumes that $\mathrm{Im}f$ satisfies a Hölder condition uniformly in x:

$$|\mathrm{Im}f(x\pm y')-\mathrm{Im}f(x)|\leq D y'^{\alpha}\quad(\alpha>0)\,,$$

where $0\leq y'\leq\lambda$ and D is a constant independent of x, then

$$\left|\frac{\mathrm{Im}f(x+y')-\mathrm{Im}f(x-y')}{y'}\right|\leq 2Dy'^{\alpha-1}\,,$$

so that using this bound in (A1) we get

$$|J_2(x)|\leq\frac{2D}{\pi}\frac{\lambda^\alpha}{\alpha}\,.$$

In the particular case where $d\mathrm{Im}f(x)/dx$ exists and is bounded, one can take $\alpha=1$ and $D=\sup_x|d\mathrm{Im}f/dx|$ in the previous bound.

At this stage one can already give bounds on $J(x)$ and assuming

$$C_1\leq\mathrm{Im}f(x)\leq C_2\,,$$

one gets

$$-\frac{C_2}{\pi}\ln x\leq J_1(x)\leq-\frac{C_1}{\pi}\ln x\,,$$

$$\frac{C_1}{\pi}\ln x\leq J_3(x)\leq\frac{C_2}{\pi}\ln x\,.$$

Hence

$$\frac{C_1-C_2}{\pi}\ln x-\frac{2D}{\pi}\frac{\lambda^\alpha}{\alpha}\leq J(x)$$

$$\leq\frac{C_2-C_1}{\pi}\ln x+\frac{2D}{\pi}\frac{\lambda^\alpha}{\alpha}\,.$$

This last result leads immediately to Eq. (26) of Ref. 8.

(2) Let us show now that if we make the "non-oscillating" assumptions

$$\lim_{x\to+\infty}\mathrm{Im}f(x)=C \text{ and } \frac{d}{dx}\mathrm{Im}f(x)=O(1)\,,$$

we get $J(x)=o(\ln x)$. Since $d\mathrm{Im}f(x)/dx=O(1)$, J_2 is bounded as shown above. J_1+J_3 can be written as

$$J_1(x)+J_3(x)=\frac{1}{\pi}\int_{x(1-\lambda)}^{x-\lambda}\frac{\mathrm{Im}f(x')-C}{x'-x}\,dx'$$

$$+\frac{1}{\pi}\int_{x+\lambda}^{x(1+\lambda)}\frac{\mathrm{Im}f(x')-C}{x'-x}\,dx'\,,$$

the contribution of the constant C being identically zero. We therefore get

$$|J_1(x)+J_3(x)|\leqslant \sup_{x'\in[x(1-\lambda),x(1+\lambda)]} |\mathrm{Im}f(x')-C|$$

$$\times\left(\frac{1}{\pi}\int_{x(1-\lambda)}^{x-\lambda}\frac{dx'}{x-x'}+\frac{1}{\pi}\int_{x+\lambda}^{x(1+\lambda)}\frac{dx'}{x'-x}\right),$$

i.e.,

$$|J_1(x)+J_3(x)|\leqslant\frac{2}{\pi}\sup_{x'\in[x(1-\lambda),x(1+\lambda)]}|\mathrm{Im}f(x')-C|\ln x,$$

which shows that $J_1+J_3=o(\ln x)$, hence $J(x)=o(\ln x)$.

If we make the further assumption that $\mathrm{Im}f(x)-C=O(1/\ln x)$ [together with $d\,\mathrm{Im}f(x)/dx=O(1)$], we get $J_1(x)+J_3(x)=O(1)$, hence $J(x)=O(1)$. What happens here is a cancellation between J_1 and J_3 (due to the opposite signs of the denominators $x'-x$ in these two integrals), which is made possible by the nonoscillatory character of $\mathrm{Im}f(x)$.

We give now a more interesting condition of "nonoscillation" which is expressed only with the derivative of $\mathrm{Im}f(x)$.

(3) Let us show that a sufficient condition to have $J(x)=O(1)$ is given by condition (12): $x\,d\,\mathrm{Im}f(x)/dx=O(1)$. By a simple change of variable $J(x)$ can be put in the form

$$J(x)=\frac{1}{\pi}\int_0^{\lambda}\frac{\mathrm{Im}f(x+xy')-\mathrm{Im}f(x-xy')}{y'}\,dy'. \qquad \text{(A2)}$$

Introducing

$$h_x(y')=\frac{\mathrm{Im}f(x+xy')-\mathrm{Im}f(x-xy')}{y'},$$

we can write, assuming $\mathrm{Im}f(x)$ is derivable,

$$h_x(y')=\frac{1}{y'}\int_{x-xy'}^{x+xy'}\frac{d}{du}\,\mathrm{Im}f(u)\,du.$$

Now using (12) we can introduce the upper bound D of $x\,d\,\mathrm{Im}f/dx$ and write

$$\left|u\,\frac{d}{du}\mathrm{Im}f\right|\leqslant D.$$

It follows that

$$|h_x(y')|\leqslant\frac{D}{y'}\int_{x-xy'}^{x+xy'}\frac{du}{u}=\frac{D}{y'}\ln\left|\frac{1+y'}{1-y'}\right|.$$

From (A2) we now get

$$|J(x)|\leqslant\frac{D}{\pi}\int_0^{\lambda}\ln\left|\frac{1+y'}{1-y'}\right|\frac{dy'}{y'},$$

which is the desired result.

We can state the intuitive meaning of condition (12) as follows: It makes the oscillations of $\mathrm{Im}f(x)$ slow enough to allow a cancellation between J_1 and J_3.

APPENDIX B

The following counterexamples are constructed to show that it is not possible to improve the bound $J(x)=O(\ln x)$ obtained in Appendix A under the assumptions $\mathrm{Im}f(x)=O(1)$, $d\,\mathrm{Im}f(x)/dx=O(1)$ without further assumptions.

Let us consider the integrand $h_x(y')$ of the right-hand side of (A2). At $y'=0$, $h_x(0)=2x\,d\,\mathrm{Im}f/dx$ (assuming the existence of the derivative at point x). For $y'\neq 0$, $|h_x(y')|\leqslant 2C/y'$ if $|\mathrm{Im}f(x)|\leqslant C$.

This suggests the construction of an example where, around a set of points $\{x_n\}$, $(x_n\to+\infty)$ one has

$$h_{x_n}(y')=\frac{2C}{y'}\ \text{ for }\ \frac{C}{x_nD_n}\leqslant y'\leqslant\lambda,$$

$$h_{x_n}(y')=2x_nD_n\ \text{ for }\ 0\leqslant y'\leqslant\frac{C}{x_nD_n},$$

where $D_n=d\,\mathrm{Im}f(x=x_n)/dx$ and D_n is assumed to be positive. Then $\pi J(x_n)$ is equal to the area under the curve drawn in Fig. 1.

We get

$$J(x_n)=\frac{1}{\pi}[2C\ln(x_nD_n)+2C(1-\ln C)],$$

hence

$$J(x_n)=O(\ln(x_nD_n)).$$

A simple choice for $\mathrm{Im}f(x)$ which satisfies the previous conditions is drawn in Fig. 2. In this example, $\mathrm{Im}f(x)$ is defined by

$$\mathrm{Im}f(x)=-C\ \text{ for }\ x_n(1-\lambda)\leqslant x\leqslant x_n-\frac{C}{D_n},$$

$$\mathrm{Im}f(x)=(x-x_n)D_n\ \text{ for }\ x_n-\frac{C}{D_n}\leqslant x\leqslant x_n+\frac{C}{D_n},$$

$$\mathrm{Im}f(x)=+C\ \text{ for }\ x_n+\frac{C}{D_n}\leqslant x\leqslant x_n(1+\lambda),$$

with $\lim_{n\to+\infty}x_n=+\infty$, and an arbitrary behavior for $\mathrm{Im}f$ can be chosen between two neighboring intervals $[x_n(1-\lambda), x_n(1+\lambda)]$ and $[x_{n+1}(1-\lambda), x_{n+1}(1+\lambda)]$. (It is assumed that the x_n are spaced in such a way that these intervals do not overlap.) Even if D_n tends to zero when x_n tends to infinity, but slower than $1/x_n$ [so that $\lim_{n\to+\infty}(x_nD_n)=+\infty$], $J(x_n)$ will tend to $+\infty$. For instance, if $D_n=(1/x_n{}^{\alpha})$ $(0<\alpha<1)$, $J(x_n)$ will increase like $\ln x_n$.

APPENDIX C

Consider the analytic function $f(z)$ defined by

$$f(z)=\frac{z}{\pi}\int_1^{+\infty}\frac{\mathrm{Im}f(x)}{x-z}\frac{dx}{x}, \qquad \text{(C1)}$$

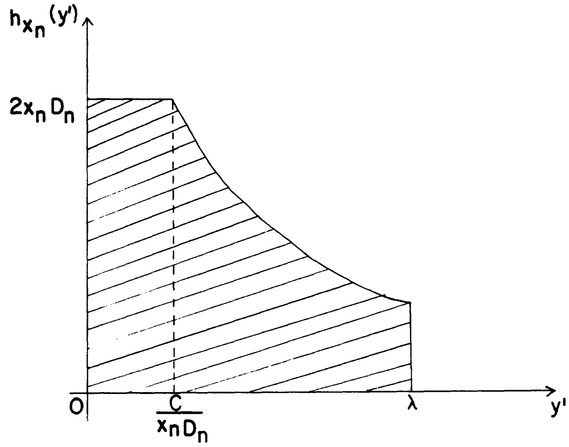

FIG. 1. A simple choice for $h_x(y')$.

where $\text{Im}f(x)=O(1)$.

We would like to show that

$$\lim_{\substack{r\to+\infty \\ \theta \text{ fixed} \neq 0}} |f(z)|=+\infty$$

(where $z=re^{i\theta}$) implies

$$\lim_{\substack{r\to+\infty \\ \theta \text{ fixed} \neq 0}} \frac{f(z)}{-(1/\pi)\int_1^r \text{Im}f(x)(dx/x)}=1. \quad \text{(C2)}$$

Proof. From (C1) one has

$$\text{Re}f(z)=\frac{r}{\pi}\int_1^{+\infty}\frac{(x\cos\theta-r)\text{Im}f(x)}{x^2+r^2-2xr\cos\theta}\frac{dx}{x},$$

$$\text{Im}f(z)=\frac{r}{\pi}\int_1^{+\infty}\frac{\sin\theta\,\text{Im}f(x)}{x^2+r^2-2xr\cos\theta}dx.$$

It is clear that $\text{Im}f(z)$ is $O(1)$ if $\text{Im}f(x)$ is $O(1)$, since

$$\text{Im}f(z)=\frac{1}{\pi}\int_{1/r}^{+\infty}\frac{\sin\theta\,\text{Im}f(yr)}{y^2-2y\cos\theta+1}dy,$$

where we have changed the variable of integration to $y=x/r$. On the other hand, one can express $\text{Re}f(z)$ as follows:

$$\text{Re}f(z)=-\frac{1}{\pi}\int_1^r \text{Im}f(x)\frac{dx}{x}$$

$$+\frac{1}{\pi}\int_1^r\frac{(x^2-xr\cos\theta)\,\text{Im}f(x)}{x^2+r^2-2xr\cos\theta}\frac{dx}{x}$$

$$+\frac{r}{\pi}\int_r^{+\infty}\frac{(x\cos\theta-r)\text{Im}f(x)}{x^2+r^2-2xr\cos\theta}\frac{dx}{x}. \quad \text{(C3)}$$

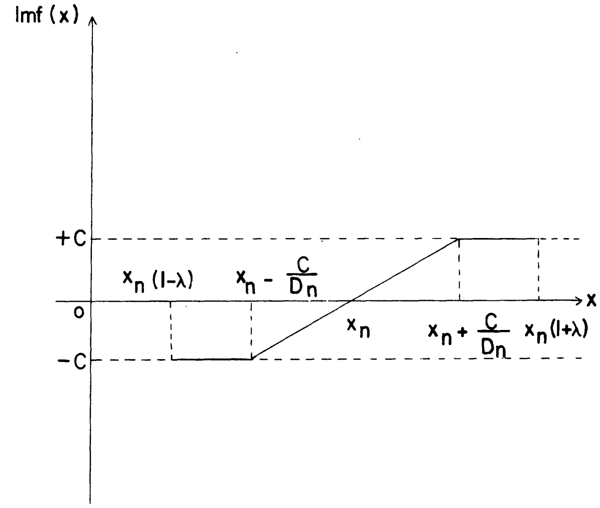

FIG. 2. A simple choice for $\text{Im}f(x)$.

The last two integrals on the right-hand side of (C3) are bounded as can be seen by changing the variable of integration as done previously and recalling that

$$\text{Im}f(x)=O(1).$$

We therefore end with the result

$$f(z)=-\frac{1}{\pi}\int_1^r \text{Im}f(x)\frac{dx}{x}+O(1),$$

$$z=re^{i\theta}(\theta\text{ fixed}\neq 0), \quad \text{(C4)}$$

which shows that if

$$\lim_{\substack{r\to+\infty \\ \theta \text{ fixed} \neq 0}} |f(z)|=+\infty,$$

(C2) is valid.

Result (C4) gives information when one goes to infinity along any radius in the complex plane (or for $\theta=\pi$ along the negative real axis). This is to be compared with Eq. (20), which gives a result on the cut ($\theta=0$):

$$g_b(r)=\frac{1}{\Delta(b)}\int_{r(1-b)}^{r(1+b)} f(x)\frac{dx}{x}$$

$$=-\frac{1}{\pi}\int_1^r \text{Im}f(x)\frac{dx}{x}+O(1). \quad \text{(C5)}$$

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$$\frac{d}{dx}\,\mathrm{Im}g(x) = \frac{\mathrm{Im}f(x)}{x}$$

we have the same result.

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