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$$\sum_{P\{i\}} (-1)^P \prod_{i=1}^{N} (x_i - y_{P\{i\}}) = \prod_{i<j} (x_i - x_j)(y_i - y_j) \prod_{i\neq j} (x_i - y_j)^{-1}.$$

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# Reggeization of elementary particles in renormalizable gauge theories: Scalars

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The question of the Reggeization of the elementary particles in gauge theories with a spontaneously broken symmetry is extended to examination of the scalar-meson states of these theories. Owing to the complexity of the problem, the study is limited to three typical gauge models, U(1), SU(2), and U(2). It is demonstrated that under special circumstances the Born approximation for some or all of the $J=0$ amplitudes can be factorized, the first time the possible Reggeization of elementary scalars has been reported. However, no model is found in which all elementary particles lie on Regge trajectories. The crucial role of an Abelian vector meson, which does not Reggeize, and a fermion, which does, is emphasized. Since the Mandelstam counting argument is inconclusive at $J=0$, our conclusions are limited to lowest order in perturbation theory.

## I. INTRODUCTION

This paper provides a continuation of the discussion of the Reggeization of elementary particles in renormalizable Yang-Mills theories begun in Ref. 1. Earlier we reported that under fairly broad conditions the non-Abelian vector mesons and fermions of these theories may Reggeize. That is, we have examined the behavior in the angular momentum plane near $J=1$ and $J=\frac{1}{2}$, and shown that the Kronecker $\delta$'s found in scattering amplitudes computed to second-order perturbation theory are turned into moving Regge poles by higher-order corrections. In this paper, we continue this program by discussing the behavior of such amplitudes near $J=0$.

We restrict ourselves to renormalizable theories,[2] so that we do not consider elementary particles with spin larger than one. In all the examples under consideration only one nonsense state occurs in each vector-vector and vector-fermion channel at $J=1$ and $J=\frac{1}{2}$, respectively. This nonsense state in turn generates at most one Regge trajectory, since, as demonstrated in Appendix A of paper I, the total number of Regge trajectories is equal to the rank of the complete nonsense-nonsense matrix. Moreover, for the theories we have considered, the non-Abelian elementary vector mesons and the fermions lie on the corresponding trajectories.

The situation near $J=0$ is much more complicated than that near $J=1$ and $J=\frac{1}{2}$. Now each vector-vector channel has two nonsense states which will generate two Regge trajectories in general. In addition to these two nonsense states, the vector-scalar and fermion-antifermion channels have one nonsense state each at $J=0$ and are capable of generating more trajectories if they are coupled to the vector-vector states. Mandelstam counting[3] indicates that the theories *need not* Reggeize at $J=0$, although they may do so for special choices of masses and coupling constants. Thus there is no reason to expect results which are as general as those obtained at $J=1$ or $J=\frac{1}{2}$. Nevertheless, given the fact that spontaneously broken gauge theories have good high-energy behavior[4] and hoping that they possess simple analytic properties in the angular momentum plane, one is strongly tempted to find some models in which all the elementary particles are Reggeized. We would like to emphasize again that Reggeization means the amplitudes exhibit Regge behavior and the elementary particles present in the Lagrangian lie on Regge trajectories. We know that in $\phi^3$ theory[5] the scattering amplitudes have Regge behavior, but the elementary scalar particle does not lie on a Regge trajectory. This property of $\phi^3$ theory can be traced to the absence of a nonsense state at $J=0$ in this theory. Nonsense states are present in gauge theories with spontaneously broken symmetries (SBSs), but to settle the question of the Reggeization of the scalar mesons, one has to do detailed calculations, especially given the absence of a conclusive Mandelstam-counting argument.

We have found that in an Abelian U(1) model with SBS in which the vector meson *does not* Reggeize, it *is possible* to Reggeize the scalar meson by coupling in a fermion-antifermion channel. This is the first time, to our knowledge, that the factorization of the Born amplitude at $J=0$ has been reported. This is in contrast to the results found by Gell-Mann *et al.*[6] In their U(1) model *without* SBS, the scalar fails to Reggeize.

In an SU(2)-gauge model, we have found that the $I=2$ amplitudes factorize at $J=0$, and the presence of (high-isospin) fermion-antifermion states does not spoil this factorization. The $I=0$ amplitudes do not factorize in this model, and the scalar particle fails to Reggeize. The situation improves when we consider a U(2) model, together with coupled fermion-antifermion states. We find that the factorization of the $I=2$ amplitude persists, and that for a special choice of the parameters of the theory, it is possible to factorize the $I=0$ amplitudes. However, this model contains an additional $I=1$ scalar meson which does not seem to lie on a Regge trajectory. We note that in this model in addition to the $I=2$ channel there are other channels which do not contain an elementary particle, but do have Kronecker $\delta$'s at $J=0$ in second-order perturbation theory because of spin in the external lines. We find the nontrivial result that in all such channels Reggeization seems to occur, the Kronecker $\delta$'s being turned into nonsense-choosing Regge poles (cf. Appendix A of paper I). We have *not* succeeded in finding a model for which the amplitudes factorize for *all* channels.

In Sec. II we give a detailed discussion of a U(1) model and a convenient method to treat the effect of additional channels. We show that with constraints on the free parameters of the theory and the aid of the fermion-antifermion channel the $J=0$ amplitudes factorize. In Sec. III, an SU(2) model is discussed with the emphasis on the $I=2$, $J=0$ amplitudes. In the same section we give a brief discussion of the $I=0$, $J=0$ amplitudes, why the factorization fails, and why we are led to the U(2) model. In Sec. IV we discuss the U(2) model and why we need the fermion again, while in Sec. V we discuss all the other channels in the U(2) model. In particular we point out that the presence of high-isospin fermions will not spoil the $I=2$ factorization found in the SU(2) model. We close in Sec. VI with some conclusions and speculative remarks.

There is considerable algebraic complexity in our calculations. We must deal with matrices of increasing size and thus find it convenient to use a super-matrix notation wherever possible. The entries in our matrices are labeled by the helicity states of the channels, with the labeling nn, sn, and ss used to denote the submatrices connecting nonsense to nonsense, sense to nonsense, and sense to sense states. For instance, in vector-vector scattering $B_{\rm nn}$ will denote the two-by-two symmetric matrix of the Born-approximation helicity amplitudes $\langle 01|T|01\rangle$, $\langle 01|T|1\,-1\rangle$, $\langle 1\,-1|T|1\,-1\rangle$.

Our strategy is basically the following: For a given system we compute at $J=0$ the sense-sense, sense-nonsense, and nonsense-nonsense matrices. If the system has an $s$-channel pole, we check the necessary condition that the nn matrix have an eigenvalue which vanishes at the position of this pole. If the condition is satisfied or if no $s$-channel pole is present, we check the actual factorization condition. This check is facilitated by the following statement which is not difficult to prove[7]: *A necessary and sufficient condition for factorization to hold is that the rank of the whole Born-approximation matrix be equal to the rank of the nn matrix.*

## II. U(1) MODEL

Consider U(1) as a local gauge group,[8] with $V_\mu$ the gauge field and $\phi$ a complex charged scalar field with charge $\frac{1}{2}g$. We arrange the interaction and mass of the scalars such that $\phi$ acquires a vacuum expectation value $\langle\phi\rangle = v/\sqrt{2}$, and write $\phi=(1/\sqrt{2})(v+\sigma+i\pi)$. In the unitary gauge, the relevant interaction Lagrangian is

$$\mathcal{L}_I=\tfrac{1}{2}gm\sigma V_\mu{}^2+\tfrac{1}{8}g^2\sigma^2V_\mu{}^2-\tfrac{1}{4}g\frac{\mu^2}{m}\sigma^3-\frac{1}{32}g^2\frac{\mu^2}{m^2}\sigma^4, \tag{2.1}$$

where $m$ and $\mu$ are the masses of vector and scalar mesons, respectively. There are three two-body channels: vector-vector, scalar-scalar, and vector-scalar. Only the first two channels are coupled, and they contain a scalar-meson pole in the $s$ channel.

Let us discuss the $V$-$V$ channel first, and denote by $S_s$, $S_t$, and $S_u$ the scalar-meson exchange terms for the $s$-, $t$-, and $u$-channel exchanges, respectively. The total contribution to the amplitude is $S_s+S_t+S_u$, but $S_s$ only contributes to the sense-sense transitions. We denote by $X_{11}$, $Y_1$, and $B$ the ss, sn, and nn matrices, respectively. They are

$$(X_{11})_{11,11}=2\frac{m^2}{s-\mu^2}+\frac{1}{2}\frac{m^2z_1}{k^2}, \tag{2.2a}$$

$$(X_{11})_{11,00}=2\frac{2E^2-m^2}{s-\mu^2}-\frac{E^2z_1}{k^2}, \tag{2.2b}$$

$$(X_{11})_{00,00}=2\frac{(2E^2-m^2)^2}{(s-\mu^2)m^2}+2\frac{E^2(E^2z_1-2k^2)}{m^2k^2}; \tag{2.2c}$$

$$(Y_1)_{11,01}=\frac{1}{\sqrt{2}}\frac{mEz_1}{k^2}, \tag{2.3a}$$

$$(Y_1)_{11,1\,-1}=-i\frac{1}{\sqrt{2}}\frac{m^2z_1}{k^2}, \tag{2.3b}$$

$$(Y_1)_{00,01}=\sqrt{2}\frac{E}{mk^2}(k^2-E^2z_1), \tag{2.3c}$$

$$(Y_1)_{00,1\,-1}=i\sqrt{2}\frac{E^2}{k^2}z_1; \tag{2.3d}$$

$$B_{01,01}=\frac{1}{2}\frac{2E^2z_1-k^2}{k^2}, \tag{2.4a}$$

$$B_{01,1\,-1}=-i\frac{mEz_1}{k^2}, \tag{2.4b}$$

$$B_{1-1,1\,-1}=-\frac{m^2z_1}{k^2}, \tag{2.4c}$$

with $k$ the c.m. momentum, $E^2=k^2+m^2$, $s=4E^2$, and $z_1=(\mu^2/2k^2)+1$. For simplicity, we shall henceforth take the gauge coupling constant $g=1$.

From Eqs. (2.4), we find that

$$\det B=\frac{m^2z_1}{2k^2}. \tag{2.5}$$

We conclude that the nonsense-nonsense matrix is of rank two and does *not* have an eigenvalue which vanishes at $s=\mu^2$, so that factorization is not satisfied. To see this explicitly, a simple calculation leads to

$$(Y_1)_{sn}(B^{-1})_{nn}(Y_1)_{ns}=(X_{11})_{ss}-(S_s)_{ss}, \tag{2.6}$$

where $S_s$ are the $s$-channel pole terms of Eq. (2.2).

We also note that the rank of the whole Born matrix is three, while that of the nonsense matrix is two. Factorization must therefore fail in this model. However, if there is a coupling to additional two-body channels (denoted collectively by $\Gamma$) which contain a nonsense state, and if the enlarged Born matrix for this coupled-channel process still has rank three, while the rank of the enlarged nonsense matrix has increased from two to three, factorization will succeed. Equivalently, we can check factorization directly as follows. Let $V$ and $U$ be the $\Gamma\to\Gamma$ and $\Gamma\to VV$ nonsense-nonsense matrices, respectively, and $Z$ the $\Gamma\to VV$ sense-nonsense matrix. The factorization of the $VV$-$VV$ channel now requires that

$$X_{11}=(Y_1Z)\begin{pmatrix}B & U^t\\ U & V\end{pmatrix}^{-1}(Y_1Z)^t, \tag{2.7}$$

which can be written as[7]

$$X_{11}=Y_1B^{-1}Y_1^t+(Y_1B^{-1}U^t-Z)(V-UB^{-1}U^t)^{-1}\times(UB^{-1}Y_1^t-Z^t). \tag{2.8}$$

If we can find a set of channels $\Gamma$ such that

$$S_s=(Y_1B^{-1}U^t-Z)(V-UB^{-1}U^t)^{-1}(UB^{-1}Y_1^t-Z^t), \tag{2.9}$$

then the factorization in the $VV$-$VV$ channel will succeed. (On the other hand, once factorization has been achieved for some system with a fixed number of channels, the addition of further channels containing nonsense states will not spoil the original factorization only if terms of the type $Y_1B^{-1}U^t-Z_1$ are zero. Note that this is what occurs in the $I=2$ sector discussed in Sec. V. For a discussion of the meaning of such a result see Ref. 7.)

In our search for suitable channels $\Gamma$, a natural candidate appeared to be a fermion-antifermion two-body state. In this U(1) model the only renormalizable interaction Lagrangian which couples in a fermion of mass $M$ is

$$\mathcal{L}_I=\alpha g\bar{\psi}\gamma_\mu\psi V_\mu, \tag{2.10}$$

where $\alpha$ is the ratio of the fermionic charge to the scalar charge. Note that in U(1) gauge theory $\alpha$ can be any *real* number.

We denote the contributions to $U$, $V$, and $Z$ from the $F\bar{F}$ channel by $\alpha^2U_1$, $\alpha^2V_1$, and $\alpha^2Z_1$. They are

$$(U_1)_{1/2\,-1/2,0\,-1}=\frac{\sqrt{2}mE^2}{Mk^2p}, \tag{2.11a}$$

$$(U_1)_{1/2\,-1/2,1\,-1}=-\frac{i\sqrt{2}E(2E^2-m^2)}{Mk^2p}, \tag{2.11b}$$

$$(V_1)_{1/2\,-1/2,1/2\,-1/2}=-\frac{1}{2}\frac{E^2}{M^2p^2}, \tag{2.12}$$

$$(Z_1)_{1/2\ -1/2,11} = \frac{m^2 E}{Mk^2 p}, \quad (2.13a)$$

$$(Z_1)_{1/2\ -1/2,00} = -\frac{2m^2 E}{Mk^2 p}, \quad (2.13b)$$

where $p$ is the c.m. momentum of the $F\bar{F}$ state.

With the observation that

$$Y_1 B^{-1} U_1^t - Z_1 = \frac{2E}{Mm^2 p}\begin{pmatrix} m^2 \\ 2E^2 - m^2 \end{pmatrix} \quad (2.14)$$

and

$$\frac{1}{\alpha^2} V_1 - U_1 B^{-1} U_1^t = \frac{2E^2}{M^2 p^2}\left[\frac{s - m^2}{m^2} + 3 - \frac{1}{4\alpha^2} - \frac{2(m^2 - \mu^2)}{m^2}\frac{z_0}{z_1}\right], \quad (2.15)$$

with $z_0 = 1 + m^2/2k^2$, we find that the conditions

$$\alpha^2 = \frac{1}{12}, \quad m^2 = \mu^2 \quad (2.16)$$

will give the desired factorization. One can always redefine $g$ so that the fermionic charge $\alpha g$ becomes an integer or $\frac{1}{3}$ of an integer to make it look more physical, but that is not our concern here.

We note that satisfying Eq. (2.9) is not sufficient, since the entire Born matrix must factorize. We have checked that the $7\times 7$ Born matrix for the coupled processes

$$\left.\begin{matrix} VV \\ F\bar{F} \\ \sigma\sigma \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} VV \\ F\bar{F} \\ \sigma\sigma \end{matrix}\right. \quad (2.17)$$

factorizes and has rank three under the conditions (2.16). We list the relevant matrix elements in the Appendix for the interested reader. Three Regge trajectories are generated in this model. One of them passes through $J=0$ at $s=m^2$ and chooses sense, while the other two choose nonsense. We encourage the reader to check this and get some appreciation of these nontrivial results.

Next, let us consider the $V\sigma$-$V\sigma$ channel. As observed in paper I, this channel has a Kronecker-$\delta$ contribution at $J=1$, but only has sense states. The $2\times 2$ sense-sense matrix has rank one, but the implication for the Reggeization of the Abelian vector meson is still an open question. This channel has also Kronecker-$\delta$ contributions at $J=0$ with one nonsense state. We found that the $2\times 2$ Born matrix $H$ for $V\sigma \to V\sigma$ factorizes at $J=0$ only under the condition $\mu^2 = m^2$. For this equal-mass case

$$H_{0,0} = \frac{m^2}{k^2}, \quad (2.18a)$$

$$H_{0,1} = \frac{1}{\sqrt{2}}\frac{mE}{k^2}, \quad (2.18b)$$

$$H_{1,1} = \frac{1}{2}\frac{E^2}{k^2}, \quad (2.18c)$$

and one nonsense-choosing trajectory is generated. The introduction of the fermion will not change this result since there is no direct coupling of the scalar and fermion in this model. It might be significant that this channel has the same quantum numbers as those of the scalar meson which has been "gauged away" in the U gauge after SBS.[9]

We have demonstrated that with the aid of the $F\bar{F}$ channel, one may Reggeize the scalar in the U(1) model. Since the fermions Reggeize in this model, if we could Reggeize the Abelian vector meson, the model would be completely Reggeized to this order. (We emphasize the fact that we have no reason to believe that Reggeization will in fact take place in higher orders since there is no Mandelstam-counting argument to support such a belief.) In Sec. III we turn to a model where the vector mesons do Reggeize and investigate the possibility of Reggeizing the scalars.

## III. SU(2) MODEL

The Lagrangian of the model is displayed in Eq. (2.1) of paper I. It describes an isotriplet of vector mesons $\vec{\rho}$ and an isosinglet scalar meson $\sigma$. The Born approximation for $\vec{\rho}$-$\vec{\rho}$ scattering contains contact terms, vector-exchange terms, and scalar-exchange terms. We call combinations of the first two terms $V_s$, $V_t$, and $V_u$ and the scalar exchanges we denote by $S_s$, $S_t$, and $S_u$ for exchange in the $s$, $t$, and $u$ channels, respectively. At $J=0$, $\vec{\rho}$-$\vec{\rho}$ scattering has contributions in both the $I=0$ and $I=2$ states. The amplitudes are given by the following combinations:

$$T_0 = (3S_s + S_t + S_u) + 2(V_t - V_u),$$
$$T_2 = (S_t + S_u) - (V_t - V_u). \quad (3.1)$$

Except for the factor of 3, the scalar-exchange contributions are those of Eqs. (2.2)–(2.4). We list below the contribution of $V_t - V_u$:

$$(V_t - V_u)_{11,11} = +m^2\,\frac{2E^2 + 3m^2}{4k^4}, \quad (3.2a)$$

$$(V_t - V_u)_{11,00} = -\frac{2E^4 - E^2 m^2 + 4m^4}{2k^4}, \quad (3.2b)$$

$$(V_t - V_u)_{00,00} = -\frac{2E^6 - 7E^4 m^2 + 4E^2 m^4 - 4m^6}{m^2 k^4}; \quad (3.2c)$$

$$(V_t - V_u)_{11,01} = \sqrt{2}\,\frac{Em(2E^2 + 3m^2)}{4k^4}, \quad (3.3a)$$

$$(V_t - V_u)_{11,1\ -1} = -i\sqrt{2}\,m^2\frac{5z_0}{2k^2}, \quad (3.3b)$$

$$(V_t - V_u)_{00,01} = -\sqrt{2}\,\frac{Em(3E^2+2m^2)}{2k^4}, \qquad (3.3c)$$

$$(V_t - V_u)_{00,1\,-1} = +i\sqrt{2}\,\frac{(E^2+4m^2)z_0}{k^2}\,; \qquad (3.3d)$$

$$(V_t - V_u)_{01,01} = +\frac{E^4+5E^2m^2-m^4}{2k^4}, \qquad (3.4a)$$

$$(V_t - V_u)_{01,1\,-1} = -i\frac{5Emz_0}{k^2}, \qquad (3.4b)$$

$$(V_t - V_u)_{1-1,1\,-1} = -\frac{8E^2-3m^2}{k^2}z_0. \qquad (3.4c)$$

Consider first the $I=2$ channel. Although the separate terms $(S_t+S_u)$ and $(V_t-V_u)$ have bad high-energy behavior, the combinations given in Eq. (3.1) satisfy unitarity bounds, and the Born approximation factorizes. The $2\times 2$ nonsense-nonsense matrix has rank two, and as discussed in Appendix A, Eq. (A11), of paper I it generates two nonsense-choosing Regge trajectories. If we set the vector and scalar masses equal (in tree approximation), the rank of the matrix reduces to one. In this case only one *nonsense-choosing* Regge trajectory with $I=2$ is generated, and it passes through $J=0$ at $s=m^2$.

Now consider the $I=0$, $J=0$ amplitudes. The sense-sense matrix has the $\sigma$-meson pole, so that as a first check of possible factorization one evaluates the determinant of the $2\times 2$ nonsense-nonsense matrix to see if it vanishes at $s=\mu^2$. It turns out this is not the case even if one set $m^2=\mu^2$. In this model, as in the U(1) model without fermions, factorization fails at $I=0$, $J=0$ and the scalar meson does not lie on a Regge trajectory. Explicit calculation shows that the whole $\vec{\rho}\vec{\rho}\to\vec{\rho}\vec{\rho}$ helicity matrix at $I=0, J=0$ has rank four, while the nonsense-nonsense matrix has rank two. Just as in the case of U(1), we must couple the $\vec{\rho}\vec{\rho}$ channel to other two-body channels to increase the rank of the new nonsense-nonsense matrix to four. This has led us to the consideration of the U(2) "extended spectrum" model of Gervais and Neveu.[9]

In the $I=1$ $\vec{\rho}\sigma$-$\vec{\rho}\sigma$ channel there are Kronecker-$\delta$ contributions at $J=0$ but no $s$-channel pole. Just as in the U(1) case, the $2\times 2$ Born helicity matrix factorizes if $m^2=\mu^2$. From now on we will discuss only the equal-mass case unless stated otherwise.

## IV. U(2) MODEL

This model is similar to the SU(2) model discussed in Sec. III, but has an additional isoscalar vector meson $\omega$ and isovector scalar meson $\vec{s}$. The interaction Lagrangian in U gauge is

$$\mathcal{L}_I = -g\,\partial_\mu\vec{\rho}_\nu\times\vec{\rho}_\mu\cdot\vec{\rho}_\nu - \tfrac{1}{4}g^2(\vec{\rho}_\mu\times\vec{\rho}_\nu)^2 + \tfrac{1}{2}gm\sigma({\omega_\mu}^2+{\vec{\rho}_\mu}^2) + gm\,\omega\vec{\rho}\cdot\vec{s} + \tfrac{1}{2}g\vec{\rho}_\mu\cdot\vec{s}\times\partial_\mu\vec{s} + \tfrac{1}{8}g^2(\sigma^2+\vec{s}^2)({\omega_\mu}^2+{\vec{\rho}_\mu}^2)$$
$$+\tfrac{1}{2}g^2\vec{\rho}_\mu\cdot\vec{s}\,\omega_\mu\sigma - \tfrac{1}{4}g\,\frac{\mu^2}{m}(\sigma^3+3\sigma\vec{s}^2) - \tfrac{1}{32}g^2\frac{\mu^2}{m^2}\left[\sigma^4+(\vec{s}^2)^2+6\sigma^2\vec{s}^2\right], \qquad (4.1)$$

where $m$ and $\mu$ are the masses of vector and scalar mesons, respectively. We have discussed the situation at $J=1$ in paper I, where we argued that the $\vec{\rho}$ Reggeizes, with no information as to the Reggeization of the $\omega$. At $I=0$, $J=0$ both the $\vec{\rho}\vec{\rho}$ and $\omega\omega$ channels have two distinct nonsense states. Let us denote by $A$, $B$, and $\sqrt{3}\,B$ the $\vec{\rho}\vec{\rho}\to\vec{\rho}\vec{\rho}$, $\omega\omega\to\omega\omega$, and $\omega\omega\to\vec{\rho}\vec{\rho}$ nonsense-nonsense matrices, respectively, so that the entire nonsense-nonsense matrix is

$$N=\begin{pmatrix} A & \sqrt{3}B \\ \sqrt{3}\,B & B\end{pmatrix}. \qquad (4.2)$$

The matrix $B$ is the same as in the U(1) model [see Eq. (2.4)]. The matrix $A$ can be read from Eq. (3.1). As we discussed earlier, factorization in the $\vec{\rho}\vec{\rho}\to\vec{\rho}\vec{\rho}$ channel demands that the determinant of $N$ have a zero at $s=\mu^2$, and that $N$ be of rank four. In fact, one obtains the desired zero for $\det N$ at $s=\mu^2$ if $\mu^2=m^2$, but the rank of $N$ is only three. Thus factorization still fails, since the rank of $N$ is less than that of the complete Born matrix. Detailed calculation shows that one can express the result in a way similar to Eq. (2.3) of the U(1) case, i.e.,

$$(Y)_{sn}(N^{-1})_{nn}(Y)_{ns}=(X)_{ss}-3(S_s)_{ss}\ , \qquad (4.3)$$

where $Y=(Y_0,\sqrt{3}\,Y_1)$, $Y_0$ and $\sqrt{3}\,Y_1$ are the nonsense-sense matrices of the $\vec{\rho}\vec{\rho}\to\vec{\rho}\vec{\rho}$ and $\omega\omega\to\vec{\rho}\vec{\rho}$ processes, respectively, and $X$ is the sense-sense matrix of the $\vec{\rho}\vec{\rho}\to\vec{\rho}\vec{\rho}$ channel. The matrix $Y_1$ is the same as in the U(1) model [see Eq. (2.3)]. The matrices $Y_0$ and $X$ can be read from Eq. (3.1). This result again suggests that we consider the coupling of a fermion-antifermion channel to this system.

To obtain (4.3) a technical note is in order. In the equal-mass case, $m^2=\mu^2$, $\det N=0$, and $N^{-1}$ will not exist. However, $Y$ annihilates the eigenvector belonging to the zero eigenvalue of $N$, so that the inversion of (4.3) can be carried out in the orthogonal subspace corresponding to the nonsingular part of $N$.

Let us introduce a fermion with mass $M$ and interaction Lagrangian

$$\mathcal{L}_I = g\bar{\psi}\gamma_\mu(\alpha\omega_\mu + \vec{T}\cdot\vec{\rho}_\mu)\psi\,, \tag{4.4}$$

where $T$ is the appropriate isotopic-spin matrix and $\alpha g$ is the coupling constant of the fermion to the $\omega$. [Note that $\alpha$ can be any real number, so that the interaction (4.4) reduces the group U(2) to SU(2)× U(1).] The complete nonsense-nonsense matrix for the $I=0$, $J=0$ part for the reaction

$$\left.\begin{matrix}\vec{\rho}\vec{\rho}\\ \omega\omega\\ F\bar{F}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\vec{\rho}\vec{\rho}\\ \omega\omega\\ F\bar{F}\end{matrix}\right.$$

is

$$V_{\rm nn} = \begin{pmatrix} N & U^t \\ U & \lambda V_1\end{pmatrix}\,, \tag{4.5}$$

where $U = (\sqrt{3}\,\beta U_1, \gamma U_1)$. The sense-nonsense matrix for $\vec{\rho}\vec{\rho} \to \{\vec{\rho}\vec{\rho},\ \omega\omega$, and $F\bar{F}\}$ is

$$V_{\rm sn} = (Y, \sqrt{3}\ \beta Z_1)\,. \tag{4.6}$$

In the above $\beta$, $\gamma$, and $\lambda$ depend on the isospin $I$ and coupling parameter $\alpha$ of the fermion in (4.4). We find

$$\begin{aligned}\beta &= \tfrac{1}{3}[I(I+1)(2I+1)^{1/2}]\,,\\ \gamma &= \alpha^2(2I+1)^{1/2}\,,\\ \lambda &= \alpha^2 - I(I+1)\,.\end{aligned} \tag{4.7}$$

We can write a result similar to Eq. (2.8):

$$\begin{aligned}V_{\rm sn}\ V_{\rm nn}^{-1}\,V_{\rm ns} = YN^{-1}Y^t &+ (YN^{-1}U^t - \sqrt{3}\,\beta Z_1)(\lambda V - UN^{-1}U^t)^{-1}\\ &\times(YN^{-1}U^t - \sqrt{3}\,\beta Z_1)^t\,.\end{aligned} \tag{4.8}$$

Now

$$\begin{aligned}YN^{-1}U^t - \sqrt{3}\,\beta Z_1 = \sqrt{3}\,[(Y_0 - 3Y_1)(A-3B)^{-1}(\beta-\gamma)\,U_1^t\\ + \alpha Y_1 B^{-1}U_1^t - \beta Z_1]\,,\end{aligned} \tag{4.9}$$

$$UN^{-1}U^t = 3(\beta-\gamma)^2 U_1(A-3B)^{-1}U_1^t + \gamma^2 U_1 B^{-1}U_1^t\,. \tag{4.10}$$

Explicit calculation gives the interesting relations

$$\begin{aligned}(Y-3Y_1)(A-3B)^{-1}U_1^t &= Z_1\,,\\ U_1(A-3B)^{-1}U_1^t &= -V_1\,.\end{aligned} \tag{4.11}$$

The second term of Eq. (4.7) therefore becomes

$$3(Y_1B^{-1}U_1^t - Z_1)\left[\frac{\lambda+3(\beta-\gamma)^2}{\gamma^2}V_1 - U_1B^{-1}U_1^t\right]^{-1} \times (Y_1B^{-1}U_1^t - Z_1)^t\,.$$

By comparing to (4.3), we see that the first term of (4.7) will reproduce all of the $\vec{\rho}\vec{\rho}$-$\vec{\rho}\vec{\rho}$ sense-sense amplitude except for the pole term $S_s$. We therefore require that (4.9) reproduce the pole term. We note [cf. (2.8)] that it already has the correct residue of the $s$-channel pole. From the requirement that it have the correct pole position, we get the condition

$$\lambda + 3(\beta-\gamma)^2 = 12\gamma^2\,. \tag{4.12}$$

For given isospin $\alpha$ can be calculated from Eqs. (4.7) and (4.11). For instance, for $I=0$, we find $\alpha=\frac{1}{3}$. By using this condition one can check (with effort) that the whole 12×12 $I=0$ matrix of

$$\left.\begin{matrix}\vec{\rho}\vec{\rho}\\ \omega\omega\\ F\bar{F}\\ \sigma\sigma\\ \vec{s}\vec{s}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\vec{\rho}\vec{\rho}\\ \omega\omega\\ F\bar{F}\\ \sigma\sigma\\ \vec{s}\vec{s}\end{matrix}\right.$$

factorizes and is of rank four. Four trajectories are generated; one choosing sense and three choosing nonsense. We will list the complete 12×12 matrix in the Appendix for the interested reader.

## V. OTHER PROCESSES IN THE U(2) MODEL

We discuss in this section the situation at $J=0$ for the various channels of the U(2) model. To begin with, we list all possible two-body states that communicate with each other in the Born approximation. There are six sectors:

$$\begin{aligned}&I=0:\ \vec{\rho}\vec{\rho},\ \omega\omega,\ F\bar{F},\sigma\sigma,\vec{s}\vec{s};\\ &I=0:\ \vec{\rho}\vec{s},\ \omega\sigma;\\ &I=1:\ \omega\vec{\rho},\vec{\rho}\vec{s},F\bar{F},\vec{s}\sigma;\\ &I=1:\ \vec{\rho}\sigma,\ \omega\vec{s};\\ &I=2:\ \vec{\rho}\vec{\rho},F\bar{F},\vec{s}\vec{s};\\ &I=2:\ \vec{\rho}\vec{s}.\end{aligned} \tag{5.1}$$

We have calculated and displayed in the Appendix the relevant Born-approximation helicity amplitudes for these sectors, and checked whether factorization holds. Here we only discuss the results.

As already discussed in Sec. IV, factorization is established for the $I=0$ $\vec{\rho}\vec{\rho}$, $\omega\omega$, $F\bar{F}$, $\sigma\sigma$, and $\vec{s}\vec{s}$ channels if the condition given by Eq. (4.11) is satisfied. The sector has a complete Born amplitude expressible as a 12×12 matrix of rank four. Four Regge trajectories are generated, with one of them choosing sense and three of them choosing nonsense. The $I=0$ $\vec{\rho}\,\vec{s}$, $\omega\sigma$ and $I=1$ $\vec{\rho}\sigma$, $\omega\vec{s}$ sectors also factorize just as in the $V\sigma$ sector of the U(1) model. The 4×4 matrix for each of these two sectors is of rank one, and one nonsense-choosing Regge trajectory is generated. The results are similar to that of Eq. (2.19). As discussed in Sec. III, the $I=2$ sector of the $\vec{\rho}\,\vec{\rho}$ amplitude also factorizes. In the unequal-mass (equal-mass) case, the nonsense-nonsense matrix is of rank two (one), and two (one) nonsense-choosing Regge trajectories are generated. In the U(2) model, with the coupling of the $F\overline{F}$ channel with fermion isospin $I \geq 1$, the $I=2$ $\vec{\rho}\,\vec{\rho}$, $F\overline{F}$, $\vec{s}\,\vec{s}$ sector is described by a 7×7 matrix, and the rank of the nonsense-nonsense matrix is increased to three (two). There is no obvious reason for the factorization to persist. However, one can show using the identities (4.11) that the factorization still holds, the 7×7 matrix is of rank three (two), and three (two) nonsense-choosing Regge trajectories are generated. This remarkable result leads us to believe that the Reggeization might persist in higher-order calculations. Similarly, in the $I=2$ $\vec{\rho}\,\vec{s}$ sector, the 2×2 Born matrix also factorizes and is of rank one (in the unequal-mass case), so that one nonsense-choosing Regge trajectory is generated. In the equal-mass case the amplitude is identically zero. However, factorization fails for the $I=1$ $\{\omega\vec{\rho}, \vec{\rho}\,\vec{s}, F\overline{F}, \vec{s}\sigma\}$ sector (for any coupling strength of the fermion). [As in the $I=2$ case, the fermion gives zero contribution in the sense that $YB^{-1}U^t - Z = 0$, cf. Eq. (2.9).] This means that the scalar meson $\vec{s}$ does not lie on a Regge trajectory. So far we do not have any general explanation to offer for these results. It therefore remains an open question whether the enormous calculational effort required to obtain these results can be encompassed by a general analysis on the order of Mandelstam's counting argument, which is so valuable in understanding Reggeization at $J=1$ and $\frac{1}{2}$.

## VI. CONCLUSION

In this paper we have discussed the Reggeization at $J=0$ of some typical renormalizable Abelian and non-Abelian gauge models. Mandelstam counting suggests that in general the scalar states need not Reggeize at $J=0$, although they might for some special values of the parameters of the theories. We have investigated this possibility and we have achieved some partial success, a summary of which follows.

(i) We find that sectors which contain elementary scalar mesons ($\sigma$ and $\vec{s}$) are difficult to Reggeize. It is possible to factorize the Born matrix in the scalar meson ($\sigma$) sector in theories *whose gauge group contains a U(1) subgroup and fermions which are coupled with a special value of the coupling constant to the Abelian gauge mesons.* We have demonstrated this explicitly in a U(1) and a U(2) model. The generalization of this result to U($n$) should not be very difficult. We have not succeeded Reggeizing the $\vec{s}$ scalar meson. Higher-order calculations should shed more light on the question of Reggeization of the $\sigma$ scalar meson and the possible persistence of our findings from lowest order.

(ii) High-isospin (e.g., $I=2$) $J=0$ sectors seem to readily Reggeize, without any constraints on the parameters of the model. We believe that similar results will be encountered when arbitrary gauge models are discussed. We also believe that calculations performed in higher order will not change the conclusions derived from lowest-order calculations.

(iii) In the sectors defined by the quantum numbers of the scalar mesons that are gauged away in the U gauge, the $J=0$ Born matrix will factorize if $\mu^2 = m^2$ in both the U(1) and U(2) models considered. This might have some significance in that it is possible that there is some relation between the Reggeization of the $\vec{\rho}\sigma$ sector at $J=0$ and $J=1$.[10] If this is the case, the analogous situation in the $\omega\sigma$ sector would suggest that $\omega$ would Reggeize at $J=1$. So far we have not been able to find nonsense states which can generate a Regge trajectory for an Abelian vector meson.

We conclude with some general comments.

(a) Recently one of us (H.S.T) and Lee, Rawls, and Wong[11] have separately arrived at the conclusion that the requirements of factorization of the Born matrix at $J=1$ give the same constraints on the Lagrangian as those found from the requirement of unitarity bounds in generalized Lagrangian theories containing vector mesons.[4] In fact, these requirements demand that the theory be of SBS type for the non-Abelian sectors of the theory. However, not all SBS theories Reggeize all vector mesons because of the absence of a sufficient number of nonsense states with the U(1)-gauge model as a trivial example. The requirement of the Regge factorization condition is thus more restrictive than that of unitarity bounds, since factorization demands that the rank of the complete Born helicity matrix be equal to that of the nonsense-nonsense matrix. There is no reason to expect that the requirement that the unitarity bound be satisfied will give some information about the rank of this matrix.

(b) We feel that our examples shed some light on the Mandelstam argument, and the two aspects it emphasizes: good high-energy behavior and counting of constraints and free parameters. At $J=1$ or $J=\frac{1}{2}$ the counting conditions indicate that Reggeization should take place, and the fact that it does in examples with renormalizable field theories and does not in nonrenormalizable ones illustrates the importance of good high-energy behavior. (However, good high-energy behavior for $J=1$ is not sufficient to guarantee Reggeization if Mandelstam counting does not hold, i.e., when there are more elementary vector mesons than the rank of the nonsense matrix.) At $J=0$ our models have good high-energy behavior but the counting conditions do not hold, and indeed we find that Reggeization will not take place in general.

The fact that we can achieve factorization for special cases (equal masses or special values of the coupling constant) might be considered an accident which will disappear when higher-order calculations are performed. Yet minor miracles take place which might support some faith in persistence of Reggeization in higher orders: It may be an accident that in the U(2) model with fermions one can factorize the $I=0\ \vec{\rho}\vec{\rho}$ part of the Born approximation. But it is remarkable and unexpected that the condition which ensures the above factorization leads to factorization of the whole 12×12 matrix for the $I=0\ \vec{\rho}\vec{\rho}$ sector. Similarly, the factorization of the $I=2\ \vec{\rho}\vec{\rho}$ amplitude would actually be destroyed by the coupling of the fermions were it not for the identities (4.8). There is no obvious reason why they should hold. Nonetheless, in the absence of Mandelstam counting, explicit higher-order calculations are required to shed more light on the question of Reggeization at $J=0$. The whole matter will be discussed further in a future publication.[7]

(c) It is not clear what relations our results have to the real world. Our finding that spin-1 and spin-$\frac{1}{2}$ particles Reggeize is consistent with a phenomenological interpretation of observed Regge trajectories. However, few scalar particles have been observed in nature, and this fact, together with the difficulty we have in Reggeizing scalars, suggests that a phenomenological interpretation is not complete. We have discussed elsewhere the relevance of our findings to the bootstrap of low-spin particles.[12]

(d) We have emphasized the role that the scalar meson plays in bringing the high-energy behavior of vector-vector scattering within unitarity bounds. It is conceivable that in theories now being considered with "color" gluons and quarks[13] one could dispense with the scalars. In such theories, masses would arise from dynamical symmetry breaking,[14] and good high-energy behavior would be achieved without scalar-meson exchange, with all observed particles composite and lying on Regge trajectories. Since this cannot be studied in perturbation theory, there is no obvious way in which our findings are related to this possibility.

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## APPENDIX

In this appendix, we list the matrices $V_{ss}$, $V_{sn}$, and $V_{nn}$ (see Appendix B of paper I for their definitions) for the processes we discussed in the main text.

### 1. U(1) model

$$V_{ss}=\begin{array}{c} \\ VV \\ F\bar{F} \\ \sigma\sigma \end{array}\!\!\begin{array}{c} \begin{array}{ccc} VV & F\bar{F} & \sigma\sigma \end{array} \\ \begin{pmatrix} X_{11} & \alpha^2X_{12} & X_{13} \\ \alpha^2X_{21} & \alpha^2X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} \end{array}, \tag{A1a}$$

$$V_{ns}=\begin{array}{c} \\ VV \\ F\bar{F} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} VV & F\bar{F} & \sigma\sigma \end{array} \\ \begin{pmatrix} Y_1 & \alpha^2Y_2 & Y_3 \\ \alpha^2Z_1 & Z_2 & Z_3 \end{pmatrix} \end{array}, \tag{A1b}$$

$$V_{nn}=\begin{array}{c} \\ VV \\ F\bar{F} \end{array}\!\!\begin{array}{c} \begin{array}{cc} VV & F\bar{F} \end{array} \\ \begin{pmatrix} B & \alpha^2U_1^\dagger \\ \alpha^2U_1 & \alpha^2V_1 \end{pmatrix} \end{array}. \tag{A1c}$$

The matrices $X_{11}$, $Y_1$, $Z_1$, and $V_{nn}$ have already been given in the text [Eqs. (2.2), (2.3), (2.4), (2.11), (2.12), and (2.13)]. Since there is no direct scalar-fermion coupling, $X_{23}=X_{32}=Z_3=0$. We list below the remaining entries (in the equal-mass case, $\mu^2=m^2$):

$$(X_{12})_{11,1/2\,1/2}=\frac{m^2}{k^2p}, \tag{A2a}$$

$$(X_{11})_{00,\,1/2\,1/2}=\frac{-2m^2}{k^2p}, \tag{A2b}$$

$$(X_{22})_{1/2\,1/2,1/2\,1/2}=\frac{1}{2p^2}, \tag{A2c}$$

$$(X_{13})_{0,11}=\frac{-3m^2}{s-m^2}+\frac{m^2}{2k^2}, \tag{A2d}$$

$$(X_{13})_{0,00}=\frac{-3(2E^2-m^2)}{s-m^2}+\frac{m^2}{k^2}, \tag{A2e}$$

$$(X_{33})=\frac{3}{2}\left(1+\frac{3m^2}{s-m^2}\right); \tag{A2f}$$

$$(Y_2)_{01,1/2\,1/2}=\frac{\sqrt{2}\,mE}{k^2p}, \tag{A3a}$$

$$(Y_2)_{1\,-1,\,1/2\,1/2}=\frac{-i\sqrt{2}\,(2E^2-m^2)}{k^2p}, \tag{A3b}$$

$$(Y_3)_{01}=\frac{mE}{\sqrt{2}\,k^2}, \tag{A3c}$$

$$(Y_3)_{1\,-1}=-i\frac{1}{\sqrt{2}}\frac{2E^2-m^2}{k^2}, \tag{A3d}$$

$$(Z_2)_{1/2\,1/2,1/2\,-1/2}=-\frac{1}{2}\frac{E}{Mp^2}. \tag{A3e}$$

### 2. U(2) model

(*a*) $I=0$ $\vec\rho\vec\rho$, $\omega\omega$, $F\bar F$, $\sigma\sigma$, $\vec s\vec s$ *sector.*

$$V_{ss}=\begin{array}{c} \\ \vec\rho\vec\rho \\ \omega\omega \\ F\bar F \\ \sigma\sigma \\ \vec s\vec s \end{array}\begin{array}{c} \begin{array}{ccccc} \vec\rho\vec\rho & \omega\omega & F\bar F & \sigma\sigma & \vec s\vec s \end{array} \\ \left[\begin{array}{ccccc} X_{00} & \sqrt{3}X_{11} & \sqrt{3}\beta X_{12} & \sqrt{3}X_{13} & 3X_{13} \\ \sqrt{3}X_{11} & X_{11} & \gamma X_{12} & X_{13} & \sqrt{3}X_{13} \\ \sqrt{3}\beta X_{21} & \gamma X_{21} & \lambda X_{22} & X_{23} & \sqrt{3}X_{23} \\ \sqrt{3}X_{31} & X_{31} & X_{32} & X_{33} & \sqrt{3}X_{33} \\ 3X_{31} & \sqrt{3}X_{31} & \sqrt{3}X_{32} & \sqrt{3}X_{32} & 3X_{33} \end{array}\right]\end{array}, \tag{A4a}$$

$$V_{ns}=\begin{array}{c} \\ \vec\rho\vec\rho \\ \omega\omega \\ F\bar F \end{array}\begin{array}{c} \begin{array}{ccccc} \vec\rho\vec\rho & \omega\omega & F\bar F & \sigma\sigma & \vec s\vec s \end{array} \\ \left(\begin{array}{ccccc} Y_0 & \sqrt{3}Y_1 & \sqrt{3}\beta Y_2 & \sqrt{3}Y_3 & 3Y_3 \\ \sqrt{3}Y_1 & Y_1 & \gamma Y_2 & Y_3 & \sqrt{3}Y_3 \\ \sqrt{3}\beta Z_1 & \gamma Z_1 & \lambda Z_2 & Z_3 & \sqrt{3}Z_3 \end{array}\right)\end{array}, \tag{A4b}$$

$$V_{nn}=\begin{array}{c} \\ \vec\rho\vec\rho \\ \omega\omega \\ F\bar F \end{array}\begin{array}{c} \begin{array}{ccc} \vec\rho\vec\rho & \omega\omega & F\bar F \end{array} \\ \left(\begin{array}{ccc} A & \sqrt{3}B & \sqrt{3}\beta U_1^t \\ \sqrt{3}B & B & \gamma U_1^t \\ \sqrt{3}\beta U_1 & \gamma U_1 & \lambda V_1 \end{array}\right)\end{array}. \tag{A4c}$$

Most of the amplitudes in this model can be readily obtained from those of the U(1) model. For most processes the only differences are because of isospin factors. The other major difference is the existence of $\vec\rho$ meson self-interaction, which gives additional contributions to the $\vec\rho\vec\rho\to\vec\rho\vec\rho$ amplitude. The matrices $A$, $Y_0$, $X_{00}$ can be obtained from Eq. (3.1).

(*b*) $I=0$ $\vec\rho\vec s$, $\omega\sigma$ *sector and* $I=1$ $\vec\rho\sigma$, $\omega\vec s$ *sector.* The complete Born matrices are

$$\begin{array}{c} \\ \vec\rho\vec s \\ \omega\sigma \end{array}\begin{array}{c} \begin{array}{cc} \vec\rho\vec s & \omega\sigma \end{array} \\ \left(\begin{array}{cc} 3H & \sqrt{3}H \\ \sqrt{3}H & H \end{array}\right)\end{array}, \tag{A5a}$$

$$\begin{array}{c} \\ \vec\rho\sigma \\ \omega\vec s \end{array}\begin{array}{c} \begin{array}{cc} \vec\rho\sigma & \omega\vec s \end{array} \\ \left(\begin{array}{cc} H & H \\ H & H \end{array}\right)\end{array}. \tag{A5b}$$

The matrix $H$ has been given in Eq. (2.19).

(*c*) $I=2$ $\vec\rho\vec\rho$, $F\bar F$, $\vec s\vec s$; $I=2$ $\vec\rho\vec s$ *sector.* The $\vec\rho\vec\rho\to\vec\rho\vec\rho$ amplitudes can be read from Eq. (3.1). $\vec\rho\vec\rho\to F\bar F$, $F\bar F\to F\bar F$ is essentially the same as in the $I=0$ case except for isospin factors. All other processes are zero in the equal-mass case.

(*d*) $I=1$, $\vec\rho\omega$, $\vec\rho\vec s$, $F\bar F$, $\vec s\sigma$ *sector.*

$$V_{ss}=\begin{array}{c} \\ \vec\rho\omega \\ \vec\rho\vec s \\ F\bar F \\ \vec s\sigma \end{array}\begin{array}{c} \begin{array}{cccc} \vec\rho\omega & \vec\rho\vec s & F\bar F & \vec s\sigma \end{array} \\ \left[\begin{array}{cccc} X_{11} & X_{14} & aX_{12} & X_{13} \\ X_{41} & X_{44} & 0 & X_{43} \\ aX_{21} & 0 & bX_{22} & 0 \\ X_{31} & X_{34} & 0 & X_{33} \end{array}\right]\end{array} \tag{A6a}$$

$$V_{ns}=\begin{array}{c} \\ \vec\rho\omega \\ \vec\rho\vec s \\ F\bar F \end{array}\begin{array}{c} \begin{array}{cccc} \vec\rho\omega & \vec\rho\vec s & F\bar F & \vec s\sigma \end{array} \\ \left[\begin{array}{cccc} Y_1 & Y_4 & aY_2 & Y_3 \\ Y_1' & Y_4' & 0 & Y_3' \\ aY_2 & 0 & bZ_2 & 0 \end{array}\right]\end{array}, \tag{A6b}$$

$$V_{nn}=\begin{array}{c} \\ \vec\rho\omega \\ \vec\rho\vec s \\ F\bar F \end{array}\begin{array}{c} \begin{array}{ccc} \vec\rho\omega & \vec\rho\vec s & F\bar F \end{array} \\ \left[\begin{array}{ccc} B & U_2^t & aU_1^t \\ U_2 & V_2 & 0 \\ aU_1 & 0 & bV_1 \end{array}\right]\end{array}, \tag{A6c}$$

where $a$ and $b$ are functions of $I$ and $\alpha$:

$$a=\alpha^2, \qquad b=\alpha^2-1+I(I+1); \tag{A7}$$

$$(X_{14})_{11,0}=\frac{4\sqrt{2}\,kE}{s-m^2}-\frac{1}{\sqrt{2}}\frac{E(2E^2-m^2)}{k^3}, \tag{A8a}$$

$$(X_{14})_{00,0}=\frac{4\sqrt{2}\,kE(2E^2-m^2)}{m^2(s-m^2)}-\frac{\sqrt{2}\,E(2E^4-5E^2m^2+4m^4)}{m^2k^3}, \tag{A8b}$$

$$(X_{44})_{0,0}=-\frac{4k^2}{s-m^2}-\frac{2(2E^2+m^2)}{k^2}, \tag{A8c}$$

$$(X_{43})_0=\frac{6\sqrt{2}\,kE}{s-m^2}-\frac{\sqrt{2}\,E}{k}; \tag{A8d}$$

$$(Y_1')_{11,1}=-\frac{m(2E^2-3m^2)}{2k^3}, \tag{A9a}$$

$$(Y_1')_{00,1}=\frac{m(E^2-2m^2)}{k^3}, \tag{A9b}$$

$$(Y_4)_{01,0}=\frac{m^3}{k^3}, \tag{A9c}$$

$$(Y_4)_{1-1,0} = -i\,\frac{E(2E^2-m^2)}{k^3}\,; \tag{A9d}$$

$$(Y_4')_{1,0} = \frac{2\sqrt{2}\,Em}{k^2}\,, \tag{A10a}$$

$$(Y_3') = \frac{m}{k}\,; \tag{A10b}$$

and

$$(U_2)_{01,1} = \frac{-E(E^2-2m^2)}{\sqrt{2}\,k^3}\,, \tag{A11a}$$

$$(U_2)_{1-1,1} = \frac{-im(2E^2-m^2)}{\sqrt{2}\,k^3}\,, \tag{A11b}$$

$$(V_2)_{1,1} = \frac{E^2+m^2}{k^2}\,. \tag{A11c}$$

All the other quantities have already been defined.

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