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Strong-coupling limit of Regge trajectories in the ϕ^3 ladder model

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We extend to $t \neq 0$ a technique, employing the Bethe-Salpeter equation, developed by Chang and Rosner to study the strong-coupling limits of ladder-graph models. We are able to calculate the first nontrivial t-dependent term in the trajectory function $\alpha(t)$ for ϕ^3 theory. The behavior of $\alpha(t)$ depends upon whether the exchanged particle is massless. We also indicate how daughter trajectories may be calculated.

I. INTRODUCTION

The strong-coupling behavior $g \to \infty$ of Regge trajectories in ϕ^3 ladder amplitudes at $t=0$ has been widely discussed. The only known analytic solution is for massless-particle exchange ($m=0$) with $t=0$ given by Wick,[1] Cutkosky,[1] and Nakanishi.[2] Large-coupling behavior has been studied by solving the Bethe-Salpeter (BS) equation approximately[3-6] and by investigating individual terms in the sum of ladder amplitudes.[7,8] Small-t behavior has been investigated by solving the BS equation numerically[4] and by examining individual ladder amplitude terms with large coupling.[8]

In a recent paper, Chang and Rosner[6] developed a technique for studying strong-coupling limits of ladder-graph models at $t=0$ using the BS equation in Euclidean coordinate space. In this paper we wish to extend their method to study ϕ^3 theory for $t \neq 0$ for both massive- and massless-particle exchange.

Briefly, Chang and Rosner's approach is the following: At $t=0$ the BS equation can be expressed in Euclidean coordinate space as a fourth-order differential equation which is O(4)-symmetric. The equation is expanded in four-dimensional spherical harmonics and the radial equation is studied. The leading Regge trajectory $\alpha(0)$ is determined by the maximum allowed angular momentum n in the t channel $[\alpha(0)=n-1]$. In the strong-coupling limit, n is large and a first approximation to the radial equation leads to a relation $n=n(r)$, where r is the radius of the orbit. The maximum $n=n_0=n(r_0)$ can be obtained, and the BS equation can then be expanded about these values to obtain inverse-g corrections to any desired order.

For $t \neq 0$ the BS equation is O(3)-symmetric rather than O(4)-symmetric. If we choose to work in the c.m. frame the equation has the usual invariance under spatial rotations, but the radial equation is now coupled to other four-dimensional angular momentum states.[9] Nevertheless we are able to calculate $\alpha(t)$ in much the same manner as was used to obtain $\alpha(0)$. More specifically, we have obtained the first nontrivial t-dependent term in the trajectory function in the strong-coupling limit. The behavior of $\alpha(t)$ depends upon whether the exchanged particle is massless. The main results are given by Eq. (3.4) ($m=0$) and (4.15) ($m \neq 0$).

Associated with the calculation of $\alpha(t)$ is the problem of determining its daughter trajectories. A consequence of the O(4) symmetry at $t=0$ is the existence of daughter trajectories with $\alpha(0)=n-2, n-3, \ldots$.[10,11] For $t \neq 0$ this symmetry is broken. The daughters do not have the same t dependence as the parent and hence they are no longer spaced one unit apart. We shall indicate how these daughters may also be calculated.

II. BETHE-SALPETER EQUATION

We begin by briefly indicating the derivation of the differential BS equation we shall study. The BS equation for a wave function is given by

$$[(\tfrac{1}{2}P+p)^2-\mu^2][(\tfrac{1}{2}P-p)^2-\mu^2]\,\phi(P,p) = \int \frac{d^4p'}{i(2\pi)^4} K(p,p')\,\phi(P,p')\,, \quad (2.1)$$

where $P^2=t$ is the square of the energy in the c.m. frame, μ is the mass of the particles forming the bound state, and $K(p,p')$ is an irreducible kernel describing the interaction. For a ϕ^3 ladder amplitude the kernel is given by (Fig. 1)

$$K(p,p')=\frac{-g^2}{(p-p')^2-m^2+i\epsilon}\,, \quad (2.2)$$

where m is the mass of the exchanged particles and g is the coupling constant.

For $t<4\mu^2$ we can perform a Wick rotation on p and p' to obtain

$$[(-p^2+\tfrac{1}{4}t-\mu^2)^2+(p\cdot P)^2]\,\phi(P,p) = \int \frac{d^4p'}{(2\pi)^4} K(p,p')\,\phi(P,p') \quad (2.3)$$

where p and p' are now Euclidean vectors. We can Fourier-transform this equation and expand the resulting coordinate-space equation in four-dimensional spherical harmonics. The detailed transformation is given in Ref. 9 and we shall not repeat it here. The term $(p\cdot P)^2$ breaks the O(4) symmetry of the equation, and so the resulting radial equation is necessarily a coupled one. However, by choosing the c.m. frame we preserve the usual invariance under spatial rotations. The resulting radial equation, derived in Ref. 9, is

$$[(-p_r{}^2+\tfrac{1}{4}t-\mu^2)^2+p_r{}^2\,t f^l_{n,n}]\,\phi^l_n(t,r)$$
$$-p_r{}^2\,t[f^l_{n,n-2}\,\phi^l_{n-2}(t,r)+f^l_{n,n+2}\,\phi^l_{n+2}(t,r)]$$
$$=V(r)\,\phi^l_n(t,r)\,, \tag{2.4}$$

where $p_r{}^2$ is the differential operator

$$p_r{}^2=-\left(\frac{d^2}{dr^2}+\frac{3}{r}\frac{d}{dr}+\frac{1-n^2}{r^2}\right), \tag{2.5}$$

n is the four-dimensional angular momentum, and l is the usual three-dimensional angular momentum. The coupling coefficients are given by

$$f^l_{n,n}=\frac{(n-1)(n+1)-l(l+1)}{2(n-1)(n+1)}\,,$$

$$f^l_{n,n+2}=\frac{1}{4(n+1)}\times\left[\frac{(n+l+2)(n+l+1)(n-l+1)(n-l)}{n(n+2)}\right]^{1/2}, \tag{2.6}$$

$$f^l_{n,n-2}=\frac{1}{4(n-1)}\times\left[\frac{(n+l)(n+l-1)(n-l-1)(n-l-2)}{n(n-2)}\right]^{1/2}.$$

The potential $V(r)$ is just the Fourier transform of the kernel $K(p,p')$:

$$V(r)=\left[\int\frac{d^4p}{(2\pi)^4}K(p,0)\,e^{-ip\cdot x}\right]_{\text{Euclidean}}$$
$$=\begin{cases}\dfrac{g^2}{4\pi^2r^2}\,, & m=0\\[2ex] g^2\,\dfrac{mK_1(mr)}{4\pi^2r}\,, & m\neq 0\,.\end{cases} \tag{2.7}$$

Equation (2.4) may be written in a form which will later be convenient:

$$\frac{1}{r^4}\left[\left(r\frac{d}{dr}\right)^2-2r\frac{d}{dr}-n^2+1-(\mu^2-\tfrac{1}{4}t)r^2\right]\left[\left(r\frac{d}{dr}\right)^2+2r\frac{d}{dr}-n^2+1-(\mu^2-\tfrac{1}{4}t)r^2\right]\phi^l_n(r)$$
$$-\frac{1}{r^2}\left[\left(r\frac{d}{dr}\right)^2+2r\frac{d}{dr}-n^2+1\right]t[f^l_{n,n}\,\phi^l_n(r)-f^l_{n,n+2}\,\phi^l_{n+2}(r)-f^l_{n,n-2}\,\phi^l_{n-2}(r)]=V(r)\,\phi^l_n(r)\,. \tag{2.8}$$

The allowed l values are $l=n-1,\ n-2,\ldots$. We analytically continue Eq. (2.8) in n and l, leaving $n-l$ integral. We then concentrate on determining the maximum n for a given t. To determine the leading parent trajectory, we maximize n with $l=n-1$. The first daughter trajectory is obtained by maximizing n with $l=n-2$ (which may result in a different n) and so on. At $t=0$ Eq. (2.8) reduces to the equation studied by Chang and Rosner.[6] In that case the equation has no l dependence and the maximum n value gives parent $[\alpha(0)=l=n-1]$ and daughter $[\alpha(0)=l=n-2,\ n-3,\ldots]$ trajectories.

We shall consider the two cases $m=0$ and $m\neq 0$ separately.

III. $m=0$

In the strong-coupling limit n is large and we obtain a first approximation to n by neglecting the differential terms relative to n^2 in Eq. (2.8).[12] For a finite $\lambda=n-l$ ($\lambda=1$ is the parent trajectory) the coupling coefficients, Eqs. (2.6), may be approximated by

$$f^{n-\lambda}_{n,n}=\frac{2\lambda-1}{2n}+O(1/n^2)\,,$$

$$f^{n-\lambda}_{n,n-2}=\frac{[(\lambda-1)(\lambda-2)]^{1/2}}{2n}+O(1/n^2)\,, \tag{3.1}$$

$$f^{n-\lambda}_{n,n+2}=\frac{[\lambda(\lambda+1)]^{1/2}}{2n}+O(1/n^2)\,.$$

Our first approximation to Eq. (2.8) now becomes

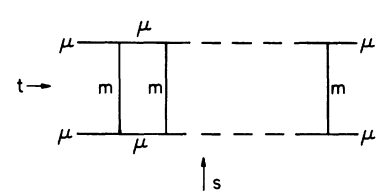

FIG. 1. Ladder amplitudes in ϕ^3 theory.

$$\left[\frac{n^2}{r^2}+(\mu^2-\tfrac{1}{4}t)\right]^2\phi_n^{n-\lambda}-V(r)\,\phi_n^{n-\lambda}=-\frac{n}{2r^2}t\{(2\lambda-1)\,\phi_n^{n-\lambda}-[(\lambda-1)(\lambda-2)]^{1/2}\,\phi_{n-2}^{n-\lambda}-[\lambda(\lambda+1)]^{1/2}\,\phi_{n+2}^{n-\lambda}\}\,. \tag{3.2}$$

The right-hand side of (3.2) may be neglected for large n provided neither $\phi_{n-2}^{n-\lambda}$ nor $\phi_{n+2}^{n-\lambda}$ is $O(n)$ larger than $\phi_n^{n-\lambda}$. We shall see later that this proves to be a self-consistent assumption. Therefore we have a relation for n as a function of r,

$$n^2(r)=r^2[V(r)]^{1/2}-(\mu^2-\tfrac{1}{4}t)\,r^2\,, \tag{3.3}$$

and for $V(r)=g^2/4\pi^2r^2$ ($m=0$) the maximum value of n is

$$n_0=\frac{g}{4\pi(\mu^2-\frac{1}{4}t)^{1/2}} \tag{3.4}$$

at

$$r_0=\frac{g}{4\pi(\mu^2-\frac{1}{4}t)}\ . \tag{3.5}$$

The above expression for n_0 was obtained for small t by Chang and Yan.[8]

We now proceed to calculate the $O(1)$ correction term to Eq. (3.4) by expanding Eq. (2.8) about n_0 and r_0. We will no longer be able to neglect the l-dependent terms and hence must deal with coupled equations. We begin by making a transformation of variables introduced by Chang and Rosner, $r=r_0e^{y/\sqrt{n_0}}$. As we shall see later, including the factor $1/\sqrt{n_0}$ makes the expectation values of y and d/dy in Eq. (3.6) of order 1. After multiplying Eq. (2.8) by r^4 and making the substitution $r\,d/dr=\sqrt{n_0}\,d/dy$, we obtain

$$\begin{aligned}&\Big(n_0^2\frac{d^4}{dy^4}-2[n^2+1+(\mu^2-\tfrac{1}{4}t)\,r_0^2e^{2y/\sqrt{n_0}}]\,n_0\frac{d^2}{dy^2}-4(\mu^2-\tfrac{1}{4}t)\,r_0^2e^{2y/\sqrt{n_0}}\sqrt{n_0}\,\frac{d}{dy}\\&\quad+[n^2-1+(\mu^2-\tfrac{1}{4}t)\,r_0^2e^{2y/\sqrt{n_0}}]^2\Big)\,\phi_n^{n-\lambda}(t,y)-r_0^4e^{4y/\sqrt{n_0}}\,V(r)\,\phi_n^{n-\lambda}(t,y)\\&\quad-tr_0^2e^{2y/\sqrt{n_0}}\Big(n_0\frac{d^2}{dy^2}+2\sqrt{n_0}\,\frac{d}{dy}+1-n^2\Big)[f_{n,n}^{n-\lambda}\,\phi_n^{n-\lambda}(t,y)-f_{n,n-2}^{n-\lambda}\,\phi_{n-2}^{n-\lambda}(t,y)-f_{n,n+2}^{n-\lambda}\,\phi_{n,n+2}^{n-\lambda}(t,y)]=0\,.\end{aligned} \tag{3.6}$$

For the moment, we assume that d/dy and d^2/dy^2 are of $O(1)$. We shall see that the resulting equation reduces to a harmonic oscillator, with the above assumption proving correct. With this assumption in mind we keep only terms of $O(n^3)$ or larger in Eq. (3.6). We substitute in (3.1), (3.4), and (3.5), and with some rearranging obtain

$$\begin{aligned}&\Big(-\frac{d^2}{dy^2}+y^2\Big)\phi_n^{n-\lambda}(t,y)+\frac{(2\lambda-1)\,t}{8(\mu^2-\frac{1}{4}t)}\,\phi_n^{n-\lambda}(t,y)\\&\quad-\frac{t}{8(\mu^2-\frac{1}{4}t)}\{[(\lambda-1)(\lambda-2)]^{1/2}\,\phi_{n-2}^{n-\lambda}(t,y)+[\lambda(\lambda+1)]^{1/2}\,\phi_{n+2}^{n-\lambda}(t,y)\}=2(n_0-n)\,\phi_n^{n-\lambda}(t,y)\,.\end{aligned} \tag{3.7}$$

Coupled to Eq. (3.7) we also have equations for $\phi_{n+K}^{n-\lambda}(t,y)$, with $K=-\lambda+1,-\lambda+3,\ldots,-2,0,2,\ldots$ for odd λ's and $K=-\lambda+2,-\lambda+4,\ldots,-2,0,2,\ldots$ for even λ's. The lower bound on K comes from the requirement that $l=n-\lambda\leq n+K-1$. Using (2.6), (3.1), and (3.6) we can write an equation for $\phi_{n+K}^{n-\lambda}(t,y)$ (see Ref. 13):

$$\begin{aligned}&\Big(-\frac{d^2}{dy^2}+y^2\Big)\phi_{n+K}^{n-\lambda}(t,y)+\frac{2(\lambda+K)-1}{8(\mu^2-\frac{1}{4}t)}\,t\,\phi_{n+K}^{n-\lambda}(t,y)\\&\quad-\frac{t}{8(\mu^2-\frac{1}{4}t)}\{[(\lambda+K-1)(\lambda+K-2)]^{1/2}\,\phi_{n+K-2}^{n-\lambda}(t,y)+[(\lambda+K)(\lambda+K+1)]^{1/2}\,\phi_{n+K+2}^{n-\lambda}(t,y)\}\\&\quad=2(n_0-n-K)\,\phi_{n+K}^{n-\lambda}(t,y),\quad K=-\lambda+1,\ldots,-1,0,1,\ldots\,.\end{aligned} \tag{3.8}$$

Equation (3.7) is just a special case of Eq. (3.8) with $K=0$. To solve Eq. (3.8) for a given n and the states coupled to it we assume that the solution is of the following separable form[14]:

$$\phi_{n+K}^{n-\lambda}(t,y)=\beta_K^{n-\lambda}(t)\,\phi_n^{n-\lambda}(0,y)\,,$$

$$K=-\lambda+1\text{ or }-\lambda+2,\ldots,-2,0,2,\ldots \tag{3.9}$$

with $\beta_0^{n-\lambda}(t)=1$ and $\beta_K^{n-\lambda}(0)=0$ for $K\neq 0$. Taking $\beta_0^{n-\lambda}(t)=1$ for all values of t fixes the normalization of the wave functions.

In writing Eq. (3.9) we have assumed that at $t=0$ $\phi_{n+K}^{n-\lambda}(0,y)=0$ for $K\neq 0$. This is not true. In fact there are nonzero solutions at $t=0$ for $n=n_{\max}-n_r$ ($n_r=0,1,\ldots,\lambda-1$), corresponding to different Lorentz poles. However, we shall show in the Appendix that these solutions for different n_r do not mix at $t\neq 0$ up to $O(1/n)$ and so we may consider the solutions for each n_r separately as we are doing here. Using Eq. (3.9), Eq. (3.7) may now be written as

$$\left(-\frac{d^2}{dy^2}+y^2\right)\phi_n^{n-\lambda}(y)=\left(2(n_0-n)+\frac{t}{8(\mu^2-\frac{1}{4}t)}\left\{[(\lambda-1)(\lambda-2)]^{1/2}\beta_{-2}^{n-\lambda}(t)+[\lambda(\lambda+1)]^{1/2}\beta_2^{n-\lambda}(t)-(2\lambda-1)\right\}\right)\phi_n^{n-\lambda}(y)\,, \quad (3.10)$$

which is just the harmonic-oscillator equation having eigenvalues

$$E_{n_r}=n_0-n+\frac{t}{16(\mu^2-\frac{1}{4}t)}\left\{[(\lambda-1)(\lambda-2)]^{1/2}\beta_{-2}^{n-\lambda}(t)+[\lambda(\lambda+1)]^{1/2}\beta_2^{n-\lambda}(t)-(2\lambda-1)\right\}=n_r+\tfrac{1}{2}\,,\quad n_r=0,1,2,\ldots\,. \quad (3.11)$$

Solving for n we obtain

$$n=\frac{g}{4\pi(\mu^2-\frac{1}{4}t)^{1/2}}+\frac{t}{16(\mu^2-\frac{1}{4}t)}\left\{[(\lambda-1)(\lambda-2)]^{1/2}\beta_{-2}^{n-\lambda}(t)+[\lambda(\lambda+1)]^{1/2}\beta_2^{n-\lambda}(t)-(2\lambda-1)\right\}-n_r-\tfrac{1}{2}+O(1/n_0)\,. \quad (3.12)$$

What we have found is a family (one parent with daughters for each n_r) of Regge poles. Notice from Eq. (3.12) that the daughter trajectories $[\alpha(t)=n-\lambda;\ \lambda\geq 2]$ are not spaced one unit apart for $t\neq 0$ because n depends on λ.

Equation (3.12) involves the unknown functions $\beta_{-2}^{n-\lambda}(t)$ and $\beta_2^{n-\lambda}(t)$, but we have not yet made use of Eq. (3.8). We can obtain a set of coupled algebraic equations for the coefficients $\beta_K^{n-\lambda}(t)$ by putting Eqs. (3.9) and (3.10) in (3.8). The resulting equations are

$$\left\{[(\lambda-1)(\lambda-2)]^{1/2}\beta_{-2}^{n-\lambda}(t)+[\lambda(\lambda+1)]^{1/2}\beta_2^{n-\lambda}(t)+\frac{16(\mu^2-\frac{1}{4}t)K}{t}+2K\right\}\beta_{K+2}^{n-\lambda}(t)=[(\lambda+K-1)(\lambda+K-2)]^{1/2}\beta_{K-2}^{n-\lambda}(t)+[(\lambda+K+1)(\lambda+K)]^{1/2}\beta_{K+2}^{n-\lambda}(t)\,,$$
$$K=-\lambda+1 \text{ or } -\lambda+2,\ldots,-2,0,2,\ldots\,. \quad (3.13)$$

In principle this set of equations is solvable. However, since they are nonlinear we are unable to obtain an exact solution. For small t, one could use these equations to generate a power series in t for $\beta_{-2}^{n-\lambda}(t)$ and $\beta_2^{n-\lambda}(t)$. We will not do that here.

IV. $m\neq 0$

We can see from Eq. (3.2) that the largest t-dependent term in n can be obtained by substituting $\mu^2-\frac{1}{4}t$ for μ^2 in the $t=0$ result. This is because the term $(2n^2/r^2)(\mu^2-\frac{1}{4}t)\,\phi_n^{n-\lambda}$ is $O(n)$ larger than the right-hand side of Eq. (3.2). Chang and Rosner found for the $m\neq 0$ case that n does not depend on μ^2 up to $O(1/n_0)$. We will not repeat that calculation here but merely quote the results so that we may use them to calculate the first t-dependent term.

For the first approximation, we maximize n in Eq. (3.3) using the potential $V(r)=g^2mK_1(mr)/4\pi^2r$. The results are

$$n_0=1.4669\,(g/4\pi m)^{1/2} \quad (4.1)$$

and

$$r_0=2.3867/m\,. \quad (4.2)$$

For the second-order correction, $r^4V(r)$ is expanded about its maximum, r_0, in terms of the variable y introduced in the last section:

$$r^4V(r)=\frac{g^2}{4\pi^2m^2}(mr)^3K_1(mr)=n_0^4\left[1-\frac{2\omega^2y^2}{n_0}+O\left(\frac{y^3}{n_0^{3/2}}\right)\right]\,, \quad (4.3)$$

where

$$\omega^2=-\frac{1}{4}\left(\frac{4\pi^2m^2}{g^2}\right)\left(r\frac{d}{dr}\right)^2[r^4V(r)]\Big|_{r=r_0}=0.5759\ .$$

Keeping terms of $O(n^3)$ and larger in Eq. (3.6), once again we obtain a harmonic-oscillator equation,

$$\left(-\frac{d^2}{dy^2}+\omega^2y^2\right)\phi_{0n}^l=\frac{1}{2}\left(n_0-\frac{n^4}{n_0^3}\right)\phi_{0n}^l\ , \tag{4.4}$$

and hence

$$E_{n_r}=\tfrac{1}{4}(n_0-n^4/n_0^3)=(n_r+\tfrac{1}{2})\,\omega$$
$$(n_r=0,1,2,\ldots\ ;\ \omega=0.75886) \tag{4.5}$$

and

$$n=1.4669\,(g/4\pi m)^{1/2}-(n_r+\tfrac{1}{2})\,\omega+O(1/n_0)\ . \tag{4.6}$$

As in the $t=0$ massless exchange case we have a Lorentz pole for each n_r.

We are now ready to calculate the $O(1/n_0)$ correction to Eq. (4.6) to obtain the first t-dependent term. We must expand about our wave function ϕ_{0n}^l in Eq. (4.4):

$$\phi_n^l(y)=\phi_{0n}^l(y)+\frac{1}{n_0}\,\phi_{1n}^l(y)\ . \tag{4.7}$$

We must also include two more terms in the expansion of the potential in Eq. (4.3). We then obtain

$$r^4V(r)=n_0^4\left[1-\frac{2\omega^2y^2}{n_0}-\frac{\omega_3^2y^2}{n_0^{3/2}}+\frac{\omega_4^2y^4}{n_0^2}+O\left(\frac{y^5}{n_0^{5/2}}\right)\right], \tag{4.8}$$

where

$$\omega_3^2=-\frac{1}{3!}\left(\frac{4\pi^2m^2}{g^2}\right)\left(r\frac{d}{dr}\right)^3[r^4V(r)]\Big|_{r=r_0}=0.4049\ ,$$

$$\omega_4^2=\frac{1}{4!}\left(\frac{4\pi^2m^2}{g^2}\right)\left(r\frac{d}{dr}\right)^4[r^4V(r)]\Big|_{r=r_0}=0.5632\ .$$

We now put (4.7) and (4.8) in (3.6) and neglect all terms of $O(n_0^{3/2})$ or lower, remembering that, in contrast with the massless case, r_0 is now finite. We obtain

$$n_0^2\frac{d^4}{dy^4}\phi_{0n}^l-2n^2n_0\frac{d^2}{dy^2}\phi_{0n}^l-2n^2\frac{d^2}{dy^2}\phi_{1n}^l+n^4\phi_{0n}^l+\frac{n^4}{n_0}\phi_{1n}^l+2n^2[(\mu^2-\tfrac{1}{4}t)r_0^2-1]\phi_{0n}^l$$
$$=n_0^4\left(1-\frac{2\omega^2y^2}{n_0}-\frac{\omega_3^2y^3}{n_0^{3/2}}+\frac{\omega_4^2y^4}{n_0^2}\right)\phi_{0n}^l+n_0^3\left(1-\frac{2\omega^2y^2}{n_0}-\frac{\omega_3^2y^3}{n_0^{3/2}}\right)\phi_{1n}^l\ . \tag{4.9}$$

Since Eq. (4.9) does not depend upon l, we shall drop the superscript. The harmonic-oscillator eigenfunctions are

$$\phi_{0n,n_r}=e^{-\omega y^2/2}H_{n_r}(\sqrt{\omega}\,y)=e^{-x^2/2}H_{n_r}(x)\ , \tag{4.10}$$

where n_r is the harmonic-oscillator quantum number and $H_{n_r}(x)$ is a Hermite polynomial. We now make another transformation of variables $x=\sqrt{\omega}\,y$ and introduce a new function:

$$\phi_{1n,n_r}=e^{-x^2/2}f_{n_r}(x)\ . \tag{4.11}$$

Putting (4.4), (4.10), and (4.11) in (4.9) we obtain a differential equation for $f_{n_r}(x)$:

$$-2\omega\frac{d^2f_{n_r}}{dx^2}+4\omega x\frac{df_{n_r}}{dx}-4n_r\,\omega f_{n_r}+\frac{\omega_3^2}{\sqrt{n_0}\,\omega^{3/2}}x^3f_{n_r}+4\omega^2x\frac{dH_{n_r}}{dx}$$
$$+\left\{\omega^2(\tfrac{5}{2}+2n_r+2n_r^2)+2[(\mu^2-\tfrac{1}{4}t)r_0^2-1]+4\delta_{n_r}-4\omega^2x^2+\frac{\omega_3^2}{\omega^{3/2}}\sqrt{n_0}\,x^3+\left(\omega^2-\frac{\omega_4^2}{\omega^2}\right)x^4\right\}H_{n_r}=0\ , \tag{4.12}$$

where

$$\delta_{n_r}=\frac{1}{n_0}\{n-[n_0-(n_r+\tfrac{1}{2})\,\omega]\}\ . \tag{4.13}$$

Equation (4.12) may be solved by assuming a power-series expansion for $f_{n_r}(x)$ and demanding that the wave function behave properly at ∞. This places an eigencondition on δ, and $f_{n_r}(x)$ becomes a finite-order polynomial. The result is

$$\delta_{n_r}=\tfrac{1}{2}-\tfrac{1}{2}(\mu^2-\tfrac{1}{4}t)r_0^2+\tfrac{3}{16}(2n_r^2+2n_r+1)\frac{\omega_4^2}{\omega^2}$$
$$-\tfrac{1}{16}(14n_r^2+14n_r+5)\,\omega^2$$
$$+\tfrac{1}{128}(11+30n_r+30n_r^2)\frac{\omega_3^2}{\omega^{3/2}}\,. \tag{4.14}$$

Using Eq. (4.13) we solve for n:

$$n=1.4669(g/4\pi m)^{1/2}-(n_r+\tfrac{1}{2})\,\omega$$
$$+\frac{1}{n_0}\left[\tfrac{1}{2}-\tfrac{1}{2}(\mu^2-\tfrac{1}{4}t)\,r_0^{\,2}+\tfrac{3}{16}(2n_r^{\,2}+2n_r+1)\frac{\omega_4^{\,2}}{\omega^2}-\tfrac{1}{16}(14n_r^{\,2}+14n_r+5)\,\omega^2+\tfrac{1}{128}(11+30n_r+30n_r^{\,2})\frac{\omega_3^{\,2}}{\omega^{3/2}}\right]. \qquad (4.15)$$

Since n does not depend upon l in Eq. (4.15), the daughter trajectories all have the same t dependence to $O(1/n_0)$. It would be necessary to calculate $\alpha(t)$ to $O(1/n_0^{\,2})$ to obtain an l dependence through the term $f^{\,l}_{n,n}p_r^{\,2}t\,\phi^l_n$. The equation for ϕ^l_n is not coupled with ϕ^l_{n+2} and ϕ^l_{n-2} until $O(1/n_0^{\,4})$. This is because $\phi^l_{n+2},\phi^l_{n-2}\sim(1/n_0^{\,2})\phi^l_n$. This suggests that for a nonzero-mass exchange and an even moderately large g it may be a good approximation to neglect the coupling to other n states. This was found to be the case in a numerical study by Wyld.[4]

Summarizing our results, we have found that $\alpha(t,\mu^2)=\alpha(t=0,\mu^2-\frac{1}{4}t)$ through the order at which the first t dependence occurs. For massless exchanges this is of $O(n_0)$ and for massive exchange it is of $O(1/n_0)$. The next order in both cases introduces l-dependent terms, indicating that daughter trajectories will no longer be spaced one unit apart for $t\neq 0$.

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APPENDIX

In this appendix we wish to justify the assumption made in Sec. III that the solutions to Eq. (3.7) at $t=0$ corresponding to different Lorentz poles do not mix at $t\neq 0$.

We begin by considering the solutions present at $t=0$ found by Chang and Rosner. In that case Eq. (3.7) becomes

$$\left(-\frac{d^2}{dy^2}+y^2\right)\phi_n^{n-\lambda}(0,y)=2(n_0-n)\,\phi_n^{n-\lambda}(0,y)\,, \qquad (A1)$$

which is just the harmonic-oscillator equation with eigenvalues

$$E_{n_r}=n_0-n=n_r+\tfrac{1}{2}\quad (n_r=0,1,2,\ldots)\,. \qquad (A2)$$

The maximum $n\equiv\bar{n}$ is of course given by $n_r=0$. The solutions to the oscillator equation are

$$\phi^{\bar{n}-\lambda}_{\bar{n}-n_r}(0,y)=e^{-y^2/2}H_{n_r}(y), \qquad (A3)$$

where $H_{n_r}(y)$ is a Hermite polynomial.

The restriction $l=\bar{n}-\lambda\leq(\bar{n}-n_r)-1$ implies that at $t=0$ we have λ nonzero solutions ($n_r=0,1,\ldots,\lambda-1$). Each n_r corresponds to a Lorentz pole with a leading Regge trajectory given by $l=n_0-(n_r+\frac{1}{2})-1$ plus integrally spaced daughters. Because of the coupling between different n states at $t\neq 0$, we might expect the solutions (A3) to become mixed at $t\neq 0$. However, we shall show that the wave functions and trajectories for different n_r do not mix at $t\neq 0$ up to the order of $1/n$ which we are considering. To illustrate this we assume, for the moment, that the wave functions for different n_r *do* mix at $t\neq 0$. We then postulate the following separable form[14] for the solutions to Eq. (3.8):

$$\phi^{\bar{n}-\lambda}_{\bar{n}+K}(t,y)=\sum_{n_r=0}^{\lambda-1}\beta^{\bar{n}-\lambda}_{n_r,K}(t)\,\phi^{\bar{n}-\lambda}_{\bar{n}-n_r}(0,y)\,, \qquad (A4)$$

with

$$\beta^{\bar{n}-\lambda}_{n_r,-n_r}(0)=1\,, \qquad (A5)$$

$$\beta^{\bar{n}-\lambda}_{n_r,K}(0)=0$$
$$(K\neq -n_r;\ K=-\lambda+1,-\lambda+2,\ldots,-1,0,1,\ldots). \qquad (A6)$$

Conditions (A 5) and (A 6) ensure that Eq. (A 4) gives the correct solutions for $t=0$. We have postulated that the solution for any n state is a linear combination of the $t=0$ wave functions with coefficients that depend on t. Note that the $\bar{n}$'s in the superscripts of the β's are $\bar{n}(t)$ but the $\bar{n}$'s in $\varphi^{\bar{n}-\lambda}_{\bar{n}-n_r}(0,y)$ are $\bar{n}(t=0)$. To see if Eq. (A4) is in fact a solution to Eq. (3.8) we combine the two and obtain

$$\left(-\frac{d^2}{dy^2}+y^2-2[n_0-\bar{n}(t)-K]+\frac{2(\lambda+K)-1}{8(\mu^2-\frac{1}{4}t)}\,t\right)\sum_{n_r=0}^{\lambda-1}\beta^{\bar{n}-\lambda}_{n_r,K}(t)\,\phi^{\bar{n}-\lambda}_{\bar{n}-n_r}(0,y)$$
$$-\frac{t}{8(\mu^2-\frac{1}{4}t)}[(\lambda+K-1)(\lambda+K-2)]^{1/2}\sum_{n_r=0}^{\lambda-1}\beta^{\bar{n}-\lambda}_{\bar{n}_r,K-2}(t)\,\phi^{\bar{n}-\lambda}_{\bar{n}-n_r}(0,y)$$
$$-\frac{t}{8(\mu^2-\frac{1}{4}t)}[(\lambda+K)(\lambda+K+1)]^{1/2}\sum_{n_r=0}^{\lambda-1}\beta^{\bar{n}-\lambda}_{n_r,K+2}(t)\,\phi^{\bar{n}-\lambda}_{n-n_r}(0,y)=0\,. \qquad (A7)$$

Making use of (A1) and (A2) we may rewrite Eq. (A7) as

$$\sum_{n_r=0}^{\lambda-1}\Bigg[\left(2n_r+1-2[n_0-\bar{n}(t)-K]+\frac{2(\lambda+K)-1}{8(\mu^2-\frac{1}{4}t)}\,t\right)\beta^{\bar{n}-\lambda}_{n_r,K}(t)$$
$$-\frac{t}{8(\mu^2-\frac{1}{4}t)}\{[(\lambda+K-1)(\lambda+K-2)]^{1/2}\,\beta^{\bar{n}-\lambda}_{n_r,K-2}(t)+[(\lambda+K)(\lambda+K+1)]^{1/2}\,\beta^{\bar{n}-\lambda}_{n_r,K+2}(t)\}\Bigg]\,\phi^{\bar{n}-\lambda}_{\bar{n}-n_r}(0,y)=0\ . \quad \text{(A8)}$$

Since the wave functions $\phi^{\bar{n}-\lambda}_{\bar{n}-n_r}(0,y)$ are orthogonal, the coefficient of each in Eq. (A8) must be equal to zero. Thus we now have the set of algebraic equations

$$\left(2n_r+1-2[n_0-\bar{n}(t)-K]+\frac{2(\lambda+K)-1}{8(\mu^2-\frac{1}{4}t)}\,t\right)\beta^{\bar{n}-\lambda}_{n_r,K}(t)$$
$$-\frac{t}{8(\mu^2-\frac{1}{4}t)}\{[(\lambda+K-1)(\lambda+K-2)]^{1/2}\,\beta^{\bar{n}-\lambda}_{n_r,K-2}(t)+[(\lambda+K)(\lambda+K+1)]^{1/2}\,\beta^{\bar{n}-\lambda}_{n_r,K+2}(t)\}=0\ . \quad \text{(A9)}$$

Different K states only couple to $K\pm 2m$ states where m is an integer, and since $\beta^{\bar{n}-\lambda}_{n_r,K}(0)=0$ for $K\neq -n_r$, we see that $\beta^{n-\lambda}_{n_r,K}(t)=0$ for $K-n_r$ odd. For a given n_r we may choose some K value (such that $K-n_r$ is even) and solve Eq. (A9) for $\bar{n}(t)$ in terms of $\beta^{\bar{n}-\lambda}_{n_r,K}(t)$, $\beta^{\bar{n}-\lambda}_{n_r,K-2}(t)$, and $\beta^{\bar{n}-\lambda}_{n_r,K+2}(t)$. We choose $K=-n_r$ to solve for $\bar{n}(t)$ and obtain

$$\bar{n}(t)=n_0-\tfrac{1}{2}-\frac{2(\lambda-n_r)-1}{16(\mu^2-\frac{1}{4}t)}\,t$$
$$+\frac{t}{16(\mu^2-\frac{1}{4}t)\,\beta^{\bar{n}-\lambda}_{n_r,-n_r}(t)}\{[(\lambda-n_r-1)(\lambda-n_r-2)]^{1/2}\,\beta^{\bar{n}-\lambda}_{n_r,-n_r-2}(t)+[(\lambda-n_r)(\lambda-n_r-1)]^{1/2}\,\beta^{\bar{n}-\lambda}_{n_r,-n_r+2}(t)\}\ . \quad \text{(A10)}$$

Equation (A10) shows that for a fixed λ, $\bar{n}(t)$ is different for each n_r.[15] This means that the proposed solution (A4) cannot be correct because it assumes that $\bar{n}(t)$ is the same for all n_r. Therefore the wave functions for different n_r are not mixed at $t\neq 0$ and we should just write

$$\phi^{\bar{n}-\lambda}_{\bar{n}+K}=\beta^{\bar{n}-\lambda}_{n_r,K}(t)\ \phi^{\bar{n}-\lambda}_{\bar{n}-n_r}(0,y)\ , \quad \text{(A11)}$$

where

$$K=-\lambda+1,\ldots,-n_r-2,-n_r,\ldots\quad (n_r-\lambda \text{ odd})\ ,$$
$$K=-\lambda+2,\ldots,-n_r-2,-n_r,\ldots\quad (n_r-\lambda \text{ even})\ ,$$

Finally, we show that Eq. (A11) is really the same as (3.9). Using $n=\bar{n}-n_r$, Eq. (A11) becomes

$$\phi^{\bar{n}-\lambda}_{\bar{n}+K}=\phi^{n-(\lambda-n_r)}_{n+(K+n_r)}(t,y)=\beta^{n-(\lambda-n_r)}_{n_r,K+n_r}(t)\ \phi^{n-(\lambda-n_r)}_{n}(0,y)\ . \quad \text{(A12)}$$

Defining $\lambda'=\lambda-n_r$ and $K'=K-n_r$ we have

$$\phi^{n-\lambda}_{\bar{n}+K}(t,y)=\phi^{n-\lambda'}_{n+K'}(t,y)=\beta^{n-\lambda'}_{n_r,K'}(t)\ \phi^{n-\lambda'}_{n}(0,y)\ , \quad \text{(A13)}$$

where

$$K'=-\lambda'+1 \text{ or } -\lambda'+2,\ldots,\ -2,0,2,\ldots\ ,$$

which is just Eq. (3.9) if we pick the normalization

$$\beta^{n-\lambda'}_{n_r,0}(t)=1\ . \quad \text{(A14)}$$

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[12]We shall see explicitly when we calculate the first correction term to the maximum n that the derivatives are a lower-order effect.

[13]The approximations $f^{n-\lambda}_{n+K,n+K}=[2(\lambda+K)-1]/2n$, etc. are valid only for $K\ll n$. Therefore these approximations must be replaced by the full f's, Eq. (2.6), in (3.8) and (3.13) when K is of order n.

[14]We were led to postulating a separable solution by studying a power-series expansion in t of the wave function.

[15]For example, in a series expansion in t of $\bar{n}(t)$ the term that goes like t is $[-2(\lambda-n_r)+1]/16\mu^2$.