PHYSICAL REVIEW D VOLUME 9, NUMBER 10 15 MAY 1974

Nonpolynomial anharmonic oscillator

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(Received 14 August 1973)

It is shown that for a nonpolynomial anharmonic correction of the form $\beta x^4/4(1+ax^2)$ to the simple harmonic oscillator with $H=p^2+\frac{1}{4}x^2$, the perturbation series in β is convergent over a finite domain $|\beta|<a$. The energy eigenvalues have a two-sheeted structure with the cut extending from $\beta=-a$ to $\beta=-\infty$, and $\beta=-a$ is the accumulation point of singularities on the second sheet along real $\beta>-a$. An estimate for the perturbation series, using dispersion relations, is presented.

I. INTRODUCTION

Anharmonic interaction is of considerable interest as a correction to the frequently assumed simple harmonic interaction. Apart from this practical importance, it is of independent interest because of the rich mathematical structure it provides in terms of the analyticity properties of the energy levels and the related problems of the convergence of the perturbation series. Some recent efforts have made important contributions in the understanding of the structure of the energy levels of the Hamiltonian

$$H=p^2+\tfrac{1}{4}x^2+\tfrac{1}{4}\beta x^4. \qquad (1)$$

It has been shown[1,2] that the energy levels of this Hamiltonian have a three-sheeted structure, with $\beta=0$ being the accumulation point of square-root branch points on the lower sheets. Furthermore, we have an asymptotic series in powers of β, so that the perturbation in terms of the anharmonic term makes some sense.

Additional interest in the problem is due to the fact that the same Hamiltonian describes the one-dimensional $\lambda\phi^4$ field theory, but without the normal ordering. One hopes that the solution to the potential problem will yield insight into the structure of quantum field theories. It must be remembered that it has long been conjectured that the quantum-electrodynamics series is probably an asymptotic perturbation series.

While an asymptotic series is better than no series, it is not an end in itself. One would like to obtain either a convergent perturbation series or a nonperturbative expression for the energy levels. Several such approaches[3] have been attempted with varying advantages and successes.

Within the last few years, it has become plausible that a class of nonpolynomial Lagrangian interactions may yield a finite perturbative approach[4] to field theory; for example, $\beta\phi^4/(1+a\phi^2)$ may yield a finite theory. However, while one can establish some general properties of these theories, such as analyticity and unitarity, one has little or no idea about the convergence or the region of convergence of the major series in powers of β. It is tempting to assume that the series converges for all values of a, though there is no basis for such hope. The problem is indeed difficult, but is one which must be solved before we can fully exploit the attractive properties of the nonpolynomial interactions. We take a step in this direction by analyzing the structure of the corresponding potential problem.

In particular, the following Hamiltonian is considered:

$$H=p^2+\tfrac{1}{4}x^2+\tfrac{1}{4}\beta x^4 f(x,a), \qquad (2)$$

where β is the major coupling constant and a is the minor coupling constant. We prove the following properties of this Hamiltonian: Let

$$f(x,a)=\frac{1}{1+ax^{2m}},$$

where a is positive, and m is a positive integer. Then:

(1) For $m\geqslant 2$, the energy levels are analytic in β for all real values of β. The same is true for $f(x,a)=e^{-\alpha x^2}$ where a is positive.

(2) For $m=1$, i.e.,

$$H=p^2+\frac{x^2}{4}+\frac{\beta}{4}\,\frac{x^4}{1+ax^2}, \qquad (3)$$

the energy levels are analytic in β for real $\beta>-a$. However, $\beta=-a$ is a singular point. Here we get a two-sheeted structure with the cut starting from $\beta=-a$, which also is conjectured to be the accumulation point of square-root branch points along real $\beta>-a$, but on the second sheet.

(3) For $m=1$, the phase of the singular points of the energy levels, on the second sheet, is 2π in the limit $\beta\to 0$, in contrast to the corresponding phase of $\frac{3}{2}\pi$ for the Hamiltonian (1). This is shown by using the dispersion techniques developed by Bender and Wu, and analytically continuing on to the second sheet. We also calculate the nth-order

term in the perturbation series in β for large n. This series converges for $|\beta|<a$ but diverges for $|\beta|\geq a$.

These results give us a word of warning about the indiscriminate use of nonpolynomial Lagrangians in field theory. If we accept the folk rule that things can become only worse when we go from potential theory to field theory, then for

$$H=\dot{\phi}^2+\frac{\phi^2}{4}+\frac{\beta\phi^4}{4(1+a\phi^2)}$$

the theory may converge for $|\beta|<a$ but will diverge for $|\beta|>a$. Therefore, at least for some types of nonpolynomial Lagrangian field theories, the theory may be finite but not for arbitrary values of the major and the minor coupling constants, even if both the coupling constants are small.

The analysis is presented in two parts. In Sec. II, the general properties of the structure of the energy eigenvalues of the Hamiltonians (2) and (3) are discussed. In Sec. III, we analyze the detailed structure of the energy levels of the Hamiltonian (3) using the dispersion techniques developed by Bender and Wu[5] and the WKB approximation. These techniques allow us to calculate the coefficient of the nth term in the perturbation series for the nonpolynomial Hamiltonian (3).

II. STRUCTURE OF ENERGY EIGENVALUES

Consider the Hamiltonian (3),

$$\begin{aligned}H&=p^2+\frac{x^2}{4}+\frac{\beta x^4}{4(1+ax^2)}\\&=p^2+\frac{(1+\beta/a)x^2}{4}-\frac{\beta x^2}{4a(1+ax^2)}\,.\end{aligned}\tag{4}$$

A. Energy eigenvalues of the Hamiltonian

We first show that the energy eigenvalues of this Hamiltonian are analytic for β real and $>-a$.

The asymptotic behavior of the wave functions is determined by the simple harmonic term $\frac{1}{4}[(1+\beta/a)x^2]$. For $\beta>-a$, the asymptotic wave function has the behavior

$$\phi(x)\underset{x\to\infty}{\sim}\exp(-\tfrac{1}{4}\lambda^{1/2}x^2),\tag{5}$$

where $\lambda=(1+\beta/a)$. Furthermore, one can take $\phi(x)$ to be real for $\beta>-a$. For a change $\delta\beta$ in β, let

$$\begin{aligned}&E(\beta)\to E(\beta)+\delta E(\beta),\\&\phi(x,\beta)\to\phi(x,\beta)+\delta\phi(x,\beta),\end{aligned}\tag{6}$$

so that

$$H\delta\phi(x,\beta)+\delta\beta\frac{x^4}{4(1+ax^2)}\,\phi(x,\beta)=\delta E(\beta)\,\phi(x,\beta)+E(\beta)\delta\phi(x,\beta).\tag{7}$$

Multiply the two sides by $\phi^*(x,\beta)$ and integrate by parts to obtain

$$\delta\beta\int\phi^*(x,\beta)\frac{x^4}{4(1+ax^2)}\,\phi(x,\beta)\,dx=\delta E(\beta)\int|\phi(x,\beta)|^2dx\tag{8}$$

or

$$\frac{\partial E(\beta)}{\partial\beta}=\frac{\int\{x^4/[4(1+ax^2)]\}|\phi(x,\beta)|^2dx}{\int|\phi(x,\beta)|^2dx}\,.$$

Hence $E(\beta)$ is analytic for β real and $>-a$. Along the same lines one can prove the analyticity of energy levels for anharmonic corrections of the types

$$H'=\frac{\beta x^4}{4(1+ax^{2n})}$$

and (9)

$$H'=\beta x^4e^{-\alpha x^2}$$

for all real β, where n is an integer >1. This, of course, does not mean that there are no singularities in the complex plane.

B. Singularity structure of the energy levels of the Hamiltonian

The singularity structure of the energy levels for the Hamiltonian (4) around $\beta=-a$ becomes transparent if we make the Symanzik scale transformation.[2] Let

$$p\to\alpha p,\quad x\to x/\alpha,\tag{10}$$

for which the corresponding unitary transformation takes the Hamiltonian (4) into

$$H\to\alpha^2\left[p^2+\frac{\lambda x^2}{4\alpha^4}-\frac{(\lambda-1)x^2}{4\alpha^4(1+ax^2/\alpha^2)}\right],\tag{11}$$

where $\lambda=\beta/a+1$. With the choice $\alpha^4=\lambda$, we get

$$H\to\lambda^{1/2}\left[p^2+\frac{x^2}{4}-\frac{(\lambda-1)x^2}{4\lambda^{1/2}(\lambda^{1/2}+ax^2)}\right],\tag{12}$$

so that for λ small

$$H\approx\lambda^{1/2}(p^2+\tfrac{1}{4}x^2)+\frac{x^2}{4(\lambda^{1/2}+ax^2)}\,.\tag{13}$$

Clearly, (12) and (13) exhibit a two-sheeted structure for energy eigenvalues. More generally, it may be observed that going around the origin twice brings us back to the same value. It is indicated that since the potential acquires poles for real x when $\lambda^{1/2}$ is negative, we come across additional singularities for λ real and positive, but on the second sheet. This we are unable to prove rigorously. However, the results of Sec. II C, based on dispersion techniques and WKB approximation, support this conjecture.

C. Proof of the singular point

One can show rigorously that $\lambda=0$, or $\beta=-a$ is a singular point.

Suppose $\lambda=0$ is a nonsingular point. Since the energy eigenvalues are analytic along positive real λ, we start from large positive λ, move along the real axis, and after circling the origin, go back to infinity. Since the scale transformation (10) is unitary, we obtain from (4) and (13)

$$\lim_{\beta\to\infty} E_n(\beta, a) \sim (\beta/a)^{1/2} E_n(0, a) \sim -(\beta/a)^{1/2} E_n(0, a), \qquad (14)$$

where the right-hand sides are just the simple harmonic energy levels, which is absurd. Hence $\lambda=0$ or equivalently $\beta=-a$ is a singular point.

D. Singularities as square-root branch points

We can apply the Bender-Wu proof,[1] for showing that the singularities for $\frac{1}{4}\beta x^4$ anharmonic corrections are likely to be square-root branch points, to the Hamiltonian (4) as well.

Let β_0 be a singular point, and $\epsilon=\beta-\beta_0$. Then

$$\left[-\frac{d^2}{dx^2}+\frac{x^2}{4}+\frac{\beta_0 x^4}{4(1+ax^2)}+\frac{\epsilon^2 x^4}{4(1+ax^2)}\right]\phi(x,\beta) = E\phi(x,\beta). \qquad (15)$$

Let

$$\phi=\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\cdots,$$
$$E(\beta)=\gamma_0+\epsilon\gamma_1+\epsilon^2\gamma_2+\cdots, \qquad (16)$$
$$\theta=-\frac{d^2}{dx^2}+\frac{x^2}{4}+\frac{\beta_0 x^4}{4(1+ax^2)},$$

in terms of which one gets

$$\theta\phi_0=\gamma_0\phi_0,$$
$$\theta\phi_1=\gamma_0\phi_1+\gamma_1\phi_0, \qquad (17)$$
$$\theta\phi_2+\frac{x^4}{4(1+ax^2)}\phi_0=\gamma_0\phi_2+\gamma_1\phi_1+\gamma_2\phi_0.$$

Multiply by ϕ_0 and integrate by parts to obtain

$$\gamma_1=\left[\int\frac{x^4}{4(1+ax^2)}\phi_0{}^2(x)\,dx\right]\Big/\int\phi_1(x)\,\phi_0(x)\,dx, \qquad (18)$$

where the integration is along a well-defined path. Therefore, unless the denominator accidentally happens to be zero, we have only square-root branch points.

III. DISPERSION RELATIONS FOR ENERGY EIGENVALUES

In this section, we analyze the detailed structure of the energy eigenvalues of the Hamiltonian (4) by means of the dispersion relations for the energy levels. The imaginary part of the eigenvalues for negative β is approximated by the WKB result, as was done for the anharmonic oscillator by Bender and Wu.[5]

A. Dispersion relations

The dispersion relations for the energy levels may be written down from the analyticity structure of the energy levels. For the Hamiltonian (4), we deduced in Sec. II that the energy levels are analytic in the cut β plane with the branch cut running from $\beta=-a$ to $\beta=-\infty$. The asymptotic behavior for $\beta\to\infty$ is easily obtained from the scale transformation $p\to\alpha p$, $x\to x/\alpha$, and

$$H\to\alpha^2\left[p^2+\frac{x^2}{4\alpha^4}+\frac{\beta x^4}{4\alpha^6(1+ax^2/\alpha^2)}\right]. \qquad (19)$$

Take $\alpha^6=\beta$ and let $\beta\to\infty$, to obtain

$$H\to\beta^{1/3}(p^2+\tfrac{1}{4}x^4), \qquad (20)$$

so that

$$E_k(p)\underset{\beta\to\infty}{\sim} c_k\beta^{1/3}, \qquad (21)$$

where c_k is a constant. Hence we write a once-subtracted dispersion relation for $E_k(\beta)$:

$$E_k(\beta)=E_k(0)+\frac{\beta}{\pi}\int_{-\infty}^{-a}\frac{\mathrm{Im}E_k(z+i\epsilon)}{z(z-\beta)}\,dz. \qquad (22)$$

Furthermore, if we write

$$E_k(\beta)=E_k(0)+\sum_{n=1}^{\infty}A_n^k\beta^n, \qquad (23)$$

we get

$$A_n^k=\frac{1}{\pi}\int_{-\infty}^{-a}\frac{\mathrm{Im}E_k(z+i\epsilon)}{z^{n+1}}\,dz. \qquad (24)$$

Therefore, given $\mathrm{Im}E_k(z)$ for negative z, relations (22) and (24) allow us not only to study the structure of the energy eigenvalues but also to calculate the perturbation series in β.

B. Calculation of the imaginary part of $E_k(\beta)$

The imaginary part of $E_k(\beta)$ is calculated for $\beta\approx-a$ by using the WKB method. Let $\lambda=\beta/a+1$, so that for $\lambda\approx0$ we have

$$H=p^2+\frac{\lambda x^2}{4}+\frac{x^2}{4(1+ax^2)}. \qquad (25)$$

For the energy level E_k, and λ small and negative, the turning points are

$$x_0=2E_k{}^{1/2}, \quad x_1=\frac{1}{|a\lambda|^{1/2}}. \qquad (26)$$

Let us also assume for simplicity that $a\ll1$, which is also quite reasonable since higher derivatives of the potential are expected to be small. Then $E_k \approx k+\frac{1}{2}$.

(i) Region $x \sim x_0$. The equation to be solved is

$$\left(-\frac{d^2}{dx^2}+\frac{x^2}{4}-k-\frac{1}{2}\right)\phi_1(x)=0 \qquad (27)$$

so that

$$\begin{aligned}\phi_1(x)&=D_k(x)\\ &=2^{-k/2}e^{-x^2/4}H_k(x/\sqrt{2})\\ &\sim x^k e^{-x^2/4} \quad \text{for large } x. \end{aligned}\qquad (28)$$

(ii) Region from x_0 *to* x_1. In this region we use the WKB solution,

$$\phi_2=c\,\frac{[4(1+ax^2)]^{1/2}}{[|\lambda a|(x^2-4E)(x^2-1/|a\lambda|)]^{1/4}}\exp\left(-\frac{|\lambda a|^{1/2}}{2}\int_{2E^{1/2}}^{x}\frac{(x^2-4E)^{1/2}(x^2-1/|a\lambda|)^{1/2}}{(1+ax^2)^{1/2}}dx\right), \qquad (29)$$

where we have suppressed the index k. For $x_0 << x << x_1$,

$$\phi_2\sim\frac{c2^{1/2}}{x^{1/2}}\exp\left[-\tfrac{1}{2}\int_{2E^{1/2}}^{x}(x^2-4E)^{1/2}dx\right], \qquad (30)$$

a comparison of which with (28) leads to

$$c=\frac{1}{2^{1/2}}\exp[\tfrac{1}{2}E(\ln E-1)]. \qquad (31)$$

We are now in a position to calculate the imaginary part of $E(z)$ by using the second method described in the Appendix of the work of Bender and Wu[5]; we have

$$\begin{aligned}\mathrm{Im}E&=\frac{J(x)}{\int_0^x\phi^*(x')\phi(x')dx'},\\ J(x)&=\tfrac{1}{2}i\left[\phi^*(x)\frac{d\phi(x)}{dx}-\phi(x)\frac{d\phi^*(x)}{dx}\right].\end{aligned} \qquad (32)$$

The denominator is well approximated by the solution (28) for all x, which gives

$$\int_0^x\phi^*(x')\phi(x')dx'\approx(\tfrac{1}{2}\pi)^{1/2}k!. \qquad (33)$$

Furthermore $J(x)$ is given by the continuation of (29) to $x>x_1$ so that

$$\begin{aligned}J(x)&=c^2\exp\left[-|\lambda a|^{1/2}\int_{x_0}^{x_1}dx\,\frac{(x^2-x_0{}^2)^{1/2}(x^2-x_1{}^2)^{1/2}}{(1+ax^2)^{1/2}}\right]\\ &=c^2\exp\left(-\frac{\pi}{4a|\lambda|^{1/2}}+\frac{1}{a}\right).\end{aligned} \qquad (34)$$

Finally,

$$\mathrm{Im}E_k=f_k\exp\left(-\frac{\pi}{4a|\lambda|^{1/2}}\right), \qquad (35)$$

where

$$f_k=\frac{\exp[E_k(\ln E_k-1)]\exp(1/a)}{(2\pi)^{1/2}k!}\ . \qquad (36)$$

C. Continuing $E_k(\beta)$ to the second sheet

We can now analytically continue $E_k(\beta)$ on to the second sheet by using (22) and (35):

$$E_k(\beta)=k+\frac{1}{2}+\frac{\beta f_k}{\pi}\int_{-\infty}^{-a}\frac{\exp\{-\pi/4a[-(z/a)-1]^{1/2}\}}{[z(z-\beta)]}dz. \qquad (37)$$

As β encircles $-a$ and goes on to the second sheet, we must distort the path of the integral. Because of the form of the imaginary part of the energy, this distortion is allowed for $\beta\to -a$, so long as

$$\mathrm{Re}\left(-1-\frac{\beta}{a}\right)^{1/2}>0. \qquad (38)$$

Writing $\beta+a=\rho e^{i\theta}$, this condition is equivalent to

$$\tfrac{1}{2}(\theta-\pi)<\tfrac{1}{2}\pi \ \text{ or } \ \theta<2\pi, \qquad (39)$$

and we meet our singular points for $\theta=2\pi$. This confirms our conjecture of Sec. II, that for $\beta\to -a$ there are singularities on the second sheet along the real axis for $\beta>-a$.

It may be noted that for the Hamiltonian (1), the imaginary part of $E_k(\beta)$ for small β has the behavior

$$\mathrm{Im}E_k(\beta)\sim\exp\left(\frac{1}{3\beta}\right), \qquad (40)$$

so that for analytic continuation, the distortion of the path is allowed for $\mathrm{Re}\beta<0$. This means that after encircling the origin, we come across the singularity at

$$\arg\beta=\tfrac{3}{2}\pi, \qquad (41)$$

as has been shown earlier by different methods.[1,2]

D. Evaluation of A_n^k

The evaluation of A_n^k is now fairly direct. From (24) and (35), one gets

$$A_n^k=\frac{f_k}{\pi}\int_{-\infty}^{-a}\frac{\exp\{-\pi/[4a|1+(z/a)|^{1/2}]\}}{z^{n+1}}dz \qquad (42)$$

$$=\frac{f_k}{\pi a^n}\int_0^\infty\frac{\exp(-y^{1/2}/a)y^{n-1}}{(1+y)^{n+1}}dy. \qquad (43)$$

It should be remembered that the imaginary part is properly represented only for $z \approx -a$, which means that our approximation becomes good for large n. Though the integral is difficult to evaluate, it can be estimated for large n by the saddle point method, which yields

$$A_n^k \approx \frac{f_k(\frac{16}{27})^{1/6}}{\pi^{1/2}a^{1/3}n^{5/6}a^n} \exp\left[-\left(\frac{27n}{4a^2}\right)^{1/3}\right], \qquad (44)$$

which also shows that the series converges for $|\beta| < a$, but diverges for $|\beta| > a$.

IV. CONCLUSIONS

The choice of anharmonic corrections to the harmonic oscillator, in the nonpolynomial form, allows one to obtain a finite perturbation series. However, in general, the energy levels are not entire functions, so that the domain of convergence of the perturbation series may be finite. We have discussed the anharmonic correction of the form $\beta x^4/(1+ax^2)$ and shown that the domain of convergence of the major series in β is determined by the value of the minor coupling constant and is given by $|\beta| < a$. We have further shown that the energy levels have a two-sheeted structure, with the cut extending from $-a$ to $-\infty$ and $\beta = -a$ being the accumulation point of singularities on the second sheet at real $\beta > -a$. It would be most interesting to find whether these results generalize to a nonpolynomial field theory, i.e., an interaction of the form $\beta\phi^4/(1+a\phi^2)$. We have also derived an approximate expression for the perturbation series for the potential by using dispersion relations for the energy levels.

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