Tidal shapes and shifts on rotating black holes*

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The shape of the tide raised on a rotating black hole by an exterior perturbation is investigated. An expression is given for the perturbed curvature of the two-surface formed by the intersection of the event horizon and a surface of constant time. The result involves only the perturbation in the Weyl tensor component ψ_0 which solves the Teukolsky equation. An application is made to the tide raised by an exterior moon outside a slowly rotating black hole. In this case the position of the tidal bulge is shifted away from the position of the moon in a way analogous to the lag of the tide in the earth-moon system.

I. INTRODUCTION

A black hole perturbed by the gravitational field of exterior matter will evolve toward a state in which either the black hole and matter are corotating or the matter is symmetric about the black hole's axis of rotation.[1-4] This effect is closely analogous to the process of tidal friction in a planet-satellite system.[2-3] For example, the rate of decrease in total angular momentum J of a slowly rotating black hole of mass M perturbed by a moon of mass μ which is at rest with respect to infinity and located a large distance R away from the black hole is

$$\frac{dJ}{dt} = -\frac{8}{5}\frac{J\mu^2 M^3}{R^6}\sin^2\Theta\,. \tag{1.1}$$

The angle Θ is the angle between the moon and the rotation axis of the black hole. This expression displays the characteristic dependence on masses, angles, and radius of the Newtonian tidal friction process. It is of interest to pursue this analogy further to see whether other aspects of the tidal-friction problem are reflected in the black-hole case. This would not only provide a more complete analogy with which to picture the interaction of a black hole with exterior matter but would also perhaps suggest answers to some of the as-yet-unresolved issues in the black-hole case.

One of the most characteristic features of the Newtonian tidal-friction process in a planet-satellite system is the lag of the peak of the tidal bulge behind the position of the perturbing moon as seen by an observer on the rotating planet (see Fig. 1). In Newtonian physics the lag is the mechanism for producing the torque on the moon necessary to increase its angular momentum to compensate for the decrease in the planet's angular momentum in the way required by over-all angular momentum conservation. It would seem reasonable to expect a shift in the position of the tidal bulge also in the black-hole-satellite case for a similar reason.

In this paper we will investigate the existence of a tidal shift in a rotating black hole perturbed by an exterior distribution of masses. This question is part of a more general one, namely, the investigation of the shape of the event horizon of a black hole acted upon by exterior perturbations. In Sec. II we resolve this more general question in principle by relating the intrinsic curvature of a slice of the horizon to the perturbations in a component of the Weyl tensor called ψ_0 in Newman-Penrose[5] notation. This component in turn can be calculated in terms of the perturbing mass distribution by solving the Teukolsky equation.[6] In Sec. III the Newtonian theory of tidal lag is reviewed. In Sec. IV the question of tidal lag in black holes is considered. We find that in general it is difficult to give precise meaning to the concept because the horizon of a rapidly rotating black hole in the presence of perturbations will suffer a considerable distortion over the nonrotating horizon, and this distortion cannot be characterized as a simple, angular shift. The net effect of a slow rotation, however, is to preserve the shape of the tide raised on the horizon of a nonrotating black hole but cause it to be rotated about the axis of rotation by a certain shift angle. In Sec. IV we calculate that angle. The magnitude of the shift corresponds to what would be expected from Newtonian theory on the basis of the tidal friction analogy. The sign of the angle, however, is opposite to one's Newtonian expectations. In a black hole the tidal bulge leads rather than lags the perturbation. The origin of this result is discussed in Sec. IV.

II. THE CURVATURE OF THE INSTANTANEOUS HORIZON

We consider a rotating Kerr black hole with mass M and specific angular momentum a perturbed by exterior matter. We will investigate the geometry of a two-surface formed from the intersection of the perturbed horizon with a spacelike hypersurface. This hypersurface will always be chosen so that in the unperturbed limit it coincides with a surface of constant time t such that $\partial/\partial t$ is a Killing vector of the Kerr geometry which is timelike at infinity. We will refer to this two-surface as the instantaneous horizon.

All the information on the shape of the instantaneous horizon is contained in the two-dimensional intrinsic scalar curvature which we denote here by $\mathcal{R}$. Expanded in powers of the perturbation, $\mathcal{R}$ will have the form

$$\mathcal{R} = R + R^{(1)} + \cdots . \tag{2.1}$$

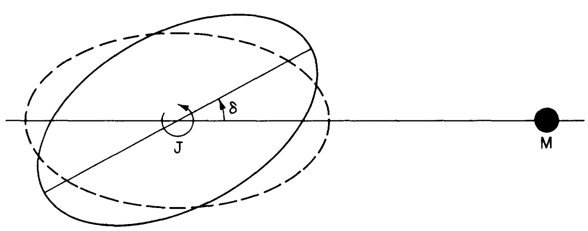

FIG. 1. Tidal lag. The presence of a stationary moon (M) outside a viscous fluid body will raise a tide on the body's surface. If the fluid does not rotate the tidal bulge (dashed surface) will point at the moon. If the body does rotate then its angular momentum (J) will decrease due to viscous dissipation. The surface will now (solid line) have the same shape as before but will be rotated by an angle δ in a positive direction about the rotation axis. The new position is such that the torque now exerted by the moon on the body equals the rate of decrease of angular momentum. The torque exerted by the body on the moon increases the moon's angular momentum thus allowing for over-all angular momentum conservation. As a consequence of the rotated position of the tidal bulge an observer riding on the fluid mass would find high tide arriving after the moon passed directly overhead. For this reason the effect is referred to as a tidal lag. The picture for black holes is exactly the same except now the solid and dashed surfaces are the instantaneous horizon of the black hole and the bulge on the future horizon leads the position of the moon.

For black holes it is not difficult using Newtonian theory to estimate the magnitude of δ from this picture and from the spin-down rate in Eq. (1.1). The torque exerted by a moon of mass μ in the equatorial plane a distance R away from a fluid mass of radius R_s for small δ is $\sim\mu$(effective mass of tidal bulge)$R^{-2} \times (R_s/R)\delta R_s$. The effective mass of the tidal bulge is the mass of the fluid M times the ratio h/R_s where h is the height of the tide. In turn, h is determined by the requirement $hM/R_s^2 \sim$(perturbing potential)$\sim \mu R_s^2/R^3$. Putting these together one arrives at a torque $\sim\mu^2\delta R_s^5/R^6$. For a black hole put $R_s \sim M$ and equate this to the spin-down rate in Eq. (1.1) to find $\delta \sim J/M^2$ independent of the parameters of the perturbation and in agreement with Eq. (4.33).

Here R is the unperturbed curvature, $R^{(1)}$ the first-order perturbation, etc. Smarr[7] has calculated R and discussed the shape of the unperturbed horizon in considerable detail. Here, we will calculate $R^{(1)}$.

The Newman-Penrose formalism[5] provides a simple and compact way to calculate $R^{(1)}$. To use this technique one sets up at each point in spacetime a tetrad consisting of two real null vectors l^μ and n^μ and two complex null vectors m^μ and $\overline{m}^\mu$ (where the overbar denotes complex conjugation). They are normalized so that $l_\mu n^\mu = 1$ and $m_\mu \overline{m}^\mu = -1$ with all other inner products vanishing.[8] The Bianchi identities and the Ricci rotation equations are then expressed in terms of the various spin coefficients and the components of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ projected onto the tetrad. The quantities which will be relevant for us are the spin coefficients

$$\begin{aligned}
\kappa &= l_{\mu;\nu} m^\mu l^\nu , \\
\pi &= -n_{\mu;\nu} \overline{m}^\mu l^\nu , \\
\rho &= l_{\mu;\nu} m^\mu \overline{m}^\nu , \\
\mu &= -n_{\mu;\nu} \overline{m}^\mu m^\nu , \\
\sigma &= l_{\mu;\nu} m^\mu m^\nu , \\
\lambda &= -n_{\mu;\nu} \overline{m}^\mu \overline{m}^\nu , \\
\epsilon &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \overline{m}^\mu l^\nu) , \\
\alpha &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu \overline{m}^\nu - m_{\mu;\nu} \overline{m}^\mu \overline{m}^\nu) , \\
\beta &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \overline{m}^\mu m^\nu)
\end{aligned} \tag{2.2}$$

and the Weyl tensor components

$$\begin{aligned}
\psi_0 &= -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta , \\
\psi_1 &= -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta , \\
\psi_2 &= -\tfrac{1}{2} C_{\alpha\beta\gamma\delta}(l^\alpha n^\beta l^\gamma n^\delta + l^\alpha n^\beta m^\gamma \overline{m}^\delta) .
\end{aligned} \tag{2.3}$$

On the horizon we choose the tetrad so that l^μ lies along the null-geodesic generators and m^μ and $\overline{m}^\mu$ lie in the instantaneous horizon. The vector l^μ is normalized so that $l^\mu t_{,\mu} = 1$. By a rotation $m^\mu \rightarrow \exp(i\theta) m^\mu$ we can enforce

$$\epsilon = \overline{\epsilon} . \tag{2.4}$$

This choice implies that on the horizon[9]

$$\begin{aligned}
&\kappa = 0, \quad \rho = \overline{\rho}, \quad \mu = \overline{\mu}, \\
&\pi = \alpha + \overline{\beta} .
\end{aligned} \tag{2.5}$$

The above choice of tetrad is convenient because by using it, the two-curvature $\mathcal{R}$ may be expressed as a simple combination of ψ_2 and the spin coefficients. To do this we employ Gauss's relation connecting the intrinsic curvature of a two-surface embedded in a four-dimensional space with its extrinsic curvature and the curvature of the embedding space. A brief derivation of this relation in Newman-Penrose form is given in the Appendix. One finds that in the vacuum when $R_{\mu\nu}=0$

$$\mathcal{R}=4\mathrm{Re}(\rho\mu-\lambda\sigma-\psi_2)\,. \tag{2.6}$$

Equation (2.6) for $\mathcal{R}$ may now be expanded in powers of the perturbation. To simplify the notation, henceforth quantities which are first order in the perturbation will be denoted by a superscript (1), viz. $\psi_2^{(1)}, \rho^{(1)}$, *while quantities without superscripts will denote the unperturbed values.* It will be sufficient to consider perturbations which are periodic in t, because a general first-order perturbation may be decomposed into a linear superposition of harmonically varying components,

$$g_{\mu\nu}^{(1)}=G_{\mu\nu}e^{-i\omega t}\,. \tag{2.7}$$

Indeed, in the interesting case when the perturbations arise from matter in stationary periodic orbits, the perturbations will already be in this form. In the following, therefore, we restrict attention to perturbations of definite frequency ω. All the first-order perturbed quantities, $R^{(1)}, \rho^{(1)}$, etc., then have the time dependence $\exp(-i\omega t)$.

On the horizon the unperturbed convergence ρ and shear σ vanish. Furthermore, $\rho^{(1)}$, the first-order perturbation in the convergence, also vanishes. An argument for this was given in Ref. 2 but is simple enough to recall here: Choose a coordinate frame corotating with the horizon so that $D=l^\mu(\partial/\partial x^\mu)$ has the unperturbed value $\partial/\partial t$. The Newman-Penrose equation for the first-order perturbation in the convergence then reads

$$\frac{\partial\rho^{(1)}}{\partial t}=2\epsilon\rho^{(1)}\,, \tag{2.8}$$

where ϵ is the unperturbed spin-coefficient having on the horizon the constant value

$$\epsilon=\frac{1}{4M}\,\frac{(M^2-a^2)^{1/2}}{M+(M^2-a^2)^{1/2}}\,. \tag{2.9}$$

The only periodic solution of Eq. (2.8) is $\rho^{(1)}=0$.

As a consequence of the vanishing of ρ, $\rho^{(1)}$, and σ, $R^{(1)}$ has the form

$$R^{(1)}=-4\,\mathrm{Re}(\psi_2^{(1)}+\lambda\sigma^{(1)})\,. \tag{2.10}$$

We must now calculate $\sigma^{(1)}$ and $\psi_2^{(1)}$. Our goal will be to express these two quantities in terms of $\psi_0^{(1)}$. This is because Teukolsky[6] has shown that the perturbation $\psi_0^{(1)}$ may be calculated in a remarkably simple way. It obeys a wave equation which can be separated into radial and angular equations each solvable by straightforward numerical techniques. An expression for $R^{(1)}$ in terms of $\psi_0^{(1)}$ may thus be regarded as essentially the solution of the problem.

In expressing $\sigma^{(1)}$ and $\psi_2^{(1)}$ in terms of $\psi_0^{(1)}$ we will always use coordinates in which $D=l^\mu(\partial/\partial x^\mu)$ is $\partial/\partial t$ on the horizon to both zeroth and first orders. Such a choice of gauge is clearly possible since it amounts to specifying the four-components l^μ to these orders and there are four gauge functions to accomplish this. Using this choice of coordinates, the above specified tetrad and the fact that $\rho^{(1)}$ vanishes, the Newman-Penrose equation for $\sigma^{(1)}$ on the horizon reads

$$\frac{\partial\sigma^{(1)}}{\partial t}=2\epsilon\sigma^{(1)}+\psi_0^{(1)}\,. \tag{2.11}$$

The solution of this for perturbations varying as $\exp(-i\omega t)$ is

$$\sigma^{(1)}=-\frac{1}{i\omega+2\epsilon}\psi_0^{(1)}\,. \tag{2.12}$$

Since ϵ is constant, $\sigma^{(1)}$ is a simple multiple of $\psi_0^{(1)}$.

An expression for $\psi_2^{(1)}$ in terms of $\psi_0^{(1)}$ can be found from two of the Bianchi identities [the first two equations of (4.5) in Ref. 5]. In writing out the perturbed versions of these equations, the above coordinates and tetrad are used as is the fact that the unperturbed quantities ψ_0 and ψ_1 vanish. (The latter is guaranteed in our tetrad by the Goldberg-Sachs theorem.[5]) One has on the horizon

$$\frac{\partial\psi_1^{(1)}}{\partial t}=2\epsilon\psi_1^{(1)}+(\bar\delta+\pi-4\alpha)\psi_0^{(1)}, \tag{2.13a}$$

$$\frac{\partial\psi_2^{(1)}}{\partial t}=(\bar\delta+2\pi-2\alpha)\psi_1^{(1)}-\lambda\psi_0^{(1)}, \tag{2.13b}$$

where $\bar\delta=\bar m^\mu(\partial/\partial x^\mu)$. For periodic perturbations the solution to these equations is

$$\psi_2^{(1)}=-\frac{i}{\omega}\left[\frac{1}{i\omega+2\epsilon}(\bar\delta+2\pi-2\alpha)(\bar\delta+\pi-4\alpha)+\lambda\right]\psi_0^{(1)}. \tag{2.14}$$

Equation (2.12) for $\sigma^{(1)}$ in terms of $\psi_0^{(1)}$ and Eq. (2.14) for $\psi_2^{(1)}$ in terms of $\psi_0^{(1)}$ may now be combined in Eq. (2.10) to give an expression for $R^{(1)}$ in terms of $\psi_0^{(1)}$. The result is

$$R^{(1)}=-4\,\mathrm{Im}\left[\frac{1}{\omega(i\omega+2\epsilon)}\mathfrak{D}\psi_0^{(1)}\right], \tag{2.15}$$

where $\mathfrak{D}$ is the operator

$$\mathfrak{D}=(\bar\delta+2\pi-2\alpha)(\bar\delta+\pi-4\alpha)+2\epsilon\lambda\,. \tag{2.16}$$

The result may be simplified by expressing the operator $\mathfrak{D}$ in terms of the Newman-Penrose[10-11] operator $\bar{\eth}$. The operator $\bar{\eth}$ is essentially a covariant derivative in the two-surface spanned by m^μ and $\bar m^\mu$. For example, from a quantity η of spin-weight 1 we can form a vector $\eta\bar m^\mu$ in the two-surface. The operator $\bar{\eth}$ is then defined by

$$\bar{\eth}\eta = (\eta\bar{m}_{\mu})_{;\nu} m^{\mu}\bar{m}^{\nu}\,, \tag{2.17}$$

which can be rewritten in terms of the spin-coefficients as

$$\bar{\eth}\eta = [\bar{\delta} - (\alpha - \bar{\beta})]\eta\,. \tag{2.18}$$

Following this procedure we define $\bar{\eth}$ acting on a general quantity η of spin-weight s by

$$\bar{\eth}\eta = [\bar{\delta} - s(\alpha - \bar{\beta})]\eta\,. \tag{2.19}$$

These definitions, the relations in Eq. (2.5) and the Newman-Penrose equation for π on the horizon [Eq. (4.2g) of Ref. 5]

$$\bar{\delta}\pi = -\pi(\pi + \alpha - \bar{\beta}) + 2\epsilon\lambda\,, \tag{2.20}$$

are enough to show that the operator $\mathfrak{D}$ is $\bar{\eth}\bar{\eth}$. The equation for $R^{(1)}$ thus becomes

$$R^{(1)} = -4\,\mathrm{Im}\left[\frac{1}{\omega(i\omega + 2\epsilon)}\,\bar{\eth}\,\bar{\eth}\psi_0^{(1)}\right]\,. \tag{2.21}$$

The operator $\bar{\eth}$ here is the unperturbed operator constructed from the Kerr metric. It is understood to be the operator in Eq. (2.19) appropriate for the spin-weight of the quantity which stands to the right of it. Each time $\bar{\eth}$ acts it lowers the spin-weight by one. Thus we pass from $\psi_0^{(1)}$, a spin-weight-two quantity, to $R^{(1)}$ a spin-weight-zero quantity.

Equation (2.21) appears to have a divergence where $\omega = 0$, i.e., when the perturbation is corotating with the black hole. This is not the case, however, because $\psi_0^{(1)}$ vanishes on the horizon when $\omega = 0$ in order that the tidal-friction spin-down rate may do so.[2]

Equation (2.21) determines the shape of the tide raised on the event horizon by an exterior perturbation having frequency ω with respect to the horizon. One could go on to calculate the form of the two-surface when embedded in a three-dimensional flat space, but this will not be necessary in order to calculate the tidal shift. In the next sections we will apply this formula to calculating the shift of the tide in a slowly rotating black hole.

III. NEWTONIAN EXPECTATIONS

In the remaining two sections we apply Eq. (2.21) to calculating the shift in the tide on a black hole expected on the basis of the analogy with a viscous fluid planet. The Newtonian theory of the tides in a viscous fluid mass goes back nearly a century to the work of Darwin.[12] In this section we review briefly and qualitatively this work as a guide to what may be expected in the relativistic case. For details the reader should consult the original papers.

We consider a sphere of radius[13] R made from an incompressible viscous fluid having density ρ and kinematic viscosity ν. The sphere is perturbed by the gravitational field of an external distribution of masses and by a slow and rigid rotation of angular velocity Ω. We work in a frame rotating with angular velocity Ω in which the perturbations due to the tide producing masses vary harmonically with frequency ω. Thus, for example, if the problem under consideration is a slowly rotating sphere being perturbed by stationary exterior masses, ω would be $-\Omega$ in this frame. The frequency ω will be assumed to be slow and quantities of second order in it, such as the acceleration of the fluid in the rotating frame, will be neglected. This assumption is not essential to the analysis but reduces the complexity of the calculation considerably and is sufficient for the relativistic cases to be considered in Sec. IV.

The equations of motion and structure for the fluid in the rotating frame are

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = \nu\nabla^2\vec{v} - \nabla\left(\frac{p}{\rho} + \Phi + \Phi_c\right)\,, \tag{3.1a}$$

$$\nabla\cdot\vec{v} = 0\,, \tag{3.1b}$$

$$\nabla^2\Phi = 4\pi\rho\,, \tag{3.1c}$$

where $\vec{v}$ is the velocity, Φ the gravitational potential, Φ_c the centrifugal potential, and p the pressure. These equations are to be solved for $\vec{v}$, p, Φ and for the shape of the surface, under the boundary condition that the stress on the surface vanish. If n^i is the normal to the surface, this latter condition is

$$\left[-p\delta_{ij} + \rho\nu\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right)\right]n^j = 0\,, \tag{3.2}$$

at the surface.

Let $\Phi_e^{(1)}$ be the gravitational potential from the external tide-producing masses. It is sufficient to consider only a particular multipole in which

$$\Phi_e^{(1)} = S^{(1)}(t) r^L Y_L^m(\theta,\phi) \tag{3.3}$$

and $S^{(1)}$ has the time dependence $\exp(-i\omega t)$. If we work to linear order in this perturbation, all other quantities will contain only angular harmonics of order (L, m). For example, the equation of the surface to first order can be written

$$r = R + R^{(1)}(t) Y_L^m(\theta,\phi)\,. \tag{3.4}$$

(Quantities such as $S^{(1)}$ and $R^{(1)}$ clearly depend on L and m, but we do not indicate this dependence explicitly.)

The total perturbation in the gravitational potential, $\Phi^{(1)}$, will be the sum of $\Phi_e^{(1)}$ and the change in the potential of the fluid mass due to the tide. Inside the radius R this change is proportional to

r^L and is linear in the shape of the surface, $R^{(1)}$, and the mass of the fluid. Simple dimensional considerations then allow $\Phi^{(1)}$ to be written

$$\Phi^{(1)} = [-C(gR^{(1)}/R^L) + S^{(1)}] r^L Y_L^m(\theta, \phi), \qquad (3.5)$$

where g is the unperturbed surface gravity and C is a dimensionless geometrical factor depending on L.

When terms quadratic in ω or quadratic in the perturbation are omitted, Eq. (3.1a) and (3.1b) for the perturbations in velocity and pressure read

$$\nu \nabla^2 \vec{v}^{(1)} = \nabla\left(\frac{p^{(1)}}{\rho} + \Phi^{(1)} + \Phi_c\right), \qquad (3.6a)$$

$$\nabla \cdot \vec{v}^{(1)} = 0 . \qquad (3.6b)$$

The solution for $\vec{v}^{(1)}$ will contain the appropriate scalar and vector spherical harmonics of order (L, m). Because of the linearity of Eqs. (3.6) and (3.2) and the form of Eq. (3.5), there will be a part of $\vec{v}^{(1)}$ proportional to $R^{(1)}$ and a part proportional to $S^{(1)}$. For example, the radial component of the velocity at the surface will have the form

$$v_r = \left(-\alpha \frac{gR^{(1)}}{R^L} + \beta S^{(1)}\right) \frac{R^{L+1}}{\nu} Y_L^m(\theta, \phi), \qquad (3.7)$$

where α and β are geometrical factors depending on L. Darwin finds for α and β

$$\alpha = \frac{L}{2(L+1)^2 + 1}, \qquad \beta = \frac{L(2L+1)}{(L-1)[2(L+1)^2+1]} . \qquad (3.8)$$

The radial velocity at the surface, however, is also $(dR^{(1)}/dt) Y_L^m(\theta, \phi)$ and so

$$\frac{dR^{(1)}}{dt} + \alpha\left(\frac{gR}{\nu}\right) R^{(1)} = \frac{\beta S^{(1)} R^{L+1}}{\nu} . \qquad (3.9)$$

If ν vanishes, the shape of the tide is given by

$$R_0^{(1)}(t) = \frac{\beta}{\alpha g} S^{(1)}(t) R^L , \qquad (3.10)$$

and the tide is in phase with the perturbation. In the presence of dissipation, however, the solution is

$$R^{(1)}(t) = \left(1 - \frac{i\omega\nu}{\alpha g R}\right)^{-1} R_0^{(1)}(t) . \qquad (3.11)$$

Viscous dissipation, therefore, causes the tide to shift behind the position of the perturbation by a phase angle δ which for small ω is

$$\delta = -\frac{\omega\nu}{\alpha g R} . \qquad (3.12)$$

There are several characteristic features of this result which are perhaps best illustrated by fixing ones attention on the tide raised on a slowly rotating fluid mass by a single stationary moon located a large distance away. In the absence of any dissipation the tide would have the familiar spheroidal shape with the bulge pointing at the moon. In the presence of dissipation Eq. (3.11) shows that the tidal bulge keeps the same angle with the direction of rotation but leads (in space) by an azimuthal angle δ. (See Fig. 2.) In the case of stationary perturbations ω is $-\Omega$ so that δ is positive and truly represents a lag. The angle δ is independent of the moon's mass, angular location, and distance away, as Eq. (3.12) shows. This seemingly surprising result can be readily understood from the magnitude of the torque necessary to reproduce the slowdown rate (cf. caption to Fig. 2) once the angle that the bulge makes with the rotation axis is understood.

IV. TIDAL SHIFTS FROM STATIONARY PERTURBATIONS OF SLOWLY ROTATING BLACK HOLES

A. The shift in the tide

In Newtonian theory tidal shifts arise in situations in which the angular momentum of a tidally dissorted body is being dissipated. The shape of the tide must adjust itself to provide the requisite torques for this dissipation and for over-all angular momentum conservation. A similar adjustment

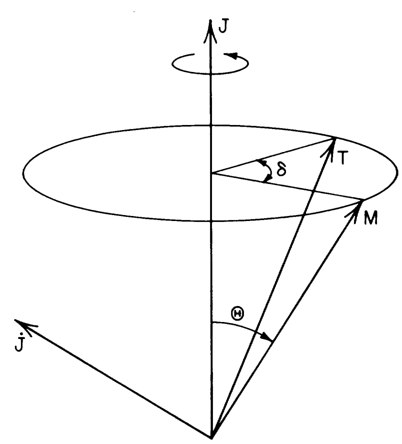

FIG. 2. Geometry of tidal lag. Shown here is the geometry of tidal lag when the perturbing moon is not in the equatorial plane. The vector $\vec{J}$ points along the direction of rotation of the black hole or fluid mass. The direction $\vec{M}$ points to the position of the moon and the direction $\vec{T}$ to the position of the tidal bulge. The direction $\vec{T}$ is simply the direction $\vec{M}$ rotated around $\vec{J}$ by the azimuthal shift angle δ. It continues to make an angle Θ with $\vec{J}$. For the black-hole case δ is negative. In Newtonian physics the torque $d\vec{J}/dt$ would point in a direction perpendicular to both $\vec{T}$ and $\vec{M}$.

in shape should take place in the instantaneous horizon of a black hole perturbed by a nonaxisymmetric distribution of exterior matter since there also the angular momentum of the black hole is changing.

In Newtonian theory the existence of tidal shifts can be demonstrated by comparing a perturbed system with dissipation to an identically structured system without dissipation. For example, in the theory of tidally distorted viscous fluid masses given in Sec. IV one compares the shape of the tide with viscosity to the shape when the viscosity has been set equal to zero.

In relativity there is no parameter like the viscosity which determines the amount of gravitational dissipation. Like gravitational radiation, gravitational tidal friction in black holes is part and parcel of the theory and there is no natural way of separating it out. In general, therefore, there is no natural configuration with which to compare a tidally distorted black hole in order to define the lag. One might consider comparing the configuration in which the black hole and the perturbation are co-rotating since the tidal friction vanishes in the latter case. However, the shapes of the instantaneous horizon in these two cases will not differ by a simple angular shift but by more complicated rotational distortions. In general cases, therefore, there seems to be no reasonable way to define tidal lag in black holes. In at least one limiting case, however, it is possible to exhibit the effect.

B. Stationary perturbations of slowly rotating black holes

Two facts make it plausible that stationary perturbations of slowly rotating black holes should exhibit the phenomenon of an angular tidal shift in a pure form. (1) To linear order in the specific angular momentum, a, the instantaneous horizon of a Kerr black hole has the same spherical shape as the horizon of a nonrotating black hole of the same mass. (2) A stationary perturbation of a nonrotating black hole induces a stationary tide with no dissipation. A comparison of a nonrotating black hole with a stationary perturbation and a slowly rotating black hole with the same perturbation is therefore a comparison of a situation with no tidal friction and a corresponding situation with tidal friction but no rotational distortion of the horizon. Here, then, we may expect to find a pure angular tidal shift. In the following we show this to be so.

To investigate the existence of a tidal shift it is necessary to evaluate Eq. (2.21) for $R^{(1)}$ to linear order in the specific angular momentum a. The quantity $\psi_0^{(1)}$ will be related to a stationary solution of the Teukolsky equation. To find the precise relation we need to know the connection between the unperturbed tetrad employed in deriving Eq. (2.21) and the Kinnersley tetrad used by Teukolsky in his paper. In the tetrad used here, on the horizon l^μ lies along the null geodesic generators and is normalized by $l^\mu t_{,\mu}=1$, while m^μ lies in the instantaneous horizon. Since the Boyer-Lindquist coordinates t', r', θ', ϕ' used by Teukolsky are singular on the horizon the time t' cannot be taken to be the Killing time t which defines the instantaneous horizon. Rather, we introduce a nonsingular coordinate system on the future horizon by the standard transformation

$$\begin{aligned} dt' &= dt - dr(r^2+a^2)/\Delta\,,\\ d\phi' &= d\phi - dra/\Delta\,,\\ r' &= r, \quad \theta'=\theta, \end{aligned} \tag{4.1}$$

and let constant t define the instantaneous horizon. Here

$$\Delta = r^2 - 2Mr + a^2 = (r-r_+)(r-r_-).$$

It is then not difficult to verify that tetrad vectors l^μ and m^μ which meet our requirements on the horizon are related to the corresponding Kinnersley tetrad vectors l_K^μ and m_K^μ used by Teukolsky through the relations

$$l^\mu = \frac{\Delta}{2(r^2+a^2)}\, l_K^\mu\,, \tag{4.2a}$$

$$m^\mu = m_K^\mu - \frac{ia\sin\theta}{\sqrt{2}\,(r+ia\cos\theta)}\, l^\mu\,. \tag{4.2b}$$

On the horizon then, if $a^\mu = (a^t, a^r, a^\theta, a^\phi)$, one has

$$l^\mu = (1,0,0,\omega_+)\,, \tag{4.3a}$$

$$m^\mu = \frac{1}{\sqrt{2}}\,\frac{1}{r+ia\cos\theta}\left(0,0,1,\frac{i}{\sin\theta} - ia\omega_+\sin\theta\right), \tag{4.3b}$$

where $\omega_+ = a/2Mr_+$ is the angular velocity of the horizon. If we denote by $\psi_{0K}^{(1)}$ the Weyl tensor component in the Kinnersley tetrad [Eq. (2.3)], then

$$\psi_0^{(1)} = \left[\frac{\Delta}{2(r^2+a^2)}\right]^2 \psi_{0K}^{(1)}\,. \tag{4.4}$$

For stationary perturbations of the Kerr geometry $\psi_{0K}^{(1)}$ may be expanded in spherical harmonics of spin-weight 2.

$$\psi_{0K}^{(1)} = \sum_{Lm} R_L^m(r)\,{}_2Y_L^m(\theta',\phi')\,. \tag{4.5}$$

The radial functions $R_L^m(r)$ satisfy the radial Teukolsky equation and are not coupled to each other. From the solutions, $\psi_0^{(1)}$ may be evaluated on the horizon $r=r_+$. One finds

$$\psi_0^{(1)}=\sum_{Lm} S^m_L\,{}_2Y^m_L(\theta,\phi), \qquad (4.6)$$

where

$$S^m_L=\lim_{r\to r_+}\left[\left(\frac{\Delta}{4Mr_+}\right)^2\left(\frac{r-r_-}{r-r_+}\right)^\gamma R^m_L(r)\right], \qquad (4.7)$$

and

$$\gamma=ima/(r_+-r_-). \qquad (4.8)$$

All the information about the exterior perturbation relevant to the shape of the instantaneous horizon is contained in the horizon multipole moments S^m_L.

In the following $R^{(1)}$ will be evaluated to linear order in a in terms of the S^m_L appropriate to a stationary exterior source. If S^m_L is expanded in powers of a,

$$S^m_L=S^{m[0]}_L+aS^{m[1]}_L+a^2S^{m[2]}_M+\cdots, \qquad (4.9)$$

this evaluation will need only the constant linear and quadratic terms in a [cf. Eq. (2.21)].

Certain general symmetry properties of these coefficients are useful in calculating $R^{(1)}$. For stationary perturbations, the constant term $S^{m[0]}_L$ must vanish identically in order that there is no tidal friction in a static Schwarzschild black hole acted on by a stationary perturbation and also so that $R^{(1)}$ remains finite in the limit of a tending to zero. Let us now consider the angular momentum and parity coupling involved in the remaining coefficients.

In the limit that a is zero, a perturbation having only a particular multipole (L,m) will become a perturbation of the Schwarzschild geometry of that multipole order. By appropriate choice of gauge[3] stationary perturbations of the Schwarzschild metric may be assumed to have parity $(-1)^L$ and to be perturbations in rotational scalars. If the Kerr geometry is expanded in powers of a, then the linear order considered as a perturbation of the Schwarzschild geometry will be a $(1,0)$ multipole having parity + while the quadratic order may be written[14] as $(0,0)$ and $(2,0)$ multipoles both having parity +.

Suppose that W^m_L is the coefficient of a typical term in the perturbation of the Schwarzschild metric of multipole order (L,m). If we let P denote the parity operation, then

$$PW^m_L=(-1)^LW^m_L, \qquad (4.10)$$

while the reality of the metric and the convention[15] $\bar{Y}^m_L=(-1)^mY^{-m}_L$ require

$$\bar{W}^m_L=(-1)^mW^{-m}_L. \qquad (4.11)$$

Quantities such as $S^{m[1]}_L$ which are linear in the perturbation and linear in a, must be formed from products of terms like W^m_L with the parts of the Kerr metric linear in a. Since $S^{m[1]}_L$ transforms under rotations like a multipole (L,m), then angular momentum composition dictates that it is of the form[16]

$$S^{m[1]}_L=A\begin{pmatrix} L & 1 & L\\ -m & 0 & m\end{pmatrix}W^m_L. \qquad (4.12)$$

Here, A is a constant linear in a and independent of m, and we have used the familiar 3-j symbol.[15]

The parity of S^m_L must be $(-1)^L$ since the Kerr metric parity is always +. To see how to enforce this, we first note that P sends $\theta\to\pi-\theta$ and $\phi\to\phi+\pi$. It follows from the form of the Kinnersley tetrad that $Pl^\mu=l^\mu$ and $Pm^\mu=\bar{m}^\mu$, whence

$$P\psi_0^{(1)}(r,\theta,\phi)=\bar{\psi}_0^{(1)}(r,\pi-\theta,\phi+\pi). \qquad (4.13)$$

Translating this into a statement about the S^m_L we find

$$PS^m_L=(-1)^{L+m}\bar{S}^{-m}_L. \qquad (4.14)$$

The 3-j symbol in Eq. (4.12) is linear in m. Therefore, in order that $S^{m[1]}_L$ have parity $(-1)^L$, the coefficient A must be pure imaginary. One then concludes that $S^{m[1]}_L$ has the general form

$$S^{m[1]}_L=imU^m_L, \qquad (4.15)$$

where

$$\bar{U}^m_L=(-1)^mU^{-m}_L. \qquad (4.16)$$

An exactly analogous argument can be made for $S^{m[2]}_L$, although it is slightly more complex. One finds

$$S^{m[2]}_L=(B+Cm^2)W^m_L, \qquad (4.17)$$

where B and C are real and are independent of m. Since tidal friction is absent in cases of pure axial symmetry ($m=0$) the coefficient B must vanish. Thus we can wirte

$$S^{m[2]}_L=\frac{m^2\xi_L}{2M}U^m_L, \qquad (4.18)$$

where ξ_L is a real constant independent of m.

In the following we calculate the tidal shift in terms of U^m_L and ξ_L. To do this, Eq. (2.21) for $R^{(1)}$ must be first written in a frame which is not rotating with respect to infinity. For stationary perturbations this simply amounts to replacing ω by $-m\omega_+$. The result must be expanded to linear order in a. In particular this involves the operator $\bar{\eth}\bar{\eth}$ on the horizon. A simple but tedious calculation shows that, when applied to a spin-weight-2 quantity on the horizon,

$$\bar{\eth}\bar{\eth}=\frac{1}{8M^2}\bar{\eth}_0\bar{\eth}_0\left(1+\frac{ia}{M}\cos\theta\right)+O(a^2). \qquad (4.19)$$

Here $\bar{\eth}_0$ is $2\sqrt{2}M$ times $\bar{\eth}$ evaluated at $a=0$ and is the lowering operator of Ref. 11 with the property

$$\bar{\eth}_0\, {}_sY_L^m = [(l+s)(l-s+1)]^{1/2}\, {}_{s-1}Y_L^m \ . \qquad (4.20)$$

Expanding the expression for $R^{(1)}$ using Eqs. (4.15), (4.18), (2.9), (4.19), and (4.20), one finds

$$R^{(1)} = R^{[0]} + aR^{[1]} + \cdots , \qquad (4.21)$$

where

$$R^{[0]} = 8M \sum_{Lm} \eta_L \operatorname{Re}[U_L^m Y_L^m(\theta, \phi)] \ , \qquad (4.22)$$

and

$$aR^{[1]} = 8M \sum_{Lm} \operatorname{Im}\left[\eta_L \frac{a}{2M} U_L^m(\xi_L - 2) m Y_L^m(\theta, \phi) - \frac{aU_L^m}{M} \bar{\eth}_0 \bar{\eth}_0 \cos\theta\, {}_2Y_L^m(\theta, \phi)\right] , \qquad (4.23)$$

with $\eta_L = [(L+2)(L+1)L(L-1)]^{1/2}$. The angular combination in the last term in Eq. (4.23) may be expanded in ordinary spherical harmonics. The relevant integral is

$$\int d\Omega\, \bar{Y}_L^m \bar{\eth}_0 \bar{\eth}_0 \cos\theta\, {}_2Y_L^m(\theta, \phi) = \eta_l \int d\Omega\, {}_2\bar{Y}_l^m \cos\theta\, {}_2Y_L^m(\theta, \phi) \ . \qquad (4.24)$$

Since the ${}_2Y_l^m$ are[11] $[(2l+1)/4\pi]^{1/2}$ times the irreducible representation of the rotation group $D^l_{-2m}(\phi, \theta, 0)$ this integral is simply an integral over three such functions. One finds finally

$$aR^{[1]} = 8M \operatorname{Im} \sum_{Lm} \left[\frac{\eta_L a}{2M}(\xi_L - 2) U_L^m m Y_L^m(\theta, \phi) - \frac{aU_L^m}{M} \sum_l \eta_l [(2L+1)(2l+1)]^{1/2} (-1)^m \begin{pmatrix} l & 1 & L \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} l & 1 & L \\ -m & 0 & m \end{pmatrix} Y_l^m(\theta, \phi)\right] . \qquad (4.25)$$

It follows from the symmetry of U_L^m [Eqs. (4.16) and (4.19)] and the parity of the 3-j symbols under change in sign of m, that the imaginary part of the second sum in Eq. (4.25) vanishes except when $l = L$. Evaluating the remaining 3-j symbols one has

$$aR^{[1]} = 8M \sum_{Lm} \left(-\zeta_L \frac{\partial}{\partial\phi}\right) \eta_L \operatorname{Re}(U_L^m Y_L^m) \ , \qquad (4.26)$$

where

$$\zeta_L = \frac{a}{2M}\ \frac{4}{L(L+1)} + \xi_L - 2 \ . \qquad (4.27)$$

This relation simply says that to first order in a each multipole leads in space (lags in time) by a coordinate angular shift ζ_L the corresponding multipole for a nonrotating black hole.

By itself a coordinate shift has no invariant significance. It could always be eliminated by a coordinate change of the form $\phi \to \phi + af(r)$. To give the shift an invariant significance an invariant connection must be given in the Kerr geometry between angular positions on the instantaneous horizon and angular positions at the source or angular positions at infinity. In order to keep the discussion general we will consider only the latter type of connection, and this will suffice for the example to be considered later where the source is near infinity. A natural way of making a connection between angular positions on the future horizon and angular positions at infinity is to say that two positions in the Kerr geometry correspond if they are connected by a zero angular momentum light ray. Such a light ray has momentum components $p_\theta = 0$ and $p_\phi = 0$ both of which are conserved to first order in a. A zero angular momentum light ray is therefore normal to the instantaneous horizon and radially directed at infinity. In the coordinate system used here [cf. Eq. (4.1) and the discussion before it] the net change in azimuthal angle ϕ of a light ray as it proceeds from infinity to the horizon is $a/2M$. This change is in the positive direction reflecting the rotational dragging. An invariant definition of tidal shift can therefore be considered to be the difference $\delta_L = \zeta_L - a/2M$, and we have

$$\delta_L = \frac{a}{2M}\left[\frac{4}{L(L+1)} + \xi_L - 3\right] . \qquad (4.28)$$

Thus in each multipole order the shape of the instantaneous horizon is the same for the slowly rotating black hole as for the nonrotating black hole, but the orientation is rotated about the axis of rotation by an azimuthal shift angle δ_L. If δ_L is positive, this corresponds to a positive angular shift or a tidal lag. The change in orientation is thus essentially the same as in the Newtonian theory illustrated in Fig. 2. The only dependence on the perturbation comes from the parameter ξ_L which is independent of m. The tidal shift therefore does not depend on the relative orientation of the perturbation and the black hole, exactly as is the case in Newtonian theory.

C. Tidal lead from a distant moon

Having established the existence of an angular shift in the tide raised on the instantaneous horizon of a slowly rotating black hole, it seems worthwhile to evaluate the shift angle [Eq. (4.28)] in one simple case—when the perturbation arises from a distant stationary moon of mass μ. This situation will also be a good approximation to the case of a distant moon in Keplerian orbit since the period of the orbit becomes arbitrarily long as the distance away from the black hole becomes large. Since the two situations are essentially indistinguishable if the moon is far away, the contribution to ξ_L of any stresses necessary to support a strictly stationary perturbation must be negligible in the large separation limit.

Teukolsky[17] has found the general stationary solutions to his equation and used them to evaluate $\psi_0^{(1)}$ for just this problem. Let the moon be located at Kerr coordinates R, Θ, Φ. When Teukolsky's result is translated into our tetrad one has on the horizon

$$S_L^m = \frac{\eta\mu}{R^{L+1}}\eta_L \frac{\Gamma(L-1)\Gamma(1+L+2\gamma)}{\Gamma(2L+2)\Gamma(-1+L+2\gamma)} \times \frac{(r_+ - r_-)^{L+2}}{(2Mr_+)^2}\bar{Y}_L^m(\Theta,\Phi)\,{}_2Y_L^m(\theta,\phi)\ , \qquad (4.29)$$

as the leading term in an expansion in powers of $1/R$ for each multipole order. The parameter γ is defined in Eq. (4.8). As might be expected, the quadrupole order ($L=2$) gives the dominant contribution to the tide at large R. Since r_+ and r_- depend quadratically on a the only difference between the linear and quadratic terms in S_2^m comes from the factorial in Eq. (4.29). One easily finds

$$U_2^m = \frac{\pi\mu}{5\sqrt{6}\,MR^3}\bar{Y}_2^m(\Theta,\Phi), \qquad (4.30)$$

and

$$\xi_2 = -1\ . \qquad (4.31)$$

For a nonrotating black hole the curvature of the instantaneous horizon becomes [Eqs. (4.22) and (2.1)]

$$\mathcal{R} = \frac{1}{2M^2} - \frac{4\mu}{R^3}P_2(\cos\chi)\ , \qquad (4.32)$$

where χ is the angle between the point of interest and the direction to the moon. A surface in a three-dimensional flat space which has this curvature to first order in the perturbation is, in polar coordinates,

$$r = 2M\left[1 + 2\frac{\mu M^2}{R^3}P_2(\cos\chi)\right]\ . \qquad (4.33)$$

This result, already established in Ref. 3, shows how the moon distorts the instantaneous horizon of a nonrotating black hole into the characteristic spheroidal shape (Fig. 1) with the tidal bulge pointing directly at the moon. The effect of a slow rotation is to preserve the shape of the instantaneous horizon but to rotate its orientation about the black hole's rotation axis by an angle [Eqs. (4.31) and (4.28)]

$$\delta_2 = -\frac{5}{3}\frac{a}{M}\ . \qquad (4.34)$$

The geometry of the reorientation is precisely that of Newtonian theory as shown in Fig. 2, but the tide lags in space rather than leads, i.e., it leads in time. As in Newtonian theory, the shift angle is independent of the parameters of the perturbation. Indeed, the relativistic shift angle has the same magnitude and form as that for a viscous fluid mass provided one takes the radius of the mass of order of magnitude of its mass and takes for the dimensionless measure of viscosity, ν/M, a number of order of magnitude unity.

The most striking departure from Newtonian theory is that the tide on the instantaneous horizon leads rather than lags. In Newtonian theory the orientation of the tidal bulge is directly connected to the sign of the torque exerted on the moon. A temporal lag is required if the torque is to have the sign required by over-all angular momentum conservation. In general relativity the connection between the orientation of the tidal bulge and the sign of the torque at infinity is much less direct. Indeed, our results show that even with an invariantly defined connection between the horizon and infinity as given here the relation between the sign of the torque and the orientation of the tidal bulge can be opposite to what might be intuitively expected from the Newtonian analogy.

The origin of the tidal lead in the black hole problem is not difficult to discover. It comes ultimately from the structure of Eq. (2.11) giving the time development of the shear. The term $2\epsilon\sigma^{(1)}$ enters as an antidamping term in the equation not present in Newtonian theory [cf. Eq. (3.9)]. The growth of the shear which this antidamping would produce if the time evolution calculation were treated as an initial-value problem is not present in the black-hole case because the defining boundary condition is imposed at future infinity. The antidamping does show up in a contribution to shift angle of opposite sign to that of the damping terms present in Newtonian theory. Thus, in the case of slow rotation the analogy between a black hole acted upon by a stationary perturbation and the tidal friction effects in a rotating viscous fluid mass is close but not complete. Once $\nu/M \sim 1$ both problems have the same form for the spin-down rate and both exhibit a tidal shift of the same order of magnitude. The shifts have opposite signs, however, because of the presence of antidamping in the black hole case and the different ways in which the temporal boundary conditions are defined. This analogy is complete enough, however, that by using it one can conjecture the answer to an as yet unsolved problem in the black-hole case. This is the problem of computing the rate of change of the *direction* of the angular momentum of a slowly rotating black hole acted upon by, say a distant moon. (Only the change in the magnitude of $\vec{J}$ was treated in Ref. 2.) From Fig. 2 one sees that in Newtonian theory whether the shift angle is positive or negative the ratio of $J_\parallel$, the component of the angular momentum parallel to the rotation axis, to $J_\perp$, the component perpendicular to the rotation axis, is

$$\frac{J_\parallel}{J_\perp}=\tan\Theta\ , \tag{4.35}$$

Here Θ is the angle between the moon and the rotation axis. On the basis of the analogy with Newtonian theory one would expect essentially the same result to hold in the black-hole case.

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APPENDIX: GAUSS'S RELATION IN NEWMAN-PENROSE NOTATION

In this appendix we express in Newman-Penrose notation Gauss's relation[18] connecting the intrinsic scalar curvature of a spacelike two-surface with its extrinsic curvature and the four-dimensional curvature of space-time. To do this, we introduce two coordinates x^A ($A=1,2$) which range over the two-surface and two orthogonal unit vectors $N_\mu^{(i)}$ ($i=1,2$) normal to the two-surface. Since the surface is spacelike one normal can be chosen spacelike and the other timelike, viz.

$$g^{\mu\nu}N_\mu^{(i)}N_\nu^{(j)}=\epsilon^{(i)}\delta^{(i)(j)}\ , \tag{A1}$$

where

$$\epsilon^{(1)}=1,\quad \epsilon^{(2)}=-1\ . \tag{A2}$$

If $\mathcal{R}_{ABCD}$ is the intrinsic curvature of the two-surface and R_{ABCD} the curvature of space-time, Gauss's relation becomes

$$\mathcal{R}_{ABCD}=R_{ABCD}+\sum_{(i)}\epsilon^{(i)}[h_{AC}^{(i)}h_{BD}^{(i)}-h_{AD}^{(i)}h_{BC}^{(i)}]\ . \tag{A3}$$

Here A, B, C, D range over 1, 2 and $h_{AB}^{(i)}$ is the extrinsic curvature

$$h_{AB}^{(i)}=-N_{A;B}^{(i)}\ . \tag{A4}$$

To form the contraction, $\mathcal{R}$, we use the fact that since m^μ and $\bar{m}^\mu$ are linearly independent vectors lying in the two-surface, the induced metric ${}^2g^{AB}$ can be expressed as

$${}^2g^{AB}=-(m^A\bar{m}^B+\bar{m}^Am^B)\ . \tag{A5}$$

Contracting Eq. (A3) one finds

$$\mathcal{R}=2\Big[R_{\alpha\beta\gamma\delta}+\sum_{(i)}\epsilon^{(i)}(h_{\alpha\gamma}^{(i)}h_{\beta\delta}^{(i)}-h_{\alpha\delta}^{(i)}h_{\beta\gamma}^{(i)})\Big]\times m^\alpha\bar{m}^\beta m^\gamma\bar{m}^\delta\ , \tag{A6}$$

where the definition of the $h_{AB}^{(i)}$ has been extended in the obvious way. Now, in the vacuum case of interest it is easy to verify [cf. Eq. (2.3)]

$$R_{\alpha\beta\gamma\delta}m^\alpha\bar{m}^\beta m^\gamma\bar{m}^\delta=-2\mathrm{Re}(\psi_2)\ . \tag{A7}$$

Further, since the normals may be expressed in terms of l^μ and n^μ by

$$N_\mu^{(1)}=2^{-1/2}(l_\mu+n_\mu)\ , \tag{A8a}$$

$$N_\mu^{(2)}=2^{-1/2}(l_\mu-n_\mu)\ , \tag{A8b}$$

the components of the extrinsic curvatures may be expressed in terms of ρ, σ, μ, and $\bar{\lambda}$.

$$h_{\mu\nu}^{(1)}m^\mu m^\nu=\sigma-\bar{\lambda}\ , \tag{A9a}$$

$$h_{\mu\nu}^{(1)}m^\mu\bar{m}^\nu=\rho-\mu\ , \tag{A9b}$$

$$h_{\mu\nu}^{(2)}m^\mu m^\nu=\sigma+\bar{\lambda}\ , \tag{A9c}$$

$$h_{\mu\nu}^{(2)}m^\mu\bar{m}^\nu=\rho+\mu\ . \tag{A9d}$$

Combining Eq. (A9) with Eq. (A7) in Eq. (A6) we have the desired result

$$\mathcal{R}=4\,\mathrm{Re}(\rho\mu-\lambda\sigma-\psi_2)\ . \tag{A10}$$

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[8]Our conventions are basically the same as those of Ref. 2 except that our Ricci tensor and scalar curvatures have the opposite sign to theirs. In particular we use signature +---; define the Riemann tensor so that for any vector A^μ, $A_{\mu;\alpha\beta}-A_{\mu;\beta\alpha}=R^\nu{}_{\mu\alpha\beta}A_\nu$ and define the Ricci tensor by $R_{\mu\nu}=R^\alpha{}_{\mu\alpha\nu}$. The two-dimensional scalar curvature of a sphere of radius a is thus $2/a^2$.

[9]See, S. Hawking, J. Math. Phys. 9, 598 (1968), or J. B. Hartle and D. C. Wilkins (unpublished) for a derivation.

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[13]Notation in this section on Newtonian physics is not to be confused with that used in previous and subsequent sections on relativity.

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[15]Our notation for angular momentum algebra and spherical harmonics follows A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, N. J., 1953).

[16]Arguments like those used in this paragraph may look unfamiliar but are really no different from ones of the form: "Given two vectors a^μ, b^μ, then the most general second-rank tensor constructed from them is of the form $Aa^\mu b^\nu + Ba^\nu b^\mu \cdots$." For expositions of these methods see E. P. Wigner, in *Quantum Theory of Angular Momentum,* edited by L. Biedenharn and H. van Dam (Academic, New York, 1965), and U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic, New York, 1959).

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