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Generalized constraints and mass spectra in classical spin theory

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The classical theory of a relativistic spinning point particle is reexamined and reformulated from a new perspective. We introduce a class of generalized constraints on the spin angular momentum tensor, whose imposition has not been previously considered. A constraint-dependent functional relationship between the observable rest mass and spin is deduced, and the implied trajectories analyzed.

I. INTRODUCTION

The purpose of this paper is to introduce a generalized version of the classical relativistic theory of a free spinning point particle originally formulated by Thomas[1] and Frenkel[2] (T-F). Let us review briefly some aspects of that original formulation. The instantaneous four-vector position of the particle is denoted by x_μ, p_μ denotes the conserved four-momentum conjugate to x_μ, and $S_{\mu\nu}$ denotes the antisymmetric spin angular momentum tensor. The total angular momentum tensor, $M_{\mu\nu}$, is given by

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu} \ , \tag{1}$$

whose conservation yields

$$\dot{S}_{\mu\nu} = p_\mu v_\nu - p_\nu v_\mu \ . \tag{2}$$

In our notation $x_\mu = (\vec{x}, ict)$, the dot denotes differentiation with respect to the proper time σ, and $v_\mu = \dot{x}_\mu$ is the instantaneous four-velocity of the particle, with $v_\mu = \gamma(\vec{v}/c, i)$ and $v_\mu v_\mu = -1$. Since v_μ need not be collinear with p_μ, $S_{\mu\nu}$ need not be separately conserved.[1-3]

In this T-F theory $S_{\mu\nu}$ is subject to the constraint

$$S_{\mu\nu} v_\nu = 0 \ , \tag{3}$$

which ensures that

$$S_{\mu\nu}\dot{S}_{\mu\nu} = \frac{d}{d\sigma}(\tfrac{1}{2}S_{\mu\nu}S_{\mu\nu}) = 0 \ . \tag{4}$$

Thus

$$S_{\mu\nu}S_{\mu\nu} \equiv 2S_0^{\,2} \tag{5}$$

is the Lorentz-invariant conserved-spin magnitude in a frame defined to be at rest with respect to the particle, i.e., $\vec{v} = 0$. We refer to this frame as the intrinsic rest frame (IRF).[4] Thus S_0 is properly termed an intrinsic particle parameter. The other intrinsic parameter is the rest mass m_0 defined by

$$v_\mu p_\mu = -m_0 c \ , \tag{6}$$

since then m_0c^2 is the energy when $\vec{v} = 0$. In an arbitrary Lorentz frame, i.e., $\vec{v} \neq 0$, we have

$$S_{\mu\nu}S_{\mu\nu} = 2(S^2 - \tau^2) = 2S_0^{\,2} \ , \tag{7}$$

where $S_{ij} = \epsilon_{ijk}S_k$ and $i\tau_j = S_{j4}$, with

$$c\vec{\tau} = \vec{v} \times \vec{S} \ . \tag{8}$$

The constraint (3) then implies that $\vec{\tau} \to 0$ as $\vec{v} \to 0$.

It has been shown previously[3] that in the momentum rest frame (MRF), defined by $p_\mu = (\vec{0}, i(E/c))$, $\vec{v}$ need not vanish, and that

$$\vec{\eta} = -c\vec{\tau}/E \tag{9}$$

is the spatial separation of the particle from its mass center. Therefore τ is real and it follows from (7) that in the MRF

$$S^2 \geq S_0^{\,2} \ . \tag{10}$$

The fact that the spin magnitude in the MRF is larger than the intrinsic rest spin is due to the internal orbital angular momentum contributions which accompany the particle gyrations about the stationary center of mass.

The T-F theory was proposed before there was much interest in hadron spectroscopy. Not surprisingly then, it was much later that the spectral content of their theory was sought. The result was disappointing but instructive. Before simply stating the result we must specify what is meant by the spectral content of a particle theory. A primary concern in any particle theory, proposed as an aid to understanding elementary particle properties, is the ability of the theory to predict relationships between the observable values of the rest energy and spin which even resemble the gross behavior of the hadron spectrum. If the theory is a classical theory of a free spinning point particle, then the spectral content, or predicted mass-spin relationship, is a trajectory, i.e., a plot of the observable mass-spin dependence. The terms "observable rest mass" and "spin values" refer, as always, to these values evaluated in the MRF.

Defining the observable rest mass, m, through the relation

$$p_\mu p_\mu = -m^2c^2 \tag{11}$$

the T-F spectral content was shown by several authors to be given by descending trajectory[3,5,6]

$$m = m_0(S_0/S)\ , \tag{12}$$

where S is the spin magnitude in the MRF. This trajectory clearly has little in common with particle data.

II. GENERALIZED CONSTRAINTS

We can generalize the T-F formulation without giving up the notion of an IRF, with respect to which intrinsic mass and spin parameters may be simultaneously assigned, by replacing (3) with

$$S_{\mu\nu}\xi_\nu = 0\ , \tag{13}$$

where

$$\xi_\mu = v_\mu + gp_\mu\ , \tag{14}$$

with g an as yet unspecified Lorentz-invariant constant of the motion. This class of constraints preserves Eqs. (4) and (5), and hence ensures the existence of a specifiable IRF. Now, however, S_0 is the Lorentz-invariant conserved spin magnitude in the frame defined by

$$\vec{v}/c + g\vec{p} = 0\ , \tag{15}$$

which frame is the new IRF. The T-F formulation corresponds to the special case, $g=0$.

Let us now deduce the functional relationship between the observable rest mass and spin implied by this class of constraints. Multiplying (2) by v_ν gives

$$\dot{S}_{\mu\nu}v_\nu = -p_\mu + m_0cv_\mu\ . \tag{16}$$

From (13) and (14) we get

$$\begin{aligned} S_{\mu\nu}\dot{\xi}_\nu &= -\dot{S}_{\mu\nu}\xi_\nu \\ &= -\dot{S}_{\mu\nu}v_\nu - g\dot{S}_{\mu\nu}p_\nu\ . \end{aligned} \tag{17}$$

Using (2) in the last term of (17) gives

$$\dot{S}_{\mu\nu}v_\nu = -S_{\mu\nu}\dot{\xi}_\nu + gm_0cp_\mu - gm^2c^2v_\mu\ . \tag{18}$$

Eliminating $\dot{S}_{\mu\nu}v_\nu$ between (16) and (18) gives

$$S_{\mu\nu}\dot{\xi}_\nu = (1+gm_0c)p_\mu - (gm^2c^2 + m_0c)v_\mu\ . \tag{19}$$

If we now multiply (19) through by $S_{\rho\sigma}S_{\sigma\mu}$ and use the identity[7]

$$2S_{\rho\sigma}S_{\sigma\mu}S_{\mu\nu} = -(S_{\alpha\beta}S_{\alpha\beta})S_{\rho\nu} \tag{20}$$

and then use the constraint equations to eliminate the explicit appearance of v_μ, we obtain

$$\begin{aligned} -\tfrac{1}{2}(S_{\alpha\beta}S_{\alpha\beta})S_{\rho\nu}\dot{\xi}_\nu = [(1+gm_0c) + g(gm^2c^2 + m_0c)] \\ \times S_{\rho\sigma}S_{\sigma\mu}p_\mu\ . \end{aligned} \tag{21}$$

Substituting for $S_{\rho\nu}\dot{\xi}_\nu$ its equivalent on the right-hand side of (19) gives

$$\begin{aligned} -\tfrac{1}{2}(S_{\alpha\beta}S_{\alpha\beta})[(1+gm_0c)p_\rho - (gm^2c^2 + m_0c)v_\rho] \\ = (1 + 2gm_0c + g^2m^2c^2)S_{\rho\sigma}S_{\sigma\mu}p_\mu\ . \end{aligned} \tag{22}$$

Multiplying through by p_ρ gives

$$\begin{aligned} -\tfrac{1}{2}(S_{\alpha\beta}S_{\alpha\beta})(-m^2c^2 + m_0{}^2c^2) \\ = (1 + 2gm_0c + g^2m^2c^2)\, S_{\rho\sigma}S_{\sigma\mu}p_\rho p_\mu\ . \end{aligned} \tag{23}$$

Notice that

$$S_{\rho\sigma}S_{\sigma\mu}p_\rho p_\mu = p_\rho p_\rho W_\mu W_\mu - \tfrac{1}{2}S_{\alpha\beta}S_{\alpha\beta}p_\rho p_\rho\ , \tag{24}$$

where

$$W_\mu = -i(p_\alpha p_\alpha)^{-1/2}\epsilon_{\mu\nu\rho\sigma}S_{\nu\rho}p_\sigma \tag{25}$$

is the Pauli-Lubanski four-vector for which $\dot{W}_\mu = 0$, $W_\mu p_\mu = 0$. Thus (23) may be equivalently written as

$$\begin{aligned} &(m^2c^2)^2(g^2W_\mu W_\mu - \tfrac{1}{2}g^2S_{\mu\nu}S_{\mu\nu}) \\ &\quad + m^2c^2(W_\mu W_\mu + 2\,gm_0cW_\mu W_\mu - gm_0cS_{\mu\nu}S_{\mu\nu}) \\ &\quad - \tfrac{1}{2}m_0{}^2c^2S_{\mu\nu}S_{\mu\nu} = 0\ . \end{aligned} \tag{26}$$

Further notice that

$$W_\mu W_\mu = \tfrac{1}{2}S'_{\mu\nu}S'_{\mu\nu}\ , \tag{27}$$

where

$$S'_{\mu\nu} = \Delta_{\mu\rho}\Delta_{\nu\sigma}S_{\rho\sigma}\ , \tag{28}$$

with

$$\Delta_{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p_\alpha p_\alpha}\ . \tag{29}$$

Thus for $S'_{\mu\nu}$ we have $\dot{S}'_{\mu\nu} = 0$, $S'_{\mu\nu}p_\nu = 0$. Hence $S'_{\mu\nu}$ is the antisymmetrical tensor equivalent of W_μ. Equation (26) may therefore be equivalently written as

$$\begin{aligned} &(m^2c^2)^2[\tfrac{1}{2}g^2(S'_{\mu\nu}S'_{\mu\nu} - S_{\mu\nu}S_{\mu\nu})] \\ &\quad + m^2c^2[gm_0c(S'_{\mu\nu}S'_{\mu\nu} - S_{\mu\nu}S_{\mu\nu}) + \tfrac{1}{2}S'_{\mu\nu}S'_{\mu\nu}] \\ &\quad - \tfrac{1}{2}m_0{}^2c^2S_{\mu\nu}S_{\mu\nu} = 0\ . \end{aligned} \tag{30}$$

If we now write

$$S'_{\mu\nu}S'_{\mu\nu} = 2S^2\ , \tag{31}$$

where S is the spin magnitude in the MRF, i.e., the observable spin, then (30) assumes the relatively simple form

$$\begin{aligned} (m^2c^2)^2g^2(S^2 - S_0{}^2) + m^2c^2[2gm_0c(S^2 - S_0{}^2) + S^2] \\ - m_0{}^2c^2S_0{}^2 = 0\ . \end{aligned} \tag{32}$$

Equation (32) constitutes the generalized constraint-dependent functional relationship between the observable rest mass and spin.

III. MASS SPECTRA

For a prescribed choice of g Eq. (32) will yield a solution, $m=m(S)$. Each solution is equivalent to a trajectory. The special case $g=0$ yields the relation (12), as expected. An interesting case arises when $g=-m_0c/m^2c^2$; this particular choice fixes $m=m_0$ for all spins and has a well-defined physical interpretation. The covariant generalization of (9),[3,8-10]

$$\eta_\mu = -\frac{S_{\mu\nu}p_\nu}{p_\sigma p_\sigma}, \tag{33}$$

is the covariant coordinatization of the center of mass of the spinning particle. Differentiating (33) with respect to the proper time gives

$$\dot{\eta}_\mu = v_\mu - \frac{m_0c}{m^2c^2}p_\mu . \tag{34}$$

Thus the choice $\xi_\mu = \dot{\eta}_\mu$ specifies $g=-m_0c/m^2c^2$. This is the one constraint which forces $\vec{p}$ and $\vec{v}$ to vanish simultaneously, i.e., the frame defined by $\dot{\vec{\eta}}=0$ is the frame in which the particle is at rest with respect to the center of mass. Thus with this constraint the model reduces to one in which there is no internal motion in the MRF, hence no contribution to the observable spin from such motion.

Equation (32) may be reduced to the form

$$h^2f^2(r)(r-1)+f(r)[2h(r-1)+r]-1=0 , \tag{35}$$

where $S^2/S_0^2=r$, $g=h/m_0c$, $m^2c^2=m_0^2c^2f(r)$. For $r=1$, we see that $f(r)=1$, so that $m^2=m_0^2$ if $S^2=S_0^2$. Therefore since $S^2 \geqslant S_0^2$ we may restrict the analysis of (35) to the region $r>1$. The form of (35) then permits a simple classification of trajectories. It is sufficient to seek the form of h for $f(r)=r^n$. Solving the quadratic equation, (35), for h yields

$$h=r^{-n}\left\{-1\pm\left[1-\left(\frac{r^{n+1}-1}{r-1}\right)\right]^{1/2}\right\} . \tag{36}$$

The solution, (36), clearly distinguishes three cases:

(a) For $n=0$ Eq. (36) gives $h=-1$, which is the special case already considered, i.e., $m^2=m_0^2$ for all spins.

(b) For all $\eta<0$ the equation yields a real value of h. Thus there exist any number of descending trajectories, distinguished by their rate of descent. To each trajectory there corresponds a physically realizable constraint of the form (13), (14).

(c) For all $n>0$, h is complex. Thus, while it is mathematically possible to obtain rising trajectories, it would appear that no physically realizable constraints of the form (13), (14) exist which will admit these rising trajectories.

IV. CONCLUSION

We have presented a generalization of the T-F theory for a classical relativistic spinning point particle. The generalization consists of the introduction of a new class of constraints which broaden the class of physical particle motions but which still allow the assignment of intrinsic particle parameters. The T-F theory then becomes a special case. Some implications of this generalization are currently under investigation. However, since rising trajectories appear to be ruled out, it is doubtful that the theory presented can aid in understanding hadron resonance properties.

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[1]L. H. Thomas, Philos. Mag. 3, 1 (1927).
[2]J. Frenkel, Z. Phys. 37, 243 (1926).
[3]H. C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day, San Francisco, 1968). An exhaustive list of references to the literature on classical, relativistic particles with spin is found in Chap. 2.
[4]K. Rafanelli, Phys. Rev. 185, 1670 (1969).
[5]K. Rafanelli, J. Math. Phys. 8, 1440 (1967).
[6]R. Schiller and L. Jacobi, Phys. Rev. 186, 1360 (1969).
[7]S. Shanmugadhasan, Can. J. Phys. 31, 1 (1953).
[8]M. H. L. Pryce, Proc. R. Soc. A195, 62 (1948).
[9]F. Halbwachs, Nuovo Cimento 36, 832 (1965).
[10]K. Rafanelli, Phys. Rev. 155, 1420 (1967).