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Gravitation and magnetic charge*

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Based on the static spherically symmetric solutions of the generalized theory of gravitation it is found that electric and magnetic charges are two fundamental constants of integration and that the corresponding electric, magnetic, and gravitational fields are regular everywhere only if the magnetic charge $\mathfrak{g} \neq 0$ ($\sim 10^{18}e$). The magnetic charge $\mathfrak{g}$ assumes an infinite spectrum of values and is an invertible function of mass. For magnetic charge $\mathfrak{g}=0$, the solutions reduce to the Nordström solution of general relativity in the limit of large r. The theory leads to elementary particles of finite self-energy $[\Delta(\pm E)=mc^2-(2\mathfrak{g}_0)^2/l_0]$ and binding energy. The structure of an elementary particle which is due to the existence of finite $\pm\mathfrak{g}$ consists of a magnetically neutral core of matter containing a distribution of magnetic charge density in stratified layers of sharply decreasing magnitude and alternating signs so that magnetic monopoles associated with a long-range field do not exist. As a consequence of the general covariance of the theory the surfaces of zero magnetic charge density in the particle core have an *indeterminacy*. These facts lead to a mass spectrum for elementary particles. In addition to charged electric and magnetic currents, the theory yields an electrically neutral current and the corresponding fields. The neutral current and the corresponding neutral field are the classical counterparts of the vacuum polarization in quantum electrodynamics. For every positive-energy solution there exists also a negative-energy solution with the corresponding electric charge. For $\mathfrak{g}=0$, the volume integral of the neutral current density diverges. The asymmetry of Maxwell's equations with regard to the absence of a magnetic current can be understood because the neutral and charged magnetic currents and the neutral part of the electric current are localized in the core of the elementary particle. Furthermore, the theory yields two lengths of the dimensions of 10^{-25} and 10^{-15} cm which could serve to differentiate between leptonic and hadronic processes. The presence of negative-energy solutions along with positive-energy solutions points to a large-scale symmetry with respect to a distribution of matter and antimatter in the universe.

I. INTRODUCTION

Gravitational and electromagnetic interactions, except in general relativity, where the electromagnetic field is incorporated into the field equations, are treated independently from one another. Despite the apparent unification of electromagnetic and gravitational fields (i.e., general relativity plus Maxwell's equations) the particles are still described as singularities of the field, and therefore the theory inherits all the major difficulties (infinite self-energy and other divergences) of classical electrodynamics. One does, of course, obtain from the field equations the mechanical law of motion of these singularities in the form of the Lorentz equations of motion. These equations of motion in electrodynamics proper have to be postulated independently from the equations of the electromagnetic field. However, the derivation of the law of motion from general relativity does not even circumvent the difficulty that the field assumes infinite values along the trajectories. Furthermore, just as in classical electrodynamics, the theory does not provide a prescription for the distribution of charge. There is no new idea for the removal of the difficulty associated with the action of the particle's own field on itself, which results in another infinity. Hence the problem of self-energy, even at the classical level, remains one of the most important unsolved problems of theoretical physics.

Einstein's general relativity accounts for the gravitational field in terms of the curvature of space. Electrodynamics, another example of a long-range field, has so far not been formulated on a geometrical basis. One of the motivating ideas in the geometrization of the gravitational forces was Mach's principle according to which the inertial properties of a particle (or more generally of energy) depend on the distribution of matter in the rest of the universe. A further profound observation was the formulation of the principle of equivalence (the equality between inertial and gravitational mass). The principle of equivalence does not apply directly to the electromagnetic field except through the gravitational field it produces as a result of its energy density in space. This is the extent of the electromagnetic field's involvement with the principle of equivalence. However, it is necessary to account fully for the action of the gravitational field on the electromagnetic field itself. The latter plays an important role in ensuring the regularity of the field everywhere. Because of the correspondence principle of the generalized theory of gravitation with respect to general relativity the principle of equivalence and Mach's principle are incorporated into the general framework of the theory in a natural way.

In the presence of the electromagnetic field the general theory of relativity is based on 10 + 6 field equations which determine the field variables $g_{\mu\nu}$ and $F_{\mu\nu}$ (the electromagnetic field). We may if we wish neglect the gravitational field and solve what is left (Maxwell's equations) for $F_{\mu\nu}$. *In the generalized theory of gravitation the 16 field equations contain the sources of the fields, and in turn these sources can only be determined through the knowledge of the fields. The basic physical reality in this theory is the field itself; all other observable quantities are derivable either as functions of these fields or as constants of integration (=constants of the motion) of the field equations.*

The fundamental premises of a theory sketched above can only be found in the unification of the forces that are well understood with respect to their over-all behavior in the asymptotic region where the origin or the structure of the particle is not included. The correct way to incorporate the structure of the elementary particle entails not only restoring a basic symmetry into the description of the electromagnetic field by introducing the axial magnetic current density, but using one that is associated only with a *short-range magnetic field*. Furthermore, the compatibility of the general covariance of the theory with an extended structure is made possible through an *indeterminacy* in the distribution of the magnetic charge density in the magnetically neutral core of an elementary particle. All of these, *inter alia*, will result from the unification of the two most fundamental theories of classical electromagnetic field and the general relativistic theory of gravitation.

One of the basic difficulties facing the nonsymmetric generalization of general relativity was its physical interpretation in terms of the familiar concepts of physics. This paper contains some progress on the physical implications of the theory and is the first of a series of papers which the author hopes to present on this subject. In Sec. II the author's version of the generalized theory of gravitation is summarized in the light of its newly established physical interpretation. In particular, the identification of the various quantities and the corresponding physical interpretation of the theory differ entirely from those contained in the earlier papers.[1-3] We present a direct assessment of its physical meaning in Secs. III and IV, where we derive the static spherically symmetric forms of the 16 field equations. The first two constants of integration, the electric and magnetic charges, play fundamental roles in the classification of the long- and short-range forces, in describing electrically neutral matter, and in ensuring the regularity of the solutions everywhere. These results and symmetries of the field are discussed in Secs. IV and V. The electrically charged and neutral currents (or polarization currents) and the corresponding fields together with their asymptotic behavior at and near the origin are discussed in Sec. VI. In Sec. VII we give an exact solution for the special case of a spherically symmetric and static field of zero magnetic charge which in the asymptotic limit of large distances reduces to the Nordström solution of general relativity. The same section contains the proof of a theorem on the absence of regular solutions for zero magnetic charge. Section VIII pertains to the neutral magnetic charge distribution in the case of an elementary particle and concludes with the "magnetic theorem" which relates the magnetic charge, the regularity of the field, the structure of the elementary particles, and the *indeterminacy* of the surfaces of zero magnetic charge in the particle core, and gives its basic role in the correspondence principle of the theory. In Sec. IX the finite self-energy and binding energy of an elementary particle are calculated. This section contains some remarks on the possible cosmological implications of the theory. The paper concludes with Sec. X, where a general discussion of the results and also a list of relevant problems for further work have been included. The same section contains a suggestion for the quantization of the theory and for the physical interpretation of the negative-energy solutions of the field equations.

II. GENERALIZED THEORY OF GRAVITATION

The theory is based on the nonsymmetric generalization of the symmetric theory (general relativity). The fundamental field variables are the components of the nonsymmetric tensor

$$\hat{g}_{\mu\nu}=\hat{g}_{\{\mu\nu\}}+q^{-1}\hat{g}_{[\mu\nu]}, \tag{2.1}$$

where the constant q is introduced in order to interpret the antisymmetric part $\hat{g}_{[\mu\nu]}$ as a generalized electromagnetic field and the symmetric part $\hat{g}_{\{\mu\nu\}}$ as the gravitational field. Thus the constant q has the dimensions of an electric field and will be calculated from the solutions of the field equations for $\hat{g}_{\mu\nu}$. For convenience we shall introduce the tensors $g_{\mu\nu}$ and $\Phi_{\mu\nu}$ by

$$\hat{g}_{\{\mu\nu\}} = \hat{g}_{\{\nu\mu\}} = g_{\mu\nu},$$
$$\hat{g}_{[\mu\nu]} = -\hat{g}_{[\nu\mu]} = \Phi_{\mu\nu}, \tag{2.2}$$

where $g_{\mu\nu}$ will assume the role of a metric tensor in space-time.

The tensor $\hat{g}_{\mu\nu}$ is reducible with respect to a transformation of the coordinates since the symmetric and antisymmetric parts of $\hat{g}_{\mu\nu}$ transform separately. This approach of Einstein's was criticized by many physicists on the basis that the generalized tensor $\hat{g}_{\mu\nu}$ was reducible and therefore gravitation and electromagnetism were not unified. However, these objections had no physical motivation. If we were to introduce some irreducible quantity to describe both fields as inseparable from one another then we would have to abandon the principle of equivalence and thereby destroy the fundamental premises of general relativity. Thus the reducibility of the generalized quantity $\hat{g}_{\mu\nu}$ is a physical necessity in order to preserve the basic differences between the two long-range forces of nature, gravitation and electromagnetism. Furthermore, besides the reducible tensor $\hat{g}_{\mu\nu}$ the generalized theory employs its inverse $\hat{g}^{\mu\nu}$, viz.,

$$\hat{g}^{\mu\rho}\hat{g}_{\nu\rho} = \hat{g}^{\rho\mu}\hat{g}_{\rho\nu} = \delta^{\mu}_{\nu}, \tag{2.3}$$

where the contractions of indices are correlated, the symmetric and antisymmetric parts of $\hat{g}^{\mu\nu}$ are given by

$$\tfrac{1}{2}(\hat{g}^{\mu\nu} + \hat{g}^{\nu\mu}) = \hat{g}^{\{\mu\nu\}} = \frac{g^{\mu\nu}(1+\Omega) - \Phi^{\mu\rho}\Phi^{\nu}{}_{\rho}}{1+\Omega-\Lambda^2}, \tag{2.4}$$

$$\tfrac{1}{2}(\hat{g}^{\mu\nu} - \hat{g}^{\nu\mu}) = \hat{g}^{[\mu\nu]} = \frac{\Phi^{\mu\nu} - \Lambda f^{\mu\nu}}{1+\Omega-\Lambda^2}, \tag{2.5}$$

$$\Omega = \tfrac{1}{2}\Phi^{\mu\nu}\Phi_{\mu\nu}, \quad \Lambda = \tfrac{1}{4}f^{\mu\nu}\Phi_{\mu\nu}, \quad f^{\mu\nu} = \frac{1}{2(-g)^{1/2}}\epsilon^{\mu\nu\rho\sigma}\Phi_{\rho\sigma}, \tag{2.6}$$

$$g = \text{Det}(g_{\mu\nu}), \quad \hat{g} = \text{Det}(\hat{g}_{\mu\nu}) = g(1+\Omega-\Lambda^2), \tag{2.7}$$

and the associated constant q has been suppressed for economy of notation. We may also define a tensor density by

$$\hat{\mathfrak{g}}^{\mu\nu} = (-\hat{g})^{1/2}\hat{g}^{\mu\nu} \tag{2.8}$$

and a fundamental symmetric tensor

$$b^{\mu\nu} = (B^{-1})^{\mu\nu}$$
$$= \frac{1}{(-g)^{1/2}}\hat{\mathfrak{g}}^{\{\mu\nu\}}$$
$$= \frac{g^{\mu\nu}(1+\Omega) - \Phi^{\mu\rho}\Phi^{\nu}{}_{\rho}}{(1+\Omega-\Lambda^2)^{1/2}}, \tag{2.9}$$

$$b_{\mu\nu} = \frac{g_{\mu\nu} + \Phi_{\mu\rho}\Phi_{\nu}{}^{\rho}}{(1+\Omega-\Lambda^2)^{1/2}}, \tag{2.10}$$

where

$$g^{\mu\rho}g_{\nu\rho} = \delta^{\mu}_{\nu}, \quad b^{\mu\rho}b_{\nu\rho} = \delta^{\mu}_{\nu}, \tag{2.11}$$

$$\text{Det}(b_{\mu\nu}) = \text{Det}(g_{\mu\nu}) = g = b. \tag{2.12}$$

The tensor indices will be raised and lowered with the aid of the metric tensor $g_{\mu\nu}$. Thus

$$\Phi^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}\Phi_{\rho\sigma}.$$

The result (2.12) can be obtained from

$$B = \tilde{K}\mathfrak{G}^{-1}K, \tag{2.13}$$

where the matrices $\mathfrak{G}$, B, and K are defined by

$$\mathfrak{G} = [g_{\mu\nu}],$$
$$\mathfrak{G}^{-1} = [g^{\mu\nu}],$$
$$B = [b_{\mu\nu}],$$
$$K = [K_{\mu\nu}],$$
$$K_{\mu\nu} = \hat{g}_{\mu\nu}(1+\Omega-\Lambda^2)^{-1/4}.$$

Hence

$$\text{Det}B = (\text{Det}K)^2\,\text{Det}(\mathfrak{G}^{-1}) = g.$$

From the above relations we see that the symmetric and antisymmetric parts of $\hat{g}^{\mu\nu}$ mix the gravitational field tensor $g_{\mu\nu}$ with the generalized electromagnetic field tensor $\Phi_{\mu\nu}$. The $\hat{g}^{\mu\nu}$ is, of course, a reducible tensor and will be used in the action principle of the theory.

The action principle of the theory can be correlated with that of general relativity. In order to achieve this we shall reformulate the action principle of general relativity by writing

$$S_G = \frac{c^3}{16\pi G}\int \mathfrak{L}_G d^4x, \tag{2.14}$$

where the Lagrangian $\mathfrak{L}_G$ is given by

$$\mathfrak{L}_G = (-g)^{1/2}g^{\mu\nu}G_{\mu\nu} + \frac{G}{c^4}(-g)^{1/2}\Phi^{\mu\nu}(\Phi_{\mu\nu} - 2F_{\mu\nu}) \tag{2.15}$$

and G is the gravitational constant. The second term in the Lagrangian contains the coupling, with the strength G/c^4, of the electromagnetic field to the gravitational field. The extra variables A_μ in

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.16}$$

are introduced in order to incorporate the special nature (derivability from a potential) of the electromagnetic tensor $\Phi_{\mu\nu}$ into a variational principle. This is a very useful device for the physical interpretation of the various quantities in the generalized theory of gravitation. The action principle of general relativity

$$\delta S_G = 0 \tag{2.17}$$

applied with respect to the variation of the 20 independent variables $g_{\mu\nu}$, $\Phi_{\mu\nu}$, and A_μ leads to the field equations

$$G_{\mu\nu}=\frac{2G}{c^4}T_{\mu\nu}, \tag{2.18}$$

$$\frac{\partial}{\partial x^\nu}[(-g)^{1/2}\Phi^{\mu\nu}]=0, \tag{2.19}$$

$$\Phi_{\mu\nu}=F_{\mu\nu}, \tag{2.20}$$

where the energy-momentum tensor $T_{\mu\nu}$ of the electromagnetic field is given by

$$T_{\mu\nu}=\tfrac{1}{4}g_{\mu\nu}\Phi^{\rho\sigma}\Phi_{\rho\sigma}-\Phi_{\mu\rho}\Phi_\nu{}^\rho. \tag{2.21}$$

Equation (2.20) implies that the electromagnetic field $\Phi_{\mu\nu}$ satisfies also the remaining Maxwell's equations,

$$\Phi_{\mu\nu,\,\rho}+\Phi_{\nu\rho,\,\mu}+\Phi_{\rho\mu,\,\nu}=0. \tag{2.22}$$

Hence the extra field $F_{\mu\nu}$ is eliminated. We must observe that the use of the tensor $\Phi_{\mu\nu}$ for the electromagnetic field in (2.14) must not be confused with its interpretation as a "generalized electromagnetic field" in the generalized theory of gravitation, where the tensor $\Phi_{\mu\nu}$ refers to the antisymmetric part of $\hat{g}_{\mu\nu}$ and is no longer derivable from a potential.

In order to construct the action principle of the generalized theory in accordance with a correspondence principle we shall rewrite (2.15) in the form

$$\mathcal{L}_G=(-g)^{1/2}(g^{\mu\nu}+q^{-1}\Phi^{\mu\nu})(G_{\mu\nu}-\tfrac{1}{2}\kappa^2q^{-1}F_{\mu\nu})+\kappa^2[\tfrac{1}{2}q^{-2}\Omega(-g)^{1/2}], \tag{2.23}$$

where the constants κ and q are related by

$$\kappa^2q^{-2}=\frac{4G}{c^4}. \tag{2.24}$$

We have thus factorized the coupling constant G/c^4 in the Lagrangian (2.15), where the universal constant

$$r_0=\sqrt{2}\,\kappa^{-1} \tag{2.25}$$

has the dimensions of a length. Because of the relation (2.24) between κ and q, the Lagrangians (2.23) and (2.15) are equal.

The Lagrangian (2.23) can now be generalized by using a one-to-one correspondence of the form

$$(-g)^{1/2}(g^{\mu\nu}+q^{-1}\Phi^{\mu\nu})\to\hat{\mathfrak{g}}^{\mu\nu},$$

$$G_{\mu\nu}\to R_{\mu\nu},$$

$$\tfrac{1}{2}q^{-2}\Omega(-g)^{1/2}\to(-\hat{g})^{1/2}-(-g)^{1/2},$$

where the expressions on the right-hand sides, on expanding and neglecting terms containing powers of q^{-1} higher than 2, reduce to the expressions on the left-hand sides. Hence the action function for the generalized theory of gravitation, based on the above correspondence, can be expressed as

$$S=\frac{q^2r_0^2}{8\pi c}\int\mathcal{L}d^4x, \tag{2.26}$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L}=\hat{\mathfrak{g}}^{\mu\nu}(R_{\mu\nu}-\tfrac{1}{2}\kappa^2q^{-1}F_{\mu\nu})+\kappa^2[(-\hat{g})^{1/2}-(-g)^{1/2}], \tag{2.27}$$

and where

$$R_{\mu\nu}=-\Gamma^\rho_{\mu\nu,\,\rho}+\Gamma^\rho_{\mu\rho,\,\nu}+\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}-\Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} \tag{2.28}$$

is a "transposition-symmetric" curvature tensor. Using the relation

$$\Gamma^\rho_{\mu\rho}=\Gamma^\rho_{\{\mu\rho\}}=\partial_\mu[\ln(-\hat{g})^{1/2}],$$

the curvature tensor $R_{\mu\nu}$ of the generalized theory can be written as

$$R_{\mu\nu}=-\Gamma^\rho_{\mu\nu,\,\rho}+\partial_\mu\partial_\nu[\ln(-\hat{g})^{1/2}]+\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}-\Gamma^\rho_{\mu\nu}\partial_\rho[\ln(-\hat{g})^{1/2}]. \tag{2.29}$$

The transposition symmetry in this theory corresponds to the charge-conjugation invariance of the quantum theory. Thus we may write

$$\hat{g}_{\mu\nu}=\text{transpose of }(\hat{g})_{\nu\mu}.$$

In a similar way, for the nonsymmetric displacement field $\Gamma^\rho_{\mu\nu}$ we have

$$\Gamma^\rho_{\mu\nu}=(\tilde{\Gamma}^\rho)_{\nu\mu},$$

where

$$(\tilde{\Gamma}^\rho)_{\mu\nu}=\Gamma^\rho_{\{\mu\nu\}}-q^{-1}\Gamma^\rho_{[\mu\nu]}, \tag{2.30}$$

where the tilde represents transposition. A vector V^μ can be displaced parallel to itself by an infinitesimal distance dx^μ, and the resulting change in its components is given by

$$\delta V^\rho=-\Gamma^\rho_{\mu\nu}dx^\mu V^\nu$$

or by its dual displacement

$$\delta V^\rho=-(\tilde{\Gamma})^\rho_{\mu\nu}dx^\mu V^\nu.$$

Thus the requirement of transposition invariance of the theory is a necessity to remove this arbitrariness of duality. For the curvature tensor $R_{\mu\nu}$, as seen from its definition (2.28), we have

$$R_{\mu\nu}(\Gamma)=R_{\nu\mu}(\tilde{\Gamma}), \tag{2.31}$$

implying the correlation of the tensor indices μ and ν appearing in the definition (2.28) as first and second indices, respectively. The extra variables A_μ, besides maintaining the correspondence with general relativity, play an important role in the transposition invariance of the theory. This role of A_μ will be used more explicitly in the derivation of the field equations from the action principle

$$\delta S=0. \tag{2.32}$$

The variation of S with respect to the 16 field variables $\hat{g}_{\mu\nu}$ and four potentials A_μ, as well as with respect to 64 displacement fields $\Gamma^\rho_{\mu\nu}$, leads to the field equations

$$R_{\{\mu\nu\}} = \tfrac{1}{2}\kappa^2(b_{\mu\nu} - g_{\mu\nu}), \tag{2.33}$$

$$R_{[\mu\nu]} = \tfrac{1}{2}\kappa^2(F_{\mu\nu} - \Phi_{\mu\nu}), \tag{2.34}$$

$$\hat{\mathfrak{g}}^{[\mu\nu]}{}_{,\nu} = 0 \tag{2.35}$$

and the transposition-invariant algebraic equations

$$\hat{g}_{\mu\nu;\,\rho} = \hat{g}_{\mu\nu,\,\rho} - \hat{g}_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \hat{g}_{\sigma\nu}\Gamma^\sigma_{\mu\rho} = 0 \tag{2.36}$$

for the $\Gamma^\rho_{\mu\nu}$. The field equations (2.35) result from variation with respect to A_μ. In the absence of (2.35) we would have the result

$$\Gamma^\rho_{[\mu\rho]} = \Gamma_\mu \neq 0$$

and the Lagrangian would not be transposition-invariant.

Now, as we did in the field equations of general relativity, we can eliminate the extra field variables $F_{\mu\nu}$ from (2.34) and rewrite the new field equations in the form

$$R_{\{\mu\nu\}} = \tfrac{1}{2}\kappa^2(b_{\mu\nu} - g_{\mu\nu}), \tag{2.37}$$

$$R_{[\mu\nu],\,\rho} + R_{[\nu\rho],\,\mu} + R_{[\rho\mu],\,\nu} + \tfrac{1}{2}\kappa^2 I_{\mu\nu\rho} = 0, \tag{2.38}$$

$$\hat{\mathfrak{g}}^{[\mu\nu]}{}_{,\nu} = 0, \tag{2.39}$$

where

$$I_{\mu\nu\rho} = \Phi_{\mu\nu,\,\rho} + \Phi_{\nu\rho,\,\mu} + \Phi_{\rho\mu,\,\nu}$$

and is an axial 4-vector. Because of the two differential identities obtainable from (2.38) and (2.39) only 16 independent field equations remain to determine 16 field variables $\hat{g}_{\mu\nu}$. The variation with respect to $\hat{g}_{\mu\nu}$ involves the relation

$$\delta[(-g)^{1/2}] = \tfrac{1}{2}g_{\mu\nu}\,\delta[(-b)^{1/2}g^{\mu\nu}] = \tfrac{1}{2}b_{\mu\nu}\,\delta\hat{\mathfrak{g}}^{\mu\nu}.$$

The fundamental significance of the extra term involving $F_{\mu\nu}$ in the general-relativistic Lagrangian, with a coupling strength G/c^4, lies in the fact that without it we could not interrelate or unify, in a physically meaningful way, the fields $g_{\mu\nu}$ and $\Phi_{\mu\nu}$ in the generalized theory. This is also clear from the simple observation that the Lagrangian (2.27) reduces, in the correspondence limit $r_0 = 0$ (or $q = \infty$), to the Lagrangian (2.15) of general relativity. We have thus established a correspondence principle for the generalized theory of gravitation, without which the theory could not possibly have a physical basis. The above results (i.e., the field equations) were also derived, previously, from a geometrical approach based on the Bianchi identities for the nonsymmetric theory.[1] Therefore, the universal constant r_0 also has a geometrical basis.

It is interesting to point out that we could, if we wished, obtain the same field equations from the Lagrangian

$$\mathfrak{L}_0 = \hat{\mathfrak{g}}^{\mu\nu}R_{\mu\nu} + \kappa^2[(-\hat{g})^{1/2} - (-g)^{1/2}], \tag{2.40}$$

where now the field variables $\hat{\mathfrak{g}}^{[\mu\nu]}$ are defined according to the equation

$$\hat{\mathfrak{g}}^{[\mu\nu]} = \mathfrak{g}^{[\mu\nu\rho]}{}_{,\rho} \tag{2.41}$$

and where $\mathfrak{g}^{[\mu\nu\rho]}$ is fully antisymmetric in μ, ν, ρ and is, therefore, an axial 4-vector. Hence the field equations (2.39) are a consequence of the definition (2.41). The potentials $\mathfrak{g}^{[\mu\nu\rho]}$ can also be defined in the form

$$\mathfrak{g}^{[\mu\nu\rho]} = \epsilon^{\mu\nu\rho\sigma}B_\sigma, \tag{2.42}$$

where the axial 4-vector B_μ generates the field $\hat{\mathfrak{g}}^{[\mu\nu]}$ according to

$$\psi_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{2.43}$$

where

$$\psi_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{\mathfrak{g}}^{[\rho\sigma]}. \tag{2.44}$$

Hence the field equations (2.39) can be replaced by

$$\psi_{\mu\nu,\,\rho} + \psi_{\nu\rho,\,\mu} + \psi_{\rho\mu,\,\nu} = 0. \tag{2.45}$$

In a similar way, the field equations (2.34) or (2.38) can be stated in the form

$$F_{\mu\nu,\,\rho} + F_{\nu\rho,\,\mu} + F_{\rho\mu,\,\nu} = 0. \tag{2.46}$$

We thus see that the theory contains a vector potential A_μ and an axial-vector potential B_μ.

The variation of the Lagrangian (2.40) with respect to the 14 field variables $\hat{\mathfrak{g}}^{\{\mu\nu\}}$ and B_μ yields the field equations (2.37) and (2.38).

We may now, by using the correspondence principle,[2] recognize the roles of the various quantities and identify them for the physical interpretation of the generalized theory. First we observe that the role of the extra variables $F_{\mu\nu}$ in general relativity is clear: They are equal (as a result of the action principle $\delta S_G = 0$) to $\Phi_{\mu\nu}$, whose divergence vanishes because of the absence of charge. Hence in the generalized theory we can follow the same path and define an electric current vector by taking the divergence of the antisymmetric tensor $\Phi_{\mu\nu} + r_0^2 R_{[\mu\nu]}$ $(=F_{\mu\nu})$, thereby eliminating the extra field variables $F_{\mu\nu}$ (once more) by defining the vector density

$$\mathcal{J}^\mu = \mathcal{J}^\mu_e + \mathcal{J}^\mu_0 = \frac{1}{4\pi}\frac{\partial}{\partial x^\nu}[(-g)^{1/2}F^{\mu\nu}] \tag{2.47}$$

as the generalized conserved electric current, where

$$\mathcal{J}^\mu_e = \frac{1}{4\pi}\frac{\partial}{\partial x^\nu}[(-g)^{1/2}\Phi^{\mu\nu}], \tag{2.48}$$

$$\mathcal{J}_0^{\mu} = \frac{r_0^2}{4\pi}\frac{\partial}{\partial x^{\nu}}[(-g)^{1/2}R^{[\mu\nu]}]. \tag{2.49}$$

In the correspondence limit $r_0 = 0$ both of these currents vanish, so that the electric currents are consequences of a finite universal length r_0. The subscripts e and 0 in the definitions (2.48) and (2.49) refer to charged and neutral currents, respectively. The statements

$$\int \mathcal{J}_e^{\mu}\, d\sigma_{\mu} = \pm e\,, \quad \int \mathcal{J}_0^{\mu}\, d\sigma_{\mu} = 0 \tag{2.50}$$

will be proved in Sec. VI. The neutral current $\mathcal{J}_0^{\mu}$ will be interpreted as a "polarization current."

The definitions (2.48) and (2.49) imply that *knowledge of the electric current* $\mathcal{J}^{\mu}$ *depends on knowledge of the field variables* $\hat{g}_{\mu\nu}$, *which, in turn, are the solutions of the field equations (2.37)–(2.39). We have now established the important facts that electric currents are determined according to the laws of the field and that the currents cannot be prescribed arbitrarily.*

The theory has also an axial-vector neutral current

$$\Phi_{\mu\nu,\,\rho} + \Phi_{\nu\rho,\,\mu} + \Phi_{\rho\mu,\,\nu} = I_{\mu\nu\rho} = 4\pi\epsilon_{\mu\nu\rho\sigma}\,\mathfrak{s}^{\sigma}, \tag{2.51}$$

or

$$\mathfrak{s}^{\mu} = (-g)^{1/2} s^{\mu} = \frac{1}{4\pi}\frac{\partial}{\partial x^{\nu}}[(-g)^{1/2} f^{\mu\nu}],$$

where the axial-vector density $\mathfrak{s}^{\mu}$ is a "magnetic current" and has no classical counterpart since, as seen from the field equations (2.38), it vanishes in the correspondence limit $r_0 = 0$. It is shown in Sec. VIII that $\mathfrak{s}^{\mu}$ represents a magnetically neutral current density (i.e., equal amounts of positive and negative magnetic charge distributions) and vanishes at distances beyond the universal length r_0, and that

$$\int \mathfrak{s}^{\mu} d\sigma_{\mu} = 0, \tag{2.52}$$

where $d\sigma_{\mu}$ are the 3-dimensional surface elements in the 4-dimensional space. Here again it is clear from the definition (2.51) and from the field equations (2.38) that the distribution of the magnetic charge density is prescribed by the laws of the field and cannot be predetermined. In order to see more explicitly the nature of the magnetic current distribution $\mathfrak{s}^{\mu}$ we may derive the linearized form of the field equations (2.38) in flat space-time in the form

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2} + \kappa^2\right)\mathfrak{s}^{\mu} = 0. \tag{2.53}$$

This equation is, of course, valid only at distances much larger than r_0, in which case, because of the large size of κ ($\sim 10^{14}\,\text{cm}^{-1}$), $\mathfrak{s}^{\mu}$ is negligibly small. However, the equation does still contain useful information on the nature of $\mathfrak{s}^{\mu}$. From a plane-wave solution $\exp(ik_{\mu}x^{\mu})$ of (2.53) we see that k^{μ} is a spacelike vector and therefore the current $\mathfrak{s}^{\mu}$ has no wavelike properties, and that the magnetic current distribution is confined to distances of the order of r_0.

Now, in addition to the neutral magnetic current density $\mathfrak{s}^{\mu}$ we may also define a charged magnetic current density ζ^{μ} as the divergence of the tensor density $(-g)^{1/2}\psi^{\mu\nu}$. Thus we write

$$\zeta^{\mu} = \frac{1}{4\pi}\frac{\partial}{\partial x^{\nu}}[(-g)^{1/2}\psi^{\mu\nu}], \tag{2.47$'$}$$

where, as follows from the case of spherical symmetry (see Secs. III and IV), we have

$$\int \zeta^{\mu} d\sigma_{\mu} = \pm \mathfrak{g}\ . \tag{2.50$'$}$$

The corresponding magnetic fields for the currents $\mathfrak{s}^{\mu}$ and ζ^{μ} are *short-range fields. Therefore, a monopole charge is associated with the short-range field.* It will be shown in Secs. VIII and IX that the range of the field associated with a monopole is of the order of the nucleon Compton wavelength. This result will be interpreted as a classical basis for the strong and weak interactions. We have thus shown that the four antisymmetric tensors $\Phi_{\mu\nu}$, $f_{\mu\nu}$, $\psi_{\mu\nu}$, $R_{[\mu\nu]}$ of the field provide two electric and two magnetic currents, where $f_{\mu\nu}$ and $r_0^2 R_{[\mu\nu]}$ refer to neutral fields.

Now, for the sake of further comparison with the classical theory, we shall cast the field equations (2.38) and (2.39) [or (2.45) and (2.46)] in a more conventional form in terms of the four generalized electromagnetic field vectors $\vec{\mathcal{E}}, \vec{\mathfrak{D}}, \vec{\mathcal{H}}, \vec{\mathfrak{B}}$ defined by

$$\begin{aligned}
\vec{\mathcal{E}} &= (\Phi_{14} + r_0^2 R_{[14]},\, \Phi_{24} + r_0^2 R_{[24]},\, \Phi_{34} + r_0^2 R_{[34]}),\\
\vec{\mathfrak{D}} &= (\hat{\mathfrak{g}}^{[14]},\, \hat{\mathfrak{g}}^{[24]},\, \hat{\mathfrak{g}}^{[34]}),\\
\vec{\mathcal{H}} &= (\hat{\mathfrak{g}}^{[23]},\, \hat{\mathfrak{g}}^{[31]},\, \hat{\mathfrak{g}}^{[12]}),\\
\vec{\mathfrak{B}} &= (\Phi_{23} + r_0^2 R_{[23]},\, \Phi_{31} + r_0^2 R_{[31]},\, \Phi_{12} + r_0^2 R_{[12]}),
\end{aligned} \tag{2.54}$$

where in identifying various components of $\psi_{\mu\nu}$ [(2.45)] and $F_{\mu\nu}$ [(2.46)] we have employed the usual polar and axial-vector symmetries of these quantities with respect to space-time transformations. Hence the field equations (2.38), (2.39) or (2.45), (2.46) can, in a local Lorentz frame of reference, be written as

$$\vec{\nabla}\cdot\vec{\mathfrak{D}} = 0, \quad \vec{\nabla}\times\vec{\mathcal{H}} = \frac{1}{c}\frac{\partial\vec{\mathfrak{D}}}{\partial t}, \tag{2.55}$$

$$\vec{\nabla}\cdot\vec{\mathfrak{B}} = 0, \quad \vec{\nabla}\times\vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial\vec{\mathfrak{B}}}{\partial t}, \tag{2.56}$$

which, in the correspondence limit $r_0=0$, reduce to Maxwell's equations for empty space where $\vec{\mathfrak{B}}=\vec{\mathfrak{H}}$, $\vec{\mathcal{E}}=\vec{\mathfrak{D}}$. The field equations (2.38), (2.39), when expressed in the form (2.55), (2.56), resemble the electrodynamics of continuous media. The Φ and R terms in the definition of the generalized electric field $\mathcal{E}$ represent charged and neutral (or polarized) fields, respectively.

For the definitions (2.47) and (2.47′) the corresponding equations in a local Lorentz frame of reference are given by

$$\vec{\nabla}\cdot\vec{\mathcal{E}}=4\pi(\mathfrak{J}_0^4+\mathfrak{J}_e^4),$$

$$-\frac{\partial\vec{\mathcal{E}}}{c\partial t}+\nabla\times\vec{\mathfrak{B}}=4\pi(\vec{\mathfrak{J}}_0+\vec{\mathfrak{J}}_e)$$

and

$$\vec{\nabla}\cdot\vec{\mathfrak{H}}=4\pi\zeta^4,$$

$$\frac{\partial\vec{\mathfrak{H}}}{c\partial t}+\vec{\nabla}\times\vec{\mathfrak{D}}=4\pi\vec{\zeta}.$$

In the same way we may rewrite the field equations (2.37) in the form

$$G_{\mu\nu}=\kappa_0{}^2\mathfrak{T}_{\mu\nu}, \tag{2.57}$$

where

$$\kappa_0{}^2=\frac{2G}{c^4},$$

$$R_{\{\mu\nu\}}=G_{\mu\nu}-S^\rho_{\mu\nu;\,\rho}+S^\rho_{\mu\rho;\,\nu}+\Gamma^\rho_{[\mu\sigma]}\Gamma^\sigma_{[\rho\nu]},$$

and where the source term $\mathfrak{T}_{\mu\nu}$ of the gravitational field is given by

$$\mathfrak{T}_{\mu\nu}=\frac{1}{2}\frac{\kappa^2}{\kappa_0{}^2}(b_{\mu\nu}-g_{\mu\nu})+\kappa_0{}^{-2}\Lambda_{\mu\nu},$$

$$\Lambda_{\mu\nu}=S^\rho_{\mu\nu;\,\rho}-S^\rho_{\mu\rho;\,\nu}-\Gamma^\sigma_{[\mu\rho]}\Gamma^\rho_{[\sigma\nu]},$$

$$S^\rho_{\mu\nu}=g^{\rho\sigma}(\Phi_{\mu\alpha}\Gamma^\alpha_{[\sigma\nu]}+\Phi_{\alpha\nu}\Gamma^\alpha_{[\mu\sigma]}), \tag{2.58}$$

$$\Gamma^\rho_{\{\mu\nu\}}=\{^{\,\rho}_{\mu\nu}\}+S^\rho_{\mu\nu},$$

$$\{^{\,\rho}_{\mu\nu}\}=\tfrac{1}{2}g^{\rho\sigma}(g_{\mu\sigma,\,\nu}+g_{\nu\sigma,\,\mu}-g_{\mu\nu,\,\sigma}).$$

The $\Lambda_{\mu\nu}$ term in $\mathfrak{T}_{\mu\nu}$ is small compared to the first term. The latter in the correspondence limit $r_0\to 0$ reduces to the energy-momentum tensor of the electromagnetic field. Hence we see that, just as in general relativity, the generalized theory also can yield Lorentz's equations of motion for point particles.[4] Higher-order corrections to these equations of motion are proportional to q^{-2}, where q is very large (~10^{39} esu).

Finally, from the action principle (2.32), via the Bianchi identities[1] of the nonsymmetric theory, we can derive the conservation laws

$$\mathfrak{T}^\nu_{\mu,\,\nu}=0, \tag{2.59}$$

where

$$-4\pi\kappa^2q^{-2}\mathfrak{T}^\nu_\mu=\hat{\mathfrak{g}}^{\nu\rho}R_{\mu\rho}+\hat{\mathfrak{g}}^{\rho\nu}R_{\rho\mu}-\delta^\nu_\mu\mathfrak{g}^{\rho\sigma}R_{\rho\sigma}+\hat{\mathfrak{g}}^{\rho\sigma}{}_{,\mu}\mathfrak{B}^\nu_{\rho\sigma}-\delta^\nu_\mu\mathfrak{B}, \tag{2.60}$$

$$\mathfrak{B}^\rho_{\mu\nu}=\delta^\rho_\mu\Gamma^\sigma_{\{\nu\sigma\}}-\Gamma^\rho_{\mu\nu},$$

$$\mathfrak{B}=\hat{\mathfrak{g}}^{\mu\nu}(\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}-\Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\{\rho\sigma\}}).$$

On substituting the field equations (2.33)–(2.35) in (2.60) we obtain

$$\mathfrak{T}^\nu_\mu=\frac{1}{4\pi}\{q^2[(-\hat{g})^{1/2}-(-g)^{1/2}]\delta^\nu_\mu-\hat{\mathfrak{g}}^{[\nu\rho]}F_{\mu\rho}\}+\tfrac{1}{2}\delta^\nu_\mu\mathfrak{T}-\frac{c^4}{16\pi G}\hat{\mathfrak{g}}^{\rho\sigma}{}_{,\mu}\mathfrak{B}^\nu_{\rho\sigma}, \tag{2.61}$$

where

$$\mathfrak{T}=\mathfrak{T}^\mu_\mu.$$

In the limit $r_0\to 0$ (2.60) and (2.61) reduce to the conventional energy tensor of general relativity. The field equations as well as the energy tensor $\mathfrak{T}^\nu_\mu$ remain invariant under the gauge transformations

$$\Gamma^\rho_{\mu\nu}\to\Gamma^\rho_{\mu\nu}+\delta^\rho_\mu\lambda_{,\nu},$$

$$A_\mu\to A_\mu+\lambda_{,\mu}, \tag{2.62}$$

$$B_\mu\to B_\mu+\lambda_{,\mu}.$$

III. STATIC SPHERICALLY SYMMETRIC EQUATIONS

The static spherically symmetric field solutions in general relativity, i.e., the Schwarzschild and Nordström solutions, have provided a satisfactory basis for deriving various physical implications of the theory. It is therefore, quite natural to adopt the same method for the generalized theory of gravitation. In the latter instance the spherically symmetric field represented by the nonsymmetric tensor $\hat{g}_{\mu\nu}$ has only five nonvanishing components. In order to see this fact we shall discuss the spherically symmetric form of the antisymmetric part $\Phi_{\mu\nu}$. The values of $\Phi_{\mu\nu}$ at, for example, the point $(0, 0, z=r, t)$ remain unchanged under the rotation of a local Lorentz frame of reference by an angle $\frac{1}{2}\pi$ around the z axis. The rotation is effected by the matrix

$$R_z(\tfrac{1}{2}\pi)=\exp(i\tfrac{1}{2}\pi M_{12}), \tag{3.1}$$

where the generator of the rotation M_{12} is given by

$$M_{12}=\begin{bmatrix}0&-i&0&0\\ i&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix}. \tag{3.2}$$

The rotation of the coordinates by $R_z(\frac{1}{2}\pi)$ produces the transformations

$$x' = -y, \quad y' = x, \quad z' = z, \quad t' = t \ .$$

Under the transformation (3.1), the tensor $\Phi_{\mu\nu}$ transforms according to the rule

$$[\Phi'_{\mu\nu}] = \tilde{R}_z(\tfrac{1}{2}\pi)[\Phi_{\mu\nu}]R_z(\tfrac{1}{2}\pi) \ . \tag{3.3}$$

By using the condition of spherical symmetry

$$\Phi'_{\mu\nu} = \Phi_{\mu\nu} \tag{3.4}$$

together with the transformation (3.3) we obtain for the points on the z axis the results

$$\Phi_{23} = \Phi_{31} = \Phi_{41} = \Phi_{42} = 0 \ .$$

There are thus, for the spherically symmetric field, only two nonvanishing components: Φ_{12} and Φ_{43}.

We may now extend the above special transformation and the resulting symmetry obtained to more general transformations pertaining to arbitrary points of space and time. Thus let us apply a new rotation to bring the point $(0, 0, r, t)$ to the point (x, y, z, t), where $r^2 = x^2 + y^2 + z^2$. We first rotate the yz plane $(x = 0)$ around the z axis by an angle ϕ to coincide with the point (x, y, z, t). The equation of the new plane is

$$a_1 x + a_2 y = 0 \ . \tag{3.5}$$

The angle of rotation ϕ is given by

$$\phi = \tan^{-1}(y/x) \ , \tag{3.6}$$

and the corresponding rotation matrix has the form

$$R_z(\phi) = \exp(i\phi M_{12}) \ . \tag{3.7}$$

The explicit form of (3.7) can be obtained by using the relation

$$\begin{aligned}\exp(i\,\vec{\omega}\cdot\vec{\mathrm{M}}) &= \exp[i\,(\omega_1 M_{23} + \omega_2 M_{31} + \omega_3 M_{12})] \\ &= 1 + i\,(\vec{\omega}\cdot\vec{\mathrm{M}})\,\frac{\sin\omega}{\omega} + (\vec{\omega}\cdot\vec{\mathrm{M}})^2\,\frac{\cos\omega - 1}{\omega^2} \ ,\end{aligned} \tag{3.8}$$

where

$$(\vec{\omega}\cdot\vec{\mathrm{M}})^3 = \omega^2(\vec{\omega}\cdot\vec{\mathrm{M}}) \ .$$

The matrix $\exp(i\,\vec{\omega}\cdot\vec{\mathrm{M}})$ represents a rotation by an angle $|\vec{\omega}|$ around the direction $\hat{\omega} = \vec{\omega}/\omega$.

We may now perform a rotation by an angle θ in the plane (3.5) around its normal direction $(a_1, a_2, 0)$ to bring the point $(0, 0, r, t)$ to the point (x, y, z, t), where

$$\theta = \tan^{-1}\frac{(x^2 + y^2)^{1/2}}{z} \ ,$$

$$\theta^2 = a_1{}^2 + a_2{}^2 \ ,$$

$$a_2 = -\frac{x}{y}\,a_1 \ ,$$

or

$$\begin{aligned} a_1 &= \frac{y\theta}{(x^2 + y^2)^{1/2}} \ , \\ a_2 &= -\frac{x\theta}{(x^2 + y^2)^{1/2}} \ . \end{aligned} \tag{3.9}$$

Hence the rotation around $(a_1, a_2, 0)$ is effected by the matrix

$$R_a(\theta) = \exp[i\,(a_1 M_{23} + a_2 M_{31})] \ . \tag{3.10}$$

From the above results it follows that the rotation matrix required to bring the point $(0, 0, r, t)$ to the point (x, y, z, t) is given by

$$R = R_a(\theta) R_z(\phi) \ . \tag{3.11}$$

Now, by applying R to the matrix $[\Phi_{\mu\nu}]$ with only two surviving components, Φ_{12} and Φ_{34}, and then transforming into spherical polar coordinates, we obtain the final result

$$[\Phi^S_{\mu\nu}] = SR[\Phi^z_{\mu\nu}]\tilde{R}\tilde{S} = \begin{bmatrix} 0 & 0 & 0 & W \\ 0 & 0 & \chi\sin\theta & 0 \\ 0 & -\chi\sin\theta & 0 & 0 \\ -W & 0 & 0 & 0 \end{bmatrix} \ , \tag{3.12}$$

where for an arbitrary function $f(r)$

$$\chi = r^2 f(r) \ , \tag{3.13}$$

and where the matrix S is given by

$$S = \left[\frac{\partial x^\mu}{\partial x'^\nu}\right], \quad x'^\mu \equiv (r, \theta, \phi, t) \ ,$$

with

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \\ t &= t' \ . \end{aligned}$$

By applying the same steps to the symmetric part $g_{\mu\nu}$ we can construct the most general spherically symmetric tensor in the form

$$[\hat{g}^{S}_{\mu\nu}] = \begin{bmatrix} -\dfrac{e^{-\mathcal{U}}}{\mathcal{V}^2} & 0 & 0 & \dfrac{1}{\mathcal{V}}\tanh\Gamma \\ 0 & -e^{\rho}\sin\Phi & e^{\rho}\cos\Phi\,\sin\theta & 0 \\ 0 & -e^{\rho}\cos\Phi\,\sin\theta & -e^{\rho}\sin\Phi\,\sin^2\theta & 0 \\ -\dfrac{1}{\mathcal{V}}\tanh\Gamma & 0 & 0 & e^{\mathcal{U}} \end{bmatrix}, \tag{3.14}$$

where the diagonal elements in agreement with the existence of a light cone are restricted by the condition

$$\sin\Phi \geqslant 0\ . \tag{3.15}$$

The condition (3.15) on the function $\Phi(r)$ will also be obtained from the field equations which do not have solutions with real time in the region excluded by (3.15). Furthermore, as will be seen later, the functional forms of $[\hat{g}^{S}_{\mu\nu}]$ for the five functions $\mathcal{U}(r)$, $\mathcal{V}(r)$, $\Phi(r)$, $\Gamma(r)$, $\rho(r)$ are imposed, in a natural way, by the spherically symmetric forms of the affine connections $\Gamma^{\rho}_{\mu\nu}(\hat{g})$.

In the course of the various manipulations of the field equations we shall need a number of algebraic results; the following is a summary of them. The inverse of $[\hat{g}^{S}_{\mu\nu}]$ is given by

$$[\hat{g}^{\mu\nu}_{S}] = \begin{bmatrix} -\mathcal{V}^2 e^{\mathcal{U}}\cosh^2\Gamma & 0 & 0 & \mathcal{V}\cosh\Gamma\sinh\Gamma \\ 0 & -e^{-\rho}\sin\Phi & -\dfrac{e^{-\rho}\cos\Phi}{\sin\theta} & 0 \\ 0 & \dfrac{e^{-\rho}\cos\Phi}{\sin\theta} & -\dfrac{e^{-\rho}\sin\Phi}{\sin^2\theta} & 0 \\ -\mathcal{V}\cosh\Gamma\sinh\Gamma & 0 & 0 & e^{-\mathcal{U}}\cosh^2\Gamma \end{bmatrix}.$$

The determinant of $[\hat{g}^{S}_{\mu\nu}]$ as follows from (3.14) is given by

$$\hat{g} = \mathrm{Det}\,[\hat{g}^{S}_{\mu\nu}] = -\frac{e^{2\rho}}{\mathcal{V}^2\cosh^2\Gamma}\sin^2\theta\ ,$$

and that of $[g^{S}_{\mu\nu}]$ by

$$g = \mathrm{Det}\,[\hat{g}^{S}_{\{\mu\nu\}}] = -\frac{e^{2\rho}}{\mathcal{V}^2}\sin^2\Phi\,\sin^2\theta\ .$$

Hence

$$(-\hat{g})^{1/2} = \frac{e^{\rho}}{\mathcal{V}\cosh\Gamma}\sin\theta\ ,$$
$$(-g)^{1/2} = \frac{e^{\rho}}{\mathcal{V}}\sin\Phi\,\sin\theta\ . \tag{3.16}$$

We may now write the tensor density $\hat{\mathfrak{g}}^{[\mu\nu]} = (-\hat{g})^{1/2}\hat{g}^{[\mu\nu]}$ as

$$[\hat{\mathfrak{g}}^{[\mu\nu]}] = \begin{bmatrix} 0 & 0 & 0 & -e^{\rho}\sinh\Gamma\sin\theta \\ 0 & 0 & \dfrac{\cos\Phi}{\mathcal{V}} & 0 \\ 0 & -\dfrac{\cos\Phi}{\mathcal{V}} & 0 & 0 \\ e^{\rho}\sinh\Gamma\sin\theta & 0 & 0 & 0 \end{bmatrix}. \tag{3.17}$$

The two fundamental invariants Ω and Λ are

$$\Omega = \tfrac{1}{2}\,\phi^{\mu\nu}\Phi_{\mu\nu} = \cot^2\Phi - \tanh^2\Gamma\ ,$$
$$\Lambda = \tfrac{1}{4}f^{\mu\nu}\Phi_{\mu\nu} = -\cot\Phi\,\tanh\Gamma\ , \tag{3.18}$$

and

$$(1+\Omega-\Lambda^2)^{1/2} = \frac{1}{\sin\Phi\,\cosh\Gamma}\ .$$

We observe that the tensor density $\hat{\mathfrak{g}}^{[\mu\nu]}$ as given by (3.17) has some interesting symmetry properties with respect to the transformations $\mathcal{V}\to-\mathcal{V}$ and $\Gamma\to-\Gamma$. Under these two transformations we obtain $\hat{\mathfrak{g}}^{[\mu\nu]}\to-\hat{\mathfrak{g}}^{[\mu\nu]}$. The physical meanings of these transformations are of great significance and are discussed in Sec. IV.

The spherically symmetric components of the fundamental symmetric tensor $b_{\mu\nu}$ are given by

$$[b^S_{\mu\nu}]=\begin{bmatrix} -\frac{e^{-\mathfrak{u}}}{\mathfrak{V}^2}\frac{\sin\Phi}{\cosh\Gamma} & 0 & 0 & 0 \\ 0 & -e^{\rho}\cosh\Gamma & 0 & 0 \\ 0 & 0 & -e^{\rho}\cosh\Gamma\sin^2\theta & 0 \\ 0 & 0 & 0 & \frac{e^{\mathfrak{u}}\sin\Phi}{\cosh\Gamma}\end{bmatrix}. \tag{3.19}$$

In this case also the restriction (3.15) on the function $\Phi(r)$ is an essential requirement.

IV. STATIC SPHERICALLY SYMMETRIC FORM OF THE FIELD EQUATIONS

We shall now discuss the first integrals of the field equations

$$R_{[\mu\nu],\rho}+R_{[\nu\rho],\mu}+R_{[\rho\mu],\nu}+\tfrac{1}{2}\kappa^2 I_{\mu\nu\rho}=0\,,$$

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu}=0\,,$$

which for the static spherically symmetric field variables, as follows from Appendix A, reduce to

$$R_{[23],1}+\tfrac{1}{2}\kappa^2\Phi_{23,1}=0\,, \tag{4.1}$$

$$\hat{\mathfrak{g}}^{[41]}{}_{,1}=0\,. \tag{4.2}$$

The components $r_0{}^2R_{[41]}+\Phi_{[41]}$ and $\hat{\mathfrak{g}}^{[23]}$ satisfy the field equations identically. Equation (4.2) can be integrated once and we obtain

$$\hat{\mathfrak{g}}^{[41]}=\text{const}\times\sin\theta\,.$$

From (3.17) we may write

$$e^{\rho}\sinh\Gamma=\pm\lambda_0{}^2\,, \tag{4.3}$$

where the constant of integration $\lambda_0{}^2$ can be expressed as

$$\lambda_0{}^2=eq^{-1}\,. \tag{4.4}$$

Since the universal constant q is positive, the constant e represents a positive electric charge. The $\pm$ signs in (4.3) are due to the invariance of the equations under the transformation $\Gamma\to-\Gamma$. The generalized electric field $\mathcal{E}$ is given by

$$\mathcal{E}=\Phi_{14}+r_0{}^2R_{[14]}\,,$$

or

$$\mathcal{E}=\frac{1}{\mathfrak{V}}q\tanh\Gamma\{1+e^{-\rho}r_0{}^2[\mathfrak{V}(\mathfrak{V}e^{\mathfrak{u}+\rho}\rho'\tanh\Gamma)'\coth\Gamma+\tfrac{1}{2}\mathfrak{V}^2e^{\mathfrak{u}+\rho}(\rho'^2+\Phi'^2)]\}\,, \tag{4.5}$$

where

$$q\tanh\Gamma=\frac{\pm e}{(e^{2\rho}+\lambda_0{}^4)^{1/2}}\,,$$

and prime indicates differentiation with respect to r. Hence we see that the theory predicts two signs for the electric charge.

The first integration of the field equation (4.1) yields the result

$$r_0{}^2R_{[23]}+q^{-1}\Phi_{23}=\pm l_0{}^2\sin\theta\,, \tag{4.6}$$

where

$$q^{-1}\phi_{23}=e^{\rho}\cos\Phi\sin\theta\,,$$

and

$$\mathcal{B}_0=qe^{\rho}\cos\Phi\sin\theta$$

represents neutral magnetic field, and where the negative sign in (4.6) can be understood by observing that if $\Phi(r)$ is a solution of (4.6) corresponding to $l_0{}^2$ then $\pi-\Phi(r)$ is another solution corresponding to $-l_0{}^2$.

In view of the axial nature of the left-hand side of (4.6), the constant of integration $l_0{}^2$ in (4.6) can be related to a magnetic charge $\mathfrak{g}$ by writing $l_0{}^2$ in the form

$$l_0{}^2=\mathfrak{g}q^{-1}\,, \tag{4.7}$$

where, as in the case of the electric charge appearing in (4.4), the constant $\mathfrak{g}$ represents a positive magnetic charge. The intrinsic charged magnetic field generated by the charge $\mathfrak{g}$ is given by

$$\mathcal{H}=q\mathfrak{g}^{[23]}=\frac{q}{\mathfrak{V}}\cos\Phi\,. \tag{4.8}$$

Because of the appearance of the function $\cos\Phi$ in the definitions of $\mathcal{B}_0$ and $\mathcal{H}$ they represent *short-range fields*.

Now, the remaining field equations to be integrated are

$$R_{11}=\tfrac{1}{2}\kappa^2\frac{e^{-\mathfrak{u}}}{\mathfrak{V}^2}\left(1-\frac{\sin\Phi}{\cosh\Gamma}\right), \tag{4.9}$$

$$R_{22}=\tfrac{1}{2}\kappa^2e^{\rho}(\sin\Phi-\cosh\Gamma)\,, \tag{4.10}$$

$$R_{33}=R_{22}\sin^2\theta\,,$$

$$R_{44}=-\tfrac{1}{2}\kappa^2e^{\mathfrak{u}}\left(1-\frac{\sin\Phi}{\cosh\Gamma}\right), \tag{4.11}$$

$$R_{[23]}=\tfrac{1}{2}\kappa^2(l_0^{\,2}-e^{\rho}\cos\Phi)\sin\theta\,. \qquad (4.12)$$

By taking the linear combinations

$$R_{[23]}\sin\Phi+R_{22}\cos\Phi\sin\theta\,,$$

$$R_{[23]}\cos\Phi-R_{22}\sin\Phi\sin\theta\,,$$

$$R_{11}-\frac{e^{-2\mathfrak{u}}}{\mathfrak{v}^2}R_{44}\,,$$

$$R_{11}+\frac{e^{-2\mathfrak{u}}}{\mathfrak{v}^2}R_{44}\,,$$

we can achieve considerable simplification and write the nonlinear differential equations in the form

$$\mathfrak{v}(\mathfrak{v}e^{\mathfrak{u}+\rho}\Phi')'+\mathfrak{v}^2\Phi'\rho'e^{\mathfrak{u}+\rho}\tanh^2\Gamma=\kappa^2(e^{\rho}\cos\Phi\cosh\Gamma\mp l_0^{\,2}\sin\Phi)-2\cos\Phi\equiv 2X\,, \qquad (4.13)$$

$$\mathfrak{v}(\mathfrak{v}e^{\mathfrak{u}+\rho}\rho')'+\mathfrak{v}^2\rho'^2e^{\mathfrak{u}+\rho}\tanh^2\Gamma=\kappa^2[e^{\rho}(1-\sin\Phi\cosh\Gamma)\mp l_0^{\,2}\cos\Phi]+2\sin\Phi\equiv 2Y\,, \qquad (4.14)$$

$$\mathfrak{v}(\mathfrak{v}e^{\mathfrak{u}+\rho}\mathfrak{u}')'+\mathfrak{v}^2e^{\mathfrak{u}+\rho}\tanh^2\Gamma(8\rho'^2\tanh^2\Gamma+3\mathfrak{u}'\rho'-3\rho'^2-\Phi'^2)\mp\kappa^2e^{\rho}\left(1-\frac{\sin\Phi}{\cosh\Gamma}\right)\equiv 2Z\,, \qquad (4.15)$$

$$\rho''+\rho'\frac{\mathfrak{v}'}{\mathfrak{v}}+\tfrac{1}{2}(\rho'^2+\Phi'^2)-\rho'^2\tanh^2\Gamma=0\,, \qquad (4.16)$$

where, as follows from (4.3), we have used the relations

$$\Gamma'=-\rho'\tanh\Gamma\,,$$

$$\Gamma''=\left(\frac{\rho'^2}{\cosh^2\Gamma}-\rho''\right)\tanh\Gamma\,,$$

and where X, Y, Z abbreviate the right-hand sides of Eqs. (4.13), (4.14), and (4.15).

In Eqs. (4.13) and (4.14) we retain $-l_0^{\,2}$ on the right-hand sides for the solution $\Phi(r)$ and $l_0^{\,2}$ for the solution $\pi-\Phi(r)$. Thus Eqs. (4.13)–(4.16) remain invariant under the transformations

$$\Phi(r)\to\pi-\Phi(r)\,,\qquad l_0^{\,2}\to -l_0^{\,2}\,. \qquad (4.17)$$

In general there exist two classes of solutions:

$$\pm 2n\pi+\Phi(r)=f_n^{\,+}\,, \qquad (4.18)$$

with positive magnetic charge in the future light cone [corresponding to retaining $-l_0^{\,2}$ in (4.13) and (4.14)] and

$$\pm(n+\tfrac{1}{2})2\pi-\Phi(r)=f_n^{\,-}\,, \qquad (4.19)$$

with negative magnetic charge in the past light cone [corresponding to retaining $+l_0^{\,2}$ in (4.13) and (4.14)], where

$$n=0,1,2,\ldots\,.$$

Hence we see that the theory predicts, for the neutral field $\mathfrak{B}_0$, the two signs for the magnetic charge simultaneously with the corresponding two sets of infinite number of solutions. These solutions for a given r represent infinitely degenerate, but, because of the nonlinearity of the equations, nontrivial solutions of the field equations. Both signs of the magnetic charge, in contrast to the electric charge, must occur at the same time for the field $\mathfrak{B}_0$. The two types of magnetic charges ($\pm\mathfrak{g}$) are not separable. We shall see that this fact implies the absence of single magnetic poles associated with a long-range field. Furthermore, the solutions (4.18) and (4.19), because of the relations $\sin(\pm 2n\pi+\Phi)=\sin\Phi$ and $\sin[\pm(2n+1)\pi-\Phi]=\sin\Phi$, are consistent with requirement (3.15) on the metrical coefficients of the field. There are no solutions for $\sin(\Phi\pm\frac{1}{2}\pi)=\pm\cos\Phi$, except when $\Phi=\frac{1}{2}\pi$. The latter possibility is discussed in Sec. VII.

The other two fundamental symmetries of the field equations refer to invariance under the transformations of electric and magnetic charge conjugation

$$\mathfrak{v}(r)\to-\mathfrak{v}(r) \qquad (4.20)$$

and electric charge reflection

$$\Gamma(r)\to-\Gamma(r)\,. \qquad (4.21)$$

The invariance under (4.20) describes, as follows from the definitions of the electric and magnetic fields by (4.5) and (4.8), both electric and magnetic charge conjugation. In fact (see Sec. IX), under (4.20) the energy of the field also changes sign. Therefore the symmetry (4.20) predicts the existence of particle (positive-energy) and antiparticle (negative-energy) pairs. The necessary requirement of positivity of the energy for particles and antiparticles will, presumably, be achieved by a possible application of quantum field theory or by some other procedure to be discovered.

The symmetry (4.21) implies merely the existence of two signs for the electric charge, which fact is contained explicitly in the definitions of the electric and magnetic fields and implicitly in Eqs. (4.13)–(4.16). The symmetry (4.21) does not effect the sign of the magnetic charge. If we apply both transformations (4.20) and (4.21) then the electric charge does not change its sign, but the sign of the energy and the sign of the magnetic charges change. This fact implies the existence of antiparticles with positive or negative electric and magnetic charges. However, if $\Gamma = 0$ then the symmetry (4.20) predicts the existence of electrically neutral particle-antiparticle pairs. There are no solutions with $\mathfrak{V} = 0$. The magnetic field $\mathcal{H}$ under (4.20) goes to $-\mathcal{H}$, and the latter under (4.19) (time reflection) is restored back to the original field. All elementary particles carry a net magnetic charge ($\pm \mathfrak{g}$) associated with a *short-range field*. Particles and antiparticles carry equal but opposite signs of magnetic charges superimposed over a magnetically neutral core.

V. SPECIAL SOLUTIONS OF THE FIELD EQUATIONS

Let us begin by solving the field equations (4.13) and (4.14) for $\cos\Phi$ and $\sin\Phi$ in terms of X and Y. Thus, setting

$$(e^{2\rho} + \lambda_0^4)^{1/2} - r_0^2 = R^2 \tag{5.1}$$

and using $\sin^2\Phi + \cos^2\Phi = 1$ we obtain

$$\cos\Phi = \frac{\pm l_0^2 \cos\alpha + R^2 \sin\alpha}{(R^4 + l_0^4)^{1/2}}, \quad \sin\Phi = \frac{\mp l_0^2 \sin\alpha + R^2 \cos\alpha}{(R^4 + l_0^4)^{1/2}}, \tag{5.2}$$

where

$$X r_0^2 = (R^4 + l_0^4)^{1/2} \sin\alpha, \tag{5.3}$$

$$e^\rho - r_0^2 Y = (R^4 + l_0^4)^{1/2} \cos\alpha, \tag{5.4}$$

$$\tan(\alpha + \Phi) = \pm \frac{R^2}{l_0^2}, \tag{5.5}$$

and $\alpha(r)$ is a function of r.

The magnetic field $\mathcal{H}$ can now be written as

$$\mathcal{H} = \frac{q}{\mathfrak{V}} \cos\Phi = \frac{\pm \mathfrak{g}}{\mathfrak{V}} \frac{\cos\alpha \pm (R^2/l_0^2)\sin\alpha}{(R^4 + l_0^4)^{1/2}}. \tag{5.6}$$

For $\Phi(r) = \pm 2n\pi$ in Eq. (4.13), using the positive sign of l_0^2 and noting that $\Phi'(r) = 0$, the left-hand side vanishes and we obtain

$$e^\rho \cosh\Gamma = r_0^2.$$

Hence in this case ρ is a constant and therefore the left-hand side of (4.14) also vanishes and we find the result

$$e^\rho = l_0^2. \tag{5.7}$$

Thus using the definition $\cosh\Gamma = e^{-\rho}(e^{2\rho} + \lambda_0^4)^{1/2}$ we obtain the relation

$$r_0^2 = (l_0^4 + \lambda_0^4)^{1/2} = q^{-1}(e^2 + \mathfrak{g}^2)^{1/2}, \tag{5.8}$$

which in conjunction with

$$q^2 r_0^2 = \frac{c^4}{2G}$$

leads to the results

$$r_0^2 = \frac{2G}{c^4}(e^2 + \mathfrak{g}^2), \quad q = \frac{(e^2 + \mathfrak{g}^2)^{1/2}}{r_0^2}. \tag{5.9}$$

We have thus obtained the particular values of the universal constant r_0^2 and the constant q of the theory in terms of the two constants of integration e and $\mathfrak{g}$. The theory, so far, does not relate e and $\mathfrak{g}$. In order to find a relation between e and $\mathfrak{g}$ we shall need an additional requirement, namely introduction of the constant $\hbar$ by a quantization of this theory. An important dimensionless number is the ratio

$$\frac{l_0^2}{\lambda_0^2} = \frac{\mathfrak{g}}{e} = f^2, \tag{5.10}$$

where the constant f, as will be seen, is a measure of the strength of the coupling between the various regions of the field at distances of the order of l_0 from the origin.

The three lengths

$$\lambda_0^2 = \frac{2G}{c^4} e(e^2 + \mathfrak{g}^2)^{1/2}, \tag{5.11}$$

$$l_0^2 = \frac{2G}{c^4} g(e^2 + \mathfrak{g}^2)^{1/2}, \tag{5.12}$$

$$r_0^2 = \frac{2G}{c^4}(e^2 + \mathfrak{g}^2), \tag{5.13}$$

are related according to

$$\lambda_0 \leq l_0 \leq r_0. \tag{5.14}$$

The lengths λ_0 and l_0 may serve to differentiate between leptonic and hadronic processes, respectively. In (5.14) the equality $\lambda_0 = l_0$ holds only for $\mathfrak{g} = e$. For $l_0 = r_0$ we must set, for an arbitrary $\mathfrak{g}$, $e = 0$.

On using (5.7) and (5.9) in the definition (5.1) we obtain $R^2 = 0$. The origin $r = 0$ is thus a point of inflection. Therefore for the solutions $\Phi(r) = \pm 2n\pi$ we have $\tan(\alpha + \Phi) = 0$ or $\alpha(r) = \pm 2n\pi$. Furthermore, for $R^2 < l_0^2$ Eq. (5.2) yields

$$\sin\phi \sim \frac{R^2}{l_0^2}, \tag{5.15}$$

which reconfirms the statement (3.15). On the other hand, for $\Phi(r) = \frac{1}{2}\pi$ (and therefore $\Phi' = 0$) Eq. (4.12) yields the result $l_0^2 = q^{-1}\mathfrak{g} = 0$, which for $q = \infty$ produces the field equations of general relativity and Maxwell's equations. However, for $q = \infty$ we have $\lambda_0^2 = 0$. In this case Eq. (5.1) reduces to

$$e^{\rho} = R^2 = r^2 . \tag{5.16}$$

Hence for $R^2 \gg l_0^2$ Eq. (5.2) gives

$$\cos\Phi \sim \frac{\pm l_0^2}{R^2}, \quad \sin\Phi \sim 1 . \tag{5.17}$$

It will be proved in Sec. VIII that in general we have the relation

$$e^{\rho} = (r^4 + l_0^4)^{1/2} . \tag{5.18}$$

Thus the function R as defined by (5.1) plays the role of an "effective radius" and in terms of r is given by

$$R^2 = (r^4 + l_0^4 + \lambda_0^4)^{1/2} - r_0^2 . \tag{5.19}$$

We may now, formally, solve Eq. (4.16) in the form

$$\frac{1}{\mathfrak{V}} = \pm \frac{r^3(r^4 + l_0^4 + \lambda_0^4)^{1/2} \exp(F)}{(r^4 + l_0^4)^{5/4}} , \tag{5.20}$$

where

$$F = \frac{1}{2} \int \frac{\Phi'^2}{\rho'} dr . \tag{5.21}$$

Let us now consider the asymptotic region

$$r \ll l_0 \quad (f > 1). \tag{5.22}$$

In this case, using (5.8), we may write

$$R \sim \frac{r^2}{r_0\sqrt{2}} , \tag{5.23}$$

From (5.15), for $r \ll l_0$, we get the approximations

$$\Phi' \sim \frac{2r^3}{r_0^2 l_0^2} , \tag{5.24}$$

$$e^{\rho} \sim l_0^2 + \frac{r^4}{2l_0^2} , \tag{5.25}$$

$$e^{-\rho} \sim \frac{1}{l_0^2}\left(1 - \frac{r^4}{2l_0^4}\right),$$

$$\rho' \sim \frac{2r^3}{l_0^4} , \tag{5.26}$$

$$(e^{2\rho} + \lambda_0^4)^{1/2} \sim r_0^2 + \frac{r^4}{2r_0^2} ,$$

$$\cosh\Gamma \sim \frac{r_0^2}{l_0^2} + \frac{1}{2}\frac{r^4}{r_0^2 l_0^2} - \frac{r^4 r_0^2}{2l_0^6} , \tag{5.27}$$

$$\sinh\Gamma \sim \pm f^{-2}\left(1 - \frac{r^4}{2l_0^4}\right), \tag{5.28}$$

$$\tanh\Gamma \sim \pm \frac{\lambda_0^2}{r_0^2}\left(1 - \frac{r^4}{2r_0^4} + \frac{r^4}{2l_0^4}\right), \tag{5.29}$$

$$F \sim \frac{r^4}{4r_0^4} , \tag{5.30}$$

$$\frac{1}{\mathfrak{V}} \sim \pm \frac{r^3 r_0^2}{l_0^5} \exp\left(\frac{r^4}{4r_0^4}\right). \tag{5.31}$$

In the asymptotic region where $r \ll l_0$ the gravitational potential, which results from setting $\rho' = \Phi' = 0$ in Eq. (4.15), is given by

$$e^{\mathfrak{U}} \sim 1 + \kappa^2 \int \frac{dr}{\mathfrak{V}} \left(\int \frac{dr}{\mathfrak{V}} \right) \tag{5.32}$$

$$\sim 1 + \left(\frac{r_0}{l_0}\right)^{10} \exp(2F) , \tag{5.33}$$

where, because of the symmetry of Eq. (4.15) under the substitution $\mathfrak{V} \to -\mathfrak{V}$, one of the constants of integration associated with the term $\int(dr/\mathfrak{V})$ is set equal to zero. The appearance of the exponential factor $\exp(2F)$, since F is positive, indicates the long-range character of the gravitational force even at distances where $r \ll l_0$.

On dividing both sides of (4.15) by v and substituting (5.32) for $\exp(\mathfrak{U})$ we easily see that (4.15) is satisfied at $r = 0$, as were (4.13) and (4.14). At $r = 0$ we obtain the exact result

$$\lim_{r=0} e^{\mathfrak{U}} = 1 + \frac{1}{c^2} V_G , \tag{5.34}$$

where

$$V_G = \left(\frac{r_0}{l_0}\right)^{10} c^2 \tag{5.35}$$

represents the value of the gravitational potential at the origin. For the vanishing magnetic charge $\mathfrak{g}$ it assumes an infinite value. Thus the regularity of the gravitational field everywhere is due to the fact that $\mathfrak{g} \neq 0$. The mass dependence or independence of (5.35) follows from the possible nature of the magnetic charge $\mathfrak{g}$. According to this theory a particle is created by a "gravitational condensation" of the electromagnetic energy density by its own gravitational field, and therefore it is quite natural to expect the mass of a particle to depend on, among other things, $\mathfrak{g}$.

The space-time line element ds^2 defined in terms of the metrical coefficients $g_{\mu\nu}$ $(= g_{\{\mu\nu\}})$ has the form

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= c^2 e^{\mathfrak{U}} dt^2 - e^{\rho} \sin\Phi (d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad - (e^{-\mathfrak{U}}/\mathfrak{V}^2) dr^2 . \end{aligned} \tag{5.36}$$

At the origin we obtain the timelike line element

$$ds_0^2 = c^2\left(1 + \frac{1}{c^2} V_G\right) dt^2 . \tag{5.37}$$

VI. ELECTRIC CHARGE DISTRIBUTION

For the present case of a spherically symmetric static field the only surviving component of the electric current defined by (2.47) is the charge density $\mathcal{J}^4$, viz.,

$$\mathcal{J}^4 = \mathcal{J}^4_e + \mathcal{J}^4_0, \tag{6.1}$$

where

$$\mathcal{J}^4_e = \frac{q}{4\pi}(e^\rho \tanh\Gamma \sin\Phi)' \sin\theta = \frac{\pm e}{4\pi}\left(\frac{\sin\Phi}{\cosh\Gamma}\right)' \sin\theta, \tag{6.2}$$

$$\mathcal{J}^4_0 = \frac{q r_0^2}{4\pi}\left[\tfrac{1}{2}\mathcal{V}^2 e^{\mathcal{U}+\rho}(\rho'^2 + \Phi'^2)\tanh\Gamma + \mathcal{V}(\mathcal{V}e^{\mathcal{U}+\rho}\rho' \tanh\Gamma)'\right]' \sin\theta = \frac{\pm e r_0^2}{4\pi}\left\{\frac{\sin\Phi}{\cosh\Gamma}\rho' \mathcal{V}^2 e^{\mathcal{U}}\left[\ln\left(\frac{e^{\mathcal{U}}}{\cosh^2\Gamma}\right)\right]'\right\}' \sin\theta, \tag{6.3}$$

and where we have employed the relations (A7), (4.3), (4.16), and

$$R_{[14]} = \mathcal{V}\Gamma' e^{\mathcal{U}}\left[\ln\left(\frac{e^{\mathcal{U}}}{\cosh^2\Gamma}\right)\right]', \tag{6.4}$$

$$q(-g)^{1/2} R^{[14]} = \pm e\rho' \mathcal{V}^2 \frac{e^{\mathcal{U}}\sin\Phi}{\cosh\Gamma}\left[\ln\left(\frac{e^{\mathcal{U}}}{\cosh^2\Gamma}\right)\right]' \sin\theta. \tag{6.5}$$

The neutral charge density[5] $\mathcal{J}^4_0$ depends on the gravitational potential $e^{\mathcal{U}}$ and therefore on the mass itself. The definitions $\mathcal{J}^4_e$ and $\mathcal{J}^4_0$ do, of course, satisfy the conservation laws (2.3), since

$$Q_e = \int \mathcal{J}^4_e \, dr\, d\theta\, d\phi = \pm e\left(\frac{\sin\Phi}{\cosh\Gamma}\right)_0^\infty = \pm e, \tag{6.6}$$

$$Q_0 = \int \mathcal{J}^4_0 \, dr\, d\theta\, d\phi = \pm e r_0^2\left\{\frac{\sin\Phi}{\cosh\Gamma}\rho' \mathcal{V}^2 e^{\mathcal{U}}\left[\ln\left(\frac{e^{\mathcal{U}}}{\cosh^2\Gamma}\right)\right]'\right\}_0^\infty = 0, \tag{6.7}$$

where we have used Eqs. (5.26)–(5.31).

A neutral particle can be defined by setting $\mathcal{J}^4_e = 0$ or, as follows from (6.2), by taking

$$\sin\Phi = A\cosh\Gamma, \tag{6.8}$$

where A is a constant. We observe that an electrically neutral particle, as defined by (6.8), does still carry a polarization charge density, and the latter is given by (6.3), where $\sin\Phi/\cosh\Gamma$ is replaced by the constant A.

The presence of the neutral charge density implies a structure for a spherically symmetric elementary particle. For $r_0 = 0$ the predicted structure reduces to the point description of the conventional theory. The charge densities $\mathcal{J}^4_e$ and $\mathcal{J}^4_0$ are derived from the generalized electric field

$$\mathcal{E} = \Phi_{14} + r_0^2 R_{[14]} = \mathcal{E}_e + \mathcal{E}_0, \tag{6.9}$$

where

$$\mathcal{E}_e = \frac{1}{\mathcal{V}}\frac{\pm e}{R^2 + r_0^2}, \tag{6.10}$$

$$\mathcal{E}_0 = \frac{(\pm e) r_0^2}{R^2 + r_0^2}\rho' \mathcal{V} e^{\mathcal{U}}\left[\ln\left(\frac{e^{\mathcal{U}}}{\cosh^2\Gamma}\right)\right]'. \tag{6.11}$$

The calculation of the electric current $\mathcal{J}^\mu$ depends on knowledge of the field variables $\Phi_{\mu\nu}$, which, in turn, are the solutions of the field equations. Thus the electric current cannot be prescribed arbitrarily but is determined from the field itself. The dependence of $\mathcal{E}_e$ on $1/\mathcal{V}$ and of $\mathcal{E}_0$ on $\mathcal{V}$ is related, as will be seen, to the range of these fields. The quantity $\mathcal{E}_e$ represents a long-range field regular everywhere; this is due to a positive or negative charge density. The field $\mathcal{E}_0$ is regular everywhere and represents a neutral short-range field caused by a neutral charge distribution. For $\Phi = \frac{1}{2}\pi$ (which corresponds to $\mathfrak{g} = 0$) Eq. (6.10), as follows from (5.20), reduces to the Coulomb[6] field $\pm e/r^2$.

In order to see the nature of the above physical quantities more explicitly we must obtain their asymptotic forms for the two regions $r < l_0$ and $r \gg r_0$. First let us consider the asymptotic behavior near the origin, where, as follows from (5.23)–(5.32), one obtains the results

$$\mathcal{J}^4_e \sim \frac{\pm e}{2\pi}\frac{r^3}{r_0^4}\sin\theta, \tag{6.12}$$

$$\mathcal{J}^4_0 \sim \frac{\pm e}{\pi}\frac{2r^3}{l_0^4}\sin\theta, \tag{6.13}$$

$$\mathcal{E}_e \sim \pm e\,\frac{(\pm 1) r^3 \exp(F)}{l_0^5}, \tag{6.14}$$

$$\mathcal{E}_0 \sim \pm e(\pm 1)\frac{4r^3}{l_0^5}\left(\frac{r_0}{l_0}\right)^4 \exp(F), \tag{6.15}$$

all of which vanish at the origin provided $\mathfrak{g} \neq 0$ or

$q^{-1}\neq 0$. The numerical factor (±1) in (6.14) and (6.15) results from particle and antiparticle conjugation.

VII. SPECIAL EXACT SOLUTIONS

In order to assess fully the significance of the magnetic charge in this theory we shall study the solutions of Eqs. (4.13)–(4.15) for the special case

$$\Phi(r)=\tfrac{1}{2}\pi . \tag{7.1}$$

This result follows also from setting $\mathfrak{g}=0$ in the relation (5.6). On substituting $\Phi=\frac{1}{2}\pi$, $\Phi'=0$ in Eq. (4.12) we obtain

$$l_0^2=0 . \tag{7.2}$$

Hence for the class of solutions (7.1) the magnetic charge $\mathfrak{g}$ must vanish so that these solutions are valid beyond the spectrum of $\mathfrak{g}$ values and beyond the distances where $\mathfrak{g}=0$. The magnetically neutral surface implied by (7.1) is, in view of the unknown value of r, *indeterminate*. This is a consequence of general covariance, according to which it is not possible to define a rigid object. From the definitions (5.18), (5.21), and (4.3) we obtain

$$e^{\rho}=r^2,$$
$$\mathfrak{V}=\pm\frac{r^2}{(r^4+\lambda_0^4)^{1/2}}, \tag{7.3}$$
$$\tanh\Gamma=\pm\frac{\lambda_0^2}{(r^4+\lambda_0^4)^{1/2}} .$$

Equations (4.13) and (4.16) are satisfied identically, and Eqs. (4.14) and (4.15) reduce to

$$\mathfrak{V}(\mathfrak{V}e^{\mathfrak{U}+\rho}\rho')'+\mathfrak{V}^2\rho'^2e^{\mathfrak{U}+\rho}\tanh^2\Gamma=\kappa^2e^{\rho}(1-\cosh\Gamma)+2, \tag{7.4}$$

$$\mathfrak{V}(\mathfrak{V}e^{\mathfrak{U}+\rho}\mathfrak{U}')' +\mathfrak{V}^2e^{\mathfrak{U}+\rho}\tanh^2\Gamma(8\rho'^2\tanh\Gamma+3\mathfrak{U}'\rho'-3\rho'^2) =\kappa^2e^{\rho}\left(1-\frac{1}{\cosh\Gamma}\right). \tag{7.5}$$

Using (7.3), Eq. (7.4) can be written as

$$\frac{d}{dr}\left(\frac{r^5e^{\mathfrak{U}}}{r^4+\lambda_0^4}\right)=1+\tfrac{1}{2}\kappa^2[r^2-(r^4+\lambda_0^4)^{1/2}], \tag{7.6}$$

and can be integrated, at once, in the form

$$e^{\mathfrak{U}}=\frac{r^4+\lambda_0^4}{r^4}\left[1-\frac{2Gm}{c^2r}+\frac{1}{3}\frac{r^2}{\lambda_0^2}-\frac{1}{3}\frac{(r^4+\lambda_0^4)^{1/2}}{\lambda_0^2} -\frac{2}{3}\frac{\lambda_0^2}{r}\mathfrak{K}(r)\right], \tag{7.7}$$

where

$$\mathfrak{K}(r)=\int\frac{dr}{(r^4+\lambda_0^4)^{1/2}}=\frac{1}{2\lambda_0}\int\frac{d\gamma}{(1-\frac{1}{2}\sin^2\gamma)^{1/2}}$$

is an elliptic integral of the first kind and where we have employed the relations

$$\gamma=\cos^{-1}\left(\frac{\lambda_0^2-r^2}{\lambda_0^2+r^2}\right),$$
$$\kappa^2\lambda_0^2=2,$$
$$\int\frac{r^4dr}{(r^4+\lambda_0^4)^{1/2}}=\tfrac{1}{3}[r(r^4+\lambda_0^4)^{1/2}-\lambda_0^4\mathfrak{K}(r)],$$
$$\int(r^4+\lambda_0^4)^{1/2}dr=\tfrac{1}{3}[2\lambda_0^4\mathfrak{K}(r)+r(r^4+\lambda_0^4)^{1/2}].$$

The solution (7.7), as can be seen by direct substitution, satisfies Eq. (7.5). Hence Eqs. (7.4) and (7.5) are compatible. The solution (7.7) is singular at $r=0$. For the charged part of the electric field we have

$$\mathcal{E}_e=\frac{q}{\mathfrak{V}}\tanh\Gamma=\frac{\pm e(\pm 1)}{r^2}, \tag{7.8}$$

which is just the usual Coulomb field, where (±1) correspond, as before, to particle and antiparticle conjugation. The neutral field can be calculated as

$$\mathcal{E}_0=q\lambda_0^2R_{[14]}=\pm e(\pm 1)\frac{4}{3r^2}\left[1-\frac{(r^4+\lambda_0^4)^{1/2}}{r^2}+\frac{\lambda_0^2}{r^3}\left(\frac{3Gm}{c^2}+\lambda_0^2\mathfrak{K}(r)\right)\right], \tag{7.9}$$

which falls off as $1/r^5$. The electric charge density

$$\mathfrak{J}_e^4=\frac{\pm e}{2\pi}\frac{\lambda_0^4r}{(r^4+\lambda_0^4)^{3/2}}\sin\theta \tag{7.10}$$

for $r\gg\lambda_0$ falls off as λ_0^4/r^5. The total charge is, of course, conserved since

$$\int\mathfrak{J}_e^4drd\theta d\phi=\left(\frac{\pm er^2}{(r^4+\lambda_0^4)^{1/2}}\right)_0^{\infty}=\pm e . \tag{7.11}$$

However, the neutral charge density $\mathfrak{J}_0^4$, in view of its singularity at $r=0$, is not conserved.[7] This illustrates the fact that the neutral charge density is held together by the neutral magnetic charge distribution in the particle itself.

Now for the asymptotic limit $r\gg\lambda_0$ the solution (7.7) reduces to

$$e^{\mathfrak{U}}\to 1-\frac{2Gm}{c^2r}+\frac{Ge^2}{c^4r^2}, \tag{7.12}$$

which is Nordström's extension of the Schwarzschild solution of general relativity in the presence of an electric field. The result (7.12) cannot be obtained for the nonvanishing magnetic charge $\mathfrak{g}$.

Theorem: There exist no regular solutions where $\mathfrak{g}=0$, *and conversely for* $\mathfrak{g}=0$ *the solutions are not regular everywhere.*

In order to prove the theorem let us consider the asymptotic solution for Eqs. (4.13)–(4.15) in the region where $r \gg \lambda_0$. From $R\sim r$, $e^\rho \sim r^2$, and from Eqs. (5.2) we obtain

$$\cos\Phi \sim \frac{\pm l_0^2}{r^2} - \frac{1}{2}\frac{l_0^2 r_0^2}{r^4}, \qquad \sin\Phi \sim 1 - \frac{l_0^4}{2r^4}, \quad \Phi' \sim \frac{2l_0^2}{r^3}. \tag{7.13}$$

Hence

$$F = \frac{1}{2}\int \frac{\Phi'^2}{\rho'}\,dr \sim -\frac{l_0^4}{4r^4}, \qquad \frac{1}{\mathfrak{v}} \sim \pm\exp\left(-\frac{l_0^4}{4r^4}\right) \sim \pm\left(1-\frac{l_0^4}{4r^4}\right), \qquad \cosh\Gamma \sim 1+\frac{\lambda_0^4}{2r^4}, \quad \tanh\Gamma \sim \frac{\pm\lambda_0^2}{r^2}. \tag{7.14}$$

Thus the only remaining unknown is the gravitational function $e^{\mathfrak{u}}$, for which we have three equations, (4.13)–(4.15). Equations (4.14) and (4.15), as in general relativity, can be solved independently. Let us consider the asymptotic form of (4.14). Using the above approximations and neglecting terms of the order $(\lambda_0/r)^5$ and higher we obtain from (4.14) the result

$$(re^{\mathfrak{u}})' = 1 - \frac{\lambda_0^4}{2r^2 r_0^2} = 1 - \frac{Ge^2}{c^4 r^2},$$

which can be integrated as

$$e^{\mathfrak{u}} = 1 - \frac{2mG}{c^2 r} + \frac{Ge^2}{c^4 r^2}. \tag{7.15}$$

Now the asymptotic form of Eq. (4.15) is given by

$$(r^2 e^{\mathfrak{u}}\mathfrak{u}')' = \frac{r_0^2}{r^2},$$

which is solved by

$$e^{\mathfrak{u}} = 1 - \frac{2mG}{c^2 r} + \frac{G(e^2+\mathfrak{g}^2)}{c^4 r^2}. \tag{7.16}$$

The two solutions (7.15) and (7.16) differ in their last terms. This incompatibility of the two equations (4.14) and (4.15) can be resolved by studying the asymptotic form of the remaining equation, (4.13). It is given by

$$l_0^2\left[\left(\frac{e^{\mathfrak{u}}}{r}\right)' + \frac{3}{2r^2} - \frac{r_0^2}{r^4}\right] = 0. \tag{7.17}$$

Hence the only solution of (7.17) which can make the solutions (7.15) and (7.16) compatible is

$$l_0^2 = 0, \tag{7.18}$$

or $\mathfrak{g}=0$. We have thus proved the theorem. The actual value of r where $\mathfrak{g}=0$ and the absence of neutral magnetic charge density beyond this value of r involve an *indeterminacy*. Thus relativistic invariance together with a structure of an elementary particle imposes an indeterminacy on the actual size of the structure. The degree of this indeterminacy may eventually be represented by the introduction of $\hbar$ (see Sec. VIII).

VIII. MAGNETIC CHARGE DISTRIBUTION

The properties of the magnetic charge predicted by this theory are novel and bear no relation to other theories on this subject.[8,9] In this theory a magnetic charge $\mathfrak{g}$ does not reside in a magnetically neutral core of an elementary particle as a doublet of positive and negative charges. Furthermore, it does not exist as a free pole carrying positive or negative magnetic charge producing a long-range field. Thus the magnetic charge density of this theory, besides playing a fundamental role in the creation of mass itself, generates only short-range fields associated with *strong as well as weak interactions*. In this theory the magnetic charge does not directly partake in electromagnetic interactions. The currents of these charges can only give rise to radiation of massive particles instead of the radiation of photons via the long-range forces of the electromagnetic field. Thus, through the magnetic charge, we have established a classical basis for *strong and weak interactions*.

From the results of the previous section we see that the "intrinsic magnetic fields" of a particle

$$\mathcal{H} = \frac{q}{\mathfrak{v}}\cos\Phi, \quad \mathcal{B}_0 = qe^\rho\cos\Phi\sin\theta \tag{8.1}$$

for $\mathfrak{g}$ vanish. The correspondence of the $\mathfrak{g}=0$ solutions to nonregular behavior of the field shows that the intrinsic magnetic fields $\mathcal{H}$ and $\mathcal{B}_0$ do not extend beyond the distribution of the neutral and charged magnetic charge densities

$$\mathfrak{s}^4 = \frac{q}{4\pi}\frac{d}{dr}(e^\rho\cos\Phi)\sin\theta, \qquad \zeta^4 = \frac{q}{4\pi}\frac{d}{dr}(e^\rho\cos\Phi\sin\Phi)\sin\theta, \tag{8.2}$$

respectively, where the total magnetic charges are given by

$$Q_0 = \int \mathfrak{s}^4\, dr d\theta d\phi = (qe^{\rho}\cos\Phi)_0^{\infty} = 0\ ,$$
$$Q_{\mathfrak{g}}^{\pm} = \int \zeta^4 dr d\theta d\phi = \pm \mathfrak{g}\ . \tag{8.3}$$

However, in the asymptotic region $r < l_0$ the magnetic field $\mathcal{H}$ has the form

$$\mathcal{H} \sim \pm \mathfrak{g}\ \frac{r^3 r_0^{\,2}}{l_0^{\,7}} \exp(F)\ , \tag{8.4}$$

where F is given by (5.30). It vanishes at the origin $r=0$.

For the static distribution, the radial part of the asymptotic equation (2.53) is solved by the spherical Bessel function

$$\mathfrak{s}^4 = \frac{1}{4\pi} j_1(\kappa r)\ , \tag{8.5}$$

where the constant κ, being large, implies that $\mathfrak{s}^4$, for $r \neq 0$, is vanishingly small. The zeros of the Bessel function $j_1(\kappa r)$ are given by

$$\tan(\kappa r) = \kappa r\ . \tag{8.6}$$

At these points the magnetic charge density changes sign and the magnitude of the distribution falls off with alternating signs. Thus at distances where $r \gg l_0$ the structure of an elementary particle appears to consist of an infinite number of constituent layers of magnetic charge densities. The charge densities of alternating signs are held together by the mutual magnetic and gravitational attractions of the layers. The absolute value of the total sum of fractional magnetic charges of fixed sign contained in the alternating layers is equal to $\mathfrak{g}$. Thus the neutral distribution of magnetic charge in the core of the elementary particle contains the quantity $+\mathfrak{g}$ of positive magnetic charge and the quantity $-\mathfrak{g}$ of negative magnetic charge. In general the distribution will depend on the centrifugal magnetic number l $(=0, 1, 2, \dots)$ which is contained in the radial part of the asymptotic equation (2.53) and is associated with the spherical Bessel functions $j_l(\kappa r)$. However, the nonlinear equation (2.38) itself may give rise to a radial magnetic number n $(=0, 1, 2, \dots)$.

In general for the points where the neutral magnetic charge density vanishes one has

$$\frac{q}{4\pi}\frac{d}{dr}(e^{\rho}\cos\Phi) = 0\ , \tag{8.7}$$

or

$$\cos\Phi = \pm l_0^{\,2} e^{-\rho}\ .$$

Now from (8.7) we also have

$$\rho' = \Phi' \tan\Phi\ , \tag{8.8}$$

where

$$\Phi \neq \tfrac{1}{2}\pi\ .$$

On setting $\lambda_0^{\,2}=0$ in Eq. (4.16) and substituting for ρ' from (8.8) we obtain the nonlinear equation

$$\Phi'' \tan\Phi + \frac{3}{2}\frac{\Phi'^2}{\cos^2\Phi} + \frac{\mathfrak{V}'}{\mathfrak{V}}\Phi'\tan\Phi = 0\ . \tag{8.9}$$

A class of solutions of this equation are given by

$$\Phi(r) = \pm n\pi\ , \quad n = 0, 1, 2, \dots \tag{8.10}$$

and the fact that they are valid at the point $r=0$. Thus to obtain the remaining solutions we must assume

$$\Phi \neq \pm n\pi$$

or

$$e^{\rho} \neq l_0^{\,2}\ .$$

In this case we can multiply Eq. (8.9) by $\cot\Phi$ and divide by Φ' to obtain

$$\frac{\Phi''}{\Phi'} + \frac{3}{2}\frac{\Phi'}{\sin\Phi\cos\Phi} + \frac{\mathfrak{V}'}{\mathfrak{V}} = 0\ . \tag{8.11}$$

Equation (8.11) is solved by

$$\frac{1}{\mathfrak{V}} = \pm\tfrac{1}{2} l_0 \Phi' (\tan\Phi)^{3/2}\ . \tag{8.12}$$

We thus see that the point of degeneracy $r=0$ (or the point of isolated regularity) and also the point r_c corresponding to the case where $\Phi(r_c) = \frac{1}{2}\pi$ do not group themselves with the remaining infinite number of interior points $(0 < r < r_c)$ (or interior neutral surfaces) where the neutral magnetic charge density vanishes. Hence we have the restrictions

$$0 < \Phi \leq \tfrac{1}{2}\pi \text{ for } l_0^{\,2} \text{ and } \Phi = \cos^{-1}(l_0^{\,2} e^{-\rho}) + \tfrac{1}{2}\pi\ ,$$
$$\tfrac{1}{2}\pi \leq \Phi < \pi \text{ for } -l_0^{\,2} \text{ and } \Phi = \cos^{-1}(-l_0^{\,2} e^{-\rho}) - \tfrac{1}{2}\pi\ ,$$

which in terms of e^{ρ} imply, for the points of zero magnetic charge density, the inequality

$$l_0^{\,2} \leq e^{\rho} < \infty\ . \tag{8.13}$$

By using the above results and the definition (5.18) for e^{ρ} in Eqs. (4.13), (4.14), and (4.15) (see Appendix B) they can be integrated at once to yield the solutions

$$\exp(\mathfrak{U}) = A_1 \frac{t^{3/2}}{(1+t^2)^{1/2}} + \frac{1}{2}\frac{t^2}{(1+t^2)^{1/2}}\left(\frac{1}{(1+t^2)^{1/2}} - \frac{\mathfrak{K}}{2\sqrt{t}}\right) + \frac{\kappa^2 l_0^{\,2}}{4}\frac{t^2}{(1+t^2)^{1/2}}\left(\frac{t}{(1+t^2)^{1/2}} - \frac{3(1+t^2)^{1/2}}{1+t} + \frac{6\mathcal{E} - 3\mathfrak{K}}{2\sqrt{t}}\right)$$
$$+ \frac{\kappa^2 l_0^{\,2}}{2}\frac{t^2}{(1+t^2)^{1/2}}\left\{1 + \frac{1}{2(2t)^{1/2}}\left[\tanh^{-1}\left(\frac{(2t)^{1/2}}{1+t}\right) - \tan^{-1}\left(\frac{(2t)^{1/2}}{1-t}\right)\right]\right\}\ , \tag{8.14}$$

$$\exp(\mathfrak{u}) = A_2 \frac{\sqrt{t}}{(1+t^2)^{1/2}} + \frac{3t}{2(1+t^2)^{1/2}}\left(\frac{\mathcal{K}-2\mathcal{E}}{2\sqrt{t}} + \frac{(1+t^2)^{1/2}}{1+t}\right) - \frac{t^2}{2(1+t^2)}$$
$$- \frac{\kappa^2 l_0^2}{4}\frac{t^3}{(1+t^2)} + \frac{5\kappa^2 l_0^2}{12}\frac{t}{(1+t^2)^{1/2}}\left((1+t^2)^{1/2} - \frac{\mathcal{K}}{2\sqrt{t}}\right)$$
$$- \frac{\kappa^2 l_0^2}{2}\frac{t}{(1+t^2)^{1/2}}\left\{\frac{t}{3} - \frac{1}{2(2t)^{1/2}}\left[\tanh^{-1}\left(\frac{(2t)^{1/2}}{1+t}\right) + \tan^{-1}\left(\frac{(2t)^{1/2}}{1-t}\right)\right]\right\}, \tag{8.15}$$

$$\frac{d}{dt}[\exp(\mathfrak{u})] = A_3 \frac{t^{3/2}}{(1+t^2)^{3/2}}$$
$$+ \frac{\kappa^2 l_0^2}{2}\frac{t^2}{(1+t^2)^{3/2}}\left\{\frac{(1+t^2)^{1/2}-t}{3} - \frac{3\mathcal{K}}{4\sqrt{t}} + \frac{1}{2(2t)^{1/2}}\left[\tanh^{-1}\left(\frac{(2t)^{1/2}}{1+t}\right) + \tan^{-1}\left(\frac{(2t)^{1/2}}{1-t}\right)\right]\right\}. \tag{8.16}$$

In these equations, because of the condition $\Phi \neq \pm n\pi$ the point $t=0$ is excluded. In fact, if $t=r^2/l_0^2$ then for t large compared to 1 Eqs. (8.15) and (8.16) with

$$A_2 = -\frac{2mG}{l_0c^2}, \quad A_3 = \frac{2mG}{l_0c^2} \tag{8.17}$$

yield the Schwarzschild solution of general relativity. Hence, as seen from Appendix B, the definition (5.18) is correct.

Now, outside $(0<r<r_c)$ the three equations (8.14)–(8.16) are compatible at only $r=\infty$ provided $l_0=0$. Hence the constant of integration $A_1=0$. We thus have three equations to determine the three unknowns t, $e^{\mathfrak{u}}$, and κl_0. By combining (8.14) and (8.15) one obtains an equation of the form

$$\frac{(2t)^{1/2}}{1-t} = \tan\gamma(t), \quad t \neq 0 \tag{8.18}$$

where $\gamma(t)$ is a function of t. In general Eq. (8.18) would have an infinite number of solutions yielding the surfaces of zero magnetic charge. Equations (8.14)–(8.16) would further yield, for each surface of zero charge, a relation between A_2, A_3 and l_0, κ. In this way we see that the constant $\mathfrak{g}$ behaves like an "eigenvalue" of the charge distribution and assumes a spectrum of values. Relativistic invariance of the theory is not compatible with a sharp boundary of neutral magnetic charge density. Therefore, general covariance of the theory, for the surfaces of zero magnetic charge density in the particle core, implies an *indeterminacy*. The degree of this indeterminacy for the surfaces of zero magnetic charge density may be given by

$$rmc = \hbar$$

or

$$m = \frac{\hbar}{cl_0\sqrt{t}}, \tag{8.19}$$

where t is a function of l_0 and m. Therefore the solution of Eqs. (8.14)–(8.19) should yield a mass spectrum.

On the basis of the above results we may now state the fundamental theorem of this theory.

Magnetic theorem. General relativity and classical electrodynamics are valid only in the region $r \gg \lambda_0$ which corresponds to the $\mathfrak{g}=0$ limit of the generalized theory of gravitation. There exist no free magnetic poles associated with a long-range field, but elementary particles are composed of stratified layers of neutral magnetic matter with or without electric charge, and the corresponding electric, magnetic and gravitational fields for $\mathfrak{g} \neq 0$ are regular everywhere. All elementary particles carry a magnetic charge $\mathfrak{g}$ (different magnitudes for different particles) associated with a short-range field. This charge is superimposed over the magnetically neutral particle core. Magnetic monopoles associated with a long-range field do not exist.

IX. SELF-ENERGY AND BINDING ENERGY OF A PARTICLE

We may now use the conserved energy-momentum tensor $\mathfrak{T}_\nu^\mu$ to calculate the binding energy of a static spherically symmetric system. We shall consider the simplest case of an electrically neutral (i.e., $\lambda_0=0$) field without polarization charge and, in analogy with general relativity, compute the volume integral of the quantity

$$\mathfrak{T}_4^4 - \mathfrak{T}_1^1 - \mathfrak{T}_2^2 - \mathfrak{T}_3^3 = \frac{q^2}{2\pi}[(-\hat{g})^{1/2} - (-g)^{1/2}]. \tag{9.1}$$

By using the definitions (3.16) we may write for the binding energy

$$\Delta E = \int (\mathfrak{T}_4^4 - \mathfrak{T}_1^1 - \mathfrak{T}_2^2 - \mathfrak{T}_3^3)drd\theta d\phi$$
$$= 2q^2 \int_0^\infty \frac{e^\rho}{\mathfrak{v}}(1-\sin\Phi)dr. \tag{9.2}$$

From the field equation (4.15) we obtain, for $\lambda_0 = 0$, the result

$$(\mathfrak{V} e^{\mathfrak{U}+\rho}\mathfrak{U}')' = \frac{\kappa^2 e^\rho}{\mathfrak{V}}(1-\sin\Phi)$$

and hence

$$\Delta E = r_0^2 q^2 (\mathfrak{V} e^{\mathfrak{U}+\rho}\mathfrak{U}')_0^\infty .$$

On substituting from (5.9) and the asymptotic solutions (5.27)–(5.33) and (5.12) we get the fundamental result

$$\Delta(\pm E) = mc^2 - \frac{(2\mathfrak{g}_0)^2}{l_0}, \tag{9.3}$$

where

$$\mathfrak{g}_0 = 2\mathfrak{g}\left(\frac{1}{\kappa l_0}\right)^3$$

and where the ± signs are due to the linear dependence of energy on $\mathfrak{V}$ and, as mentioned before, are interpreted as pertaining to the classical counterparts of particles and antiparticles. The constant of integration m, obtained earlier, is the gravitational mass of the particle. The second term

$$E_S = \frac{(2\mathfrak{g}_0)^2}{l_0} \tag{9.4}$$

represents the total self-energy (or magnetic potential energy) of a totally neutral particle arising from the magnetic attraction between the layers of magnetic charge densities of alternating sign. The factor 2 is due to the two possible signs of a layer of magnetic charge density. Thus E_S may be interpreted as the total rest energy of the constituents of a particle. From (9.4) we see that for $\mathfrak{g}=0$ the self-energy becomes infinite.

For an estimate of the self-energy or the binding energy of an elementary particle we need a reasonable value for the magnetic charge $\mathfrak{g}$. One possibility is to assume that the length l_0 is of the order of the nucleon Compton wavelength. Another possibility is to relate the gravitational potential energy of a uniform homogeneous spherical nucleon to the dimensionless number $e^2/\mathfrak{g}^2$ by writing

$$\frac{1}{4\pi}\left(\frac{2}{5}\,\frac{Gm^2}{e^2}\right) = \frac{e^2}{\mathfrak{g}^2}, \tag{9.5}$$

where if we take m to be the nucleon mass then one of the values of the magnetic charge is given by

$$\mathfrak{g} = 1 \text{ coulomb} = 3\times10^9 \text{ esu}$$

or

$$\mathfrak{g} = 6.24\times10^{18} e . \tag{9.6}$$

The above value for $\mathfrak{g}$ is, of course, only a guess based on the assumption that the length l_0 ought to be of the order of nucleon Compton wavelength. The actual value of $\mathfrak{g}$ may have to come from the quantization of the theory. However, the value (9.6) appears to be quite reasonable even though it is 17 orders of magnitude larger than the value of the magnetic charge obtained by Dirac for a free monopole associated with a long-range field.

For the value (9.6) of $\mathfrak{g}$ the corresponding values of the hadronic (l_0) and leptonic (λ_0) lengths are given by

$$l_0 \cong \frac{(2G)^{1/2}}{c^2}\,\mathfrak{g} = 1.2\times10^{-15}\text{ cm}, \tag{9.7}$$

$$\lambda_0 \cong \frac{(2G)^{1/2}}{c^2}(e\mathfrak{g})^{1/2} = 4.8\times10^{-25}\text{ cm}. \tag{9.8}$$

The lengths λ_0 and l_0 as given by (5.11) and (5.12) are equal only if $\mathfrak{g}=e$. In this case, as seen from (5.35), the gravitational potential assumes its maximum value. The corresponding value of the length is given by

$$l_m = l_0 = \lambda_0 = \frac{2^{3/4}}{c^2} e\sqrt{G} = 2.3\times10^{-34}\text{ cm}. \tag{9.9}$$

A possible speculation on the origin of the length l_m can be based on the assumption that in the "primordial field" the energy density and the corresponding gravitational field were high enough for the particle to consume its binding energy (i.e., its magnetic potential energy) and thus collapse to a size of the order l_m where

$$mc^2 = \frac{e^2+\mathfrak{g}^2}{r_0} \tag{9.10}$$

or

$$m = \frac{e}{\sqrt{G}} = 7.2\times10^{-7}\text{ g}. \tag{9.11}$$

Thus l_m is the smallest size into which particles could have collapsed in the primeval time. The binding energy of such "micro black holes" and "anti micro black holes" is of the order of 10^{15} ergs. However, the binding energy of a particle or antiparticle, as follows from (9.4), is of the order of 10^{33} ergs. Thus the minimum temperature in the primordial field would have been of the order $T_0 \sim 10^{50}\,^\circ$K. Thus T_0 is the minimum temperature required to put all the magnetic charge layers together to produce a nucleon. The large size of the self-energy confirms the earlier conclusion of the theory that there exist no free monopoles associated with a long-range field.

For $\mathfrak{g}=0$, except for the factor ±1, we obtain the result of general relativity provided that the range of integration in (9.2) does not extend beyond the size of the material system with Schwarzschild radius $2mG/c^2$. In this case $E = mc^2$ represents the total energy of matter in terms of the energy-momentum tensor of the matter alone.

The above, from the point of view of the classical field theory, is a solution of the self-energy problem. The presence of negative-energy solutions in a classical theory is a pleasant surprise and not necessarily a vice. The negative-energy solutions imply the necessity for a quantum-field-theoretical formulation of the classical theory.

The above value of the order of 10^{33} cgs units for the binding energy can also be regarded as the degree of conservation of baryon or lepton charge.

X. CONCLUSIONS

From the classical point of view the results of this paper show that the generalized theory of gravitation, which is based on a correspondence principle, lays the foundations for regular and divergence-free electromagnetic, gravitational, and short-range interactions. In this theory the new ideas are the existence of a magnetic charge $\mathfrak{g}$ (which assumes a spectrum of values) and the idea that the regularity of the solutions is due to the finite value of $\mathfrak{g}$ since for $\mathfrak{g}=0$ the spherically symmetric solutions reduce to Nordström's solution of general relativity. An elementary particle has a magnetically neutral core of matter containing a distribution of alternating positive and negative magnetic charge densities over the stratified layers of the core. Furthermore, every particle carries an excess $\pm\mathfrak{g}$ (different for different particles) and generates a short-range field. A novel result is the emergence of an electrically neutral current, in addition to a charged current, and a corresponding neutral field which appears to be a classical version of the vacuum polarization in quantum electrodynamics. From (6.12) and (6.13) we see that the ratio of the neutral charge density $\mathcal{J}_0^4$ to the electric charge density in the neighborhood of the origin is given by $(r_0/l_0)^4$ and becomes infinite for $\mathfrak{g}=0$.

A very interesting consequence of the above results is the finiteness of the self-energy, which in turn yields for an elementary particle a finite binding energy. Furthermore, the classical counterparts of the strong and weak interactions are represented by short-range fields due to a spectrum of $\mathfrak{g}$ values superimposed over the neutral magnetic charge density in the core of the particle. An important difference of the magnetic charge from the electric charge is the fact that the former depends on the mass itself, and this is the basic reason for the short-range nature of the corresponding field. The appearance of both positive- and negative-energy solutions with corresponding electric charges is very surprising. In this context it is necessary to study the plane-wave solutions of the field equations to see the nature of the negative-energy solutions in this case. If $g^+_{\mu\nu}$ represent a set of solutions with positive energy and $g^-_{\mu\nu}$ the corresponding set with negative energy then the superimposed quantities $g^+_{\mu\nu}+g^-_{\mu\nu}$ do not yield an approximate solution, though each of them is an exact solution. However, if the fields in $g^\pm_{\mu\nu}$ are small compared to q then we may call them weak fields, and in this case the field $g^+_{\mu\nu}+g^-_{\mu\nu}$ is an approximate solution. The approximation can further be improved to higher orders. In particular, one would like to discuss the classical aspects of the scattering of light by light to understand further the relationship between positive- and negative-energy solutions.

The origin of the negative-energy solutions, in this theory, is presumably due to the presence in the theory of square-root terms containing both electromagnetic and gravitational variables. In order to elucidate the negative-energy problem we shall calculate the extremum action function of the theory by substituting the field equations (2.33)–(2.35) in the action function (2.26) and obtaining

$$S_{\text{ext}} = -\frac{q^2}{4\pi}\int[(-\hat{g})^{1/2}-(-g)^{1/2}]\,d^4x\,. \qquad (10.1)$$

Hence for each solution of the field equations there exists a nonvanishing extremum action function corresponding to particles represented by the field. The corresponding limit (i.e., $q\to\infty$) of (10.1) yields the extremum value for the action function of general relativity as

$$S_{G\ \text{ext}} = -\frac{1}{16\pi c}\int(-g)^{1/2}\Phi^{\mu\nu}\Phi_{\mu\nu}\,d^4x\,, \qquad (10.2)$$

which is the action function of a pure electromagnetic field in its own gravitational field.

A Dirac type of linearization (in terms of Dirac matrices) of the action (10.1) was carried out by this author[3] earlier, and in view of the then unknown physical interpretation and the corresponding solutions of the field equations no useful results were obtained. We are planning to reconsider the older[3] approach in light of this paper's results. The above suggested procedure may turn out to be a simpler way to quantize the theory. Quantization is, presumably, the only way to discover a connection between g and e

and also to associate negative-energy solutions with real particles. However, on a cosmological scale the classical positive- and negative-energy solutions may imply a large-scale symmetry between the distribution of matter and antimatter in the universe. On a classical level one may regard the total energy content of the universe as being zero. In a similar way the total hadron number N_h, total lepton number N_L, total electric charge Q_e, and total magnetic charge Q_g of the universe must vanish. In this connection another classical problem that needs early attention is the study of the time-dependent spherically symmetric fields.

This paper might have, perhaps, been written over 20 years ago. The mathematical formulation for the theory has not changed. However, there are some fundamental reasons for the delay. One of the stumbling blocks in the development of all three versions[1,4,10] of the theory has been the misinterpretation of the axial-vector $\mathfrak{s}^\mu$ [(2.44)] as the electric current density. We know now that this was an erroneous assumption. Another important consideration was the absence of a correspondence principle, which would, of course, have demonstrated the true nature of $\mathfrak{s}^\mu$. In the affirmative one can cite the recent proliferation of elementary-particle models, particularly those describing possible constituents of the particles, none of which have been entirely satisfactory. Under these circumstances an elementary-particle model based on a fundamental theory, even if a classical one, should be given very serious consideration.

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APPENDIX A: SPHERICALLY SYMMETRIC STATIC AFFINE FIELDS

The static spherically symmetric components of the 64-component affine connections $\Gamma^\rho_{\mu\nu}$ can be obtained by a long but straightforward process which leads to an algebraic solution of the equations $\hat{g}_{\mu\nu;\,\rho}=0$. For the time-independent case there are only 17 nonvanishing components:

$$\begin{aligned}
\Gamma^1_{11} &= -(\tfrac{1}{2}\mathfrak{U}'+\mathfrak{V}'/\mathfrak{V}),\\
\Gamma^1_{22} &= \frac{1}{\sin^2\theta}\Gamma^1_{33}\\
&= \tfrac{1}{2}\mathfrak{V}^2 e^{\mathfrak{U}+\rho}(\Phi'\cos\Phi-\rho'\sin\Phi),\\
\Gamma^1_{44} &= \mathfrak{V}^2 e^{2\mathfrak{U}}(\tfrac{1}{2}\mathfrak{U}'-2\Gamma'\tanh\Gamma),\\
\Gamma^2_{\{12\}} &= \Gamma^3_{\{13\}} = \tfrac{1}{2}\rho',\\
\Gamma^2_{33} &= -\sin\theta\cos\theta,\\
\Gamma^3_{\{23\}} &= \cot\theta,\\
\Gamma^4_{\{14\}} &= \tfrac{1}{2}\mathfrak{U}'-\Gamma'\tanh\Gamma,\\
\Gamma^2_{\{34\}} &= \tfrac{1}{2}\mathfrak{V}\Phi' e^{\mathfrak{U}}\tanh\Gamma\sin\theta,\\
\Gamma^3_{\{24\}} &= -\frac{1}{2}\frac{\mathfrak{V}e^{\mathfrak{U}}}{\sin\theta}\Phi'\tanh\Gamma,\\
\Gamma^1_{[23]} &= \tfrac{1}{2}\mathfrak{V}^2 e^{\mathfrak{U}+\rho}(\Phi'\sin\Phi+\rho'\cos\Phi)\sin\theta,\\
\Gamma^2_{[31]} &= -\tfrac{1}{2}\Phi'\sin\theta,\\
\Gamma^3_{[12]} &= -\frac{1}{2}\frac{\Phi'}{\sin\theta},\\
\Gamma^1_{[41]} &= \mathfrak{V}e^{\mathfrak{U}}\Gamma',\\
\Gamma^2_{[42]} &= \Gamma^3_{[43]} = \tfrac{1}{2}\mathfrak{V}e^{\mathfrak{U}}\rho'\tanh\Gamma,
\end{aligned} \tag{A1}$$

where a prime indicates differentiation with respect to r.

The only nonvanishing components of $R_{\{\mu\nu\}}$ and $R_{[\mu\nu]}$ are given by

$$\begin{aligned}
R_{11} &= \rho''+\rho'\frac{\mathfrak{V}'}{\mathfrak{V}}+\tfrac{1}{2}(\Phi'^2+\rho'^2)+\tfrac{1}{2}\left(\mathfrak{U}''+\mathfrak{U}'^2+\mathfrak{U}'\frac{\mathfrak{V}'}{\mathfrak{V}}+\mathfrak{U}'\rho'\right)\\
&\quad+\Gamma'^2\left(\tanh^2\Gamma-\frac{1}{\cosh^2\Gamma}\right)-\left(\Gamma''+\frac{\mathfrak{V}'}{\mathfrak{V}}\Gamma'+\tfrac{3}{2}\mathfrak{U}'\Gamma'\right)\tanh\Gamma,
\end{aligned} \tag{A2}$$

$$\begin{aligned}
R_{22} &= -1+\tfrac{1}{2}\mathfrak{V}[\mathfrak{V}e^{\mathfrak{U}+\rho}(\rho'\sin\Phi-\Phi'\cos\Phi)]'-\tfrac{1}{2}\Phi'\mathfrak{V}^2e^{\mathfrak{U}+\rho}(\Phi'\sin\Phi+\rho'\cos\Phi)\\
&\quad-\tfrac{1}{2}\mathfrak{V}^2\Gamma' e^{\mathfrak{U}+\rho}(\rho'\sin\Phi-\Phi'\cos\Phi)\tanh\Gamma,
\end{aligned} \tag{A3}$$

$$R_{33} = R_{22}\sin^2\theta, \tag{A4}$$

$$R_{44}=\mathcal{V}^2e^{2\mathcal{U}}\left[\Gamma'^2-\tfrac{1}{2}\left(\mathcal{U}''+\mathcal{U}'^2+\mathcal{U}'\frac{\mathcal{V}'}{\mathcal{V}}+\mathcal{U}'\rho'\right)-\tfrac{1}{2}(\Phi'^2+\rho'^2)\tanh^2\Gamma\right.$$
$$\left.+\tanh\Gamma\left(2\Gamma''+2\Gamma'\frac{\mathcal{V}'}{\mathcal{V}}+2\Gamma'\rho'+\tfrac{3}{2}\mathcal{U}'\Gamma'\right)\right], \tag{A5}$$

$$\frac{1}{\sin\theta}R_{[23]}=-\tfrac{1}{2}\mathcal{V}[\mathcal{V}e^{\mathcal{U}+\rho}(\Phi'\sin\Phi+\rho'\cos\Phi)]'$$
$$+\tfrac{1}{2}\mathcal{V}^2e^{\mathcal{U}+\rho}[\Phi'(\Phi'\cos\Phi-\rho'\sin\Phi)+\Gamma'\tanh\Gamma(\Phi'\sin\Phi+\rho'\cos\Phi)], \tag{A6}$$

$$R_{[14]}=-e^{-\rho}[(\mathcal{V}e^{\mathcal{U}+\rho}\rho'\tanh\Gamma)'+\tfrac{1}{2}\mathcal{V}e^{\mathcal{U}+\rho}(\rho'^2+\Phi'^2)\tanh\Gamma]. \tag{A7}$$

APPENDIX B: OSCILLATIONS OF THE MAGNETIC CHARGE DENSITY

By using the results (8.8) and (8.12) in Eqs. (4.13)–(4.15) we obtain, for the points of zero magnetic charge density, the equations

$$[(e^{2\rho}-l_0^4)^{-3/4}e^\rho e^{\mathcal{U}}]'=\tfrac{1}{4}\rho'[\kappa^2(e^{2\rho}-l_0^4)^{1/4}-\kappa^2e^{-\rho}(e^{2\rho}-l_0^4)^{3/4}-2e^{-\rho}(e^{2\rho}-l_0^4)^{1/4}], \tag{B1}$$

$$[(e^{2\rho}-l_0^4)^{-1/4}e^\rho e^{\mathcal{U}}]'=\tfrac{1}{4}\rho'(e^{2\rho}-l_0^4)^{1/4}\kappa^2[e^\rho-(e^{2\rho}-l_0^4)^{1/2}-l_0^4e^{-\rho}]+\tfrac{1}{2}\rho'(e^{2\rho}-l_0^4)^{3/4}e^{-\rho}, \tag{B2}$$

$$[(e^{2\rho}-l_0^4)^{-1/4}(\mathcal{U}'/\rho')e^\rho e^{\mathcal{U}}]'=\tfrac{1}{4}\rho'\kappa^2(e^{2\rho}-l_0^4)^{1/4}[e^\rho-(e^{2\rho}-l_0^4)^{1/2}]. \tag{B3}$$

On dividing by ρ' and using the substitution

$$e^\rho=l_0^2(1+t^2)^{1/2},$$

the above equations can be replaced by

$$\frac{d}{dt}\left(\frac{(1+t^2)^{1/2}}{t^{3/2}}e^{\mathcal{U}}\right)=\frac{1}{4}\frac{t^{3/2}}{1+t^2}\left[\kappa^2l_0^2\left(1-\frac{t}{(1+t^2)^{1/2}}\right)-\frac{2}{(1+t^2)^{1/2}}\right], \tag{B4}$$

$$\frac{d}{dt}\left(\frac{(1+t^2)^{1/2}}{t^{1/2}}e^{\mathcal{U}}\right)=-\frac{1}{4}\frac{t^{5/2}}{1+t^2}\left[\kappa^2l_0^2\left(1-\frac{t}{(1+t^2)^{1/2}}\right)-\frac{2}{(1+t^2)^{1/2}}\right], \tag{B5}$$

$$\frac{d}{dt}\left(\frac{(1+t^2)^{3/2}}{t^{3/2}}\frac{de^{\mathcal{U}}}{dt}\right)=\tfrac{1}{4}\kappa^2l_0^2\frac{t^{3/2}}{1+t^2}[(1+t^2)^{1/2}-t]. \tag{B6}$$

The integrals for the integration of these equations are given below:

(i) $\displaystyle\int\frac{t^{3/2}}{1+t^2}dt=2t^{1/2}+\frac{1}{\sqrt{2}}\left[-\tan^{-1}\left(\frac{(2t)^{1/2}}{1-t}\right)+\tanh^{-1}\left(\frac{(2t)^{1/2}}{1+t}\right)\right],$

(ii) $\displaystyle\int\frac{t^{5/2}}{(1+t^2)^{3/2}}dt=-\frac{t^{3/2}}{(1+t^2)^{1/2}}+\frac{3}{2}\int\frac{t^{1/2}}{(1+t^2)^{1/2}}dt,$

where

$$\int\frac{t^{1/2}}{(1+t^2)^{1/2}}dt=\mathcal{K}\left(\alpha,\frac{1}{\sqrt{2}}\right)-2\mathcal{E}\left(\alpha,\frac{1}{\sqrt{2}}\right)+2\frac{t^{1/2}(1+t^2)^{1/2}}{1+t}.$$

The first- and second-kind elliptic integrals are defined by

$$\mathcal{K}\left(\alpha,\frac{1}{\sqrt{2}}\right)=\int^\alpha\frac{d\gamma}{(1-\frac{1}{2}\sin^2\gamma)^{1/2}},$$

$$\mathcal{E}\left(\alpha,\frac{1}{\sqrt{2}}\right)=\int^\alpha(1-\tfrac{1}{2}\sin^2\gamma)^{1/2}d\gamma,$$

respectively, where

$$\alpha=\cos^{-1}\left(\frac{1-t}{1+t}\right),$$

$$\frac{d\mathcal{K}}{dt}=\frac{1}{t^{1/2}(1+t^2)^{1/2}}.$$

The remaining integrals are

(iii) $\int \frac{t^{3/2}}{(1+t^2)^{3/2}}dt = -\frac{t^{1/2}}{(1+t^2)^{1/2}} + \frac{1}{2}\mathcal{K}\left(\alpha, \frac{1}{\sqrt{2}}\right)$,

(iv) $\int \frac{t^{3/2}}{(1+t^2)^{1/2}}dt = \frac{2}{3}\left[t^{1/2}(1+t^2)^{1/2} - \frac{1}{2}\mathcal{K}\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$,

(v) $\int \frac{t^{5/2}}{t^2+1}dt = \frac{2}{3}t^{3/2} - \frac{1}{\sqrt{2}}\left[\tanh^{-1}\left(\frac{(2t)^{1/2}}{1+t}\right) + \tan^{-1}\left(\frac{(2t)^{1/2}}{1-t}\right)\right]$,

(vi) $\int \frac{t^{7/2}}{(1+t^2)^{3/2}}dt = -\frac{t^{5/2}}{(1+t^2)^{1/2}} + \frac{5}{3}\left[t^{1/2}(1+t^2)^{1/2} - \frac{1}{2}\mathcal{K}\left(\alpha, \frac{1}{\sqrt{2}}\right)\right]$.

For t large compared to 1 the elliptic functions $\mathcal{K}$ and $\mathcal{E}$ tend to $-2/\sqrt{t}$.

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[4]The Einstein and Schrödinger versions of the generalized theory of gravitation, because of the absence of a correspondence principle in them, do not yield Eqs. (2.57). In fact the Einstein and Schrödinger (Ref. 10) theories are obtained from the present one by setting $r_0 = \infty$. Hence these theories cannot yield Lorentz's equations of motion. For the above-mentioned theories see A. Einstein, Can. J. Math. 2, 120 (1950); B. Kaufman, Helv. Phys. Acta Supp. 4, 227 (1956); A. Einstein and B. Kaufman, Ann. Math. 62, 128 (1955); A. Papapetrou, Proc. R. Irish Acad. A LII, 69 (1948).

[5]The presence of a short-range neutral charge density $\mathcal{J}_0^4$ may be thought of as the classical version of the vacuum polarization in quantum electrodynamics.

[6]In this theory the correspondence with general relativity plus Maxwell's equations is based on setting a physical constant, such as the magnetic charge $\mathfrak{g}$, equal to zero, while in quantum theory correspondence with classical mechanics is obtained by setting $\hbar = 0$. However, if a relation between the $\mathfrak{g}$ of this theory and $\hbar$ can be established then in this theory also the correspondence principle can be satisfied by setting $\hbar = 0$ everywhere.

[7]The nonconservation of the neutral charge density for $\mathfrak{g} = 0$ can be compared to the divergence of the vacuum polarization in quantum electrodynamics.

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