PHYSICAL REVIEW D VOLUME 9, NUMBER 10 15 MAY 1974

Solution of radiation-reaction problem for the uniform magnetic field

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(Received 12 February 1973; revised manuscript received 2 August 1973)

An approximate solution to the classical radiation reaction for the motion of a Dirac classical particle in a uniform magnetic field is presented. The approximation involves restrictions on the magnitude of the magnetic field and on the energy associated with motion transverse and parallel to the field. Within these restrictions, the solution is valid over all proper times.

I. INTRODUCTION

The solution to the equation of motion of a Dirac classical particle in a uniform electric field has been known analytically ever since Dirac[1] did his original work on the radiation-reaction problem. No analytic solution to the complete set of differential equations of motion in a uniform magnetic field has been found, although various approximate solutions exist in the literature. Sen Gupta[2] has derived an analytic solution to the motion parallel to the magnetic field line. But, when solving for the transverse motion, he assumed a system in which the particle was at rest parallel to the field line. The solution given below yields the same results as Sen Gupta for motion in the plane orthogonal to the field line. Herrera,[3] apparently independently of Sen Gupta, obtained another approximate solution to the equation of motion on a plane perpendicular to the direction of the uniform magnetic field, but the solution is not valid for all proper time. The solution presented here is more general than that given by Sen Gupta or Herrera since it treats the motion along and transverse to the field lines together. Shen[4] considered effects on the spectral distribution of synchrotron radiation due to the presence of intense magnetic fields, including strong radiation damping and quantum effects. However, Shen's results do not hold in the nonrelativistic limit. Shen suggested evaluating the invariant radiated power in the rest frame, a procedure used in this work, but did so to a different approximation. Jaffe[5] considered the power in the same approximation as presented here, but a different form resulted because he assumed a constant pitch angle—the angle between the instantaneous velocity and the direction of the magnetic field. The Jaffe solution for the energy decays inversely with time for large time rather than exponentially given in the results below; this solution is valid only as long as the exponential can be expanded in a linear approximation. Mitchell, Chirivella, and Lingerfelt[6] developed a solution to the radiation-reaction problem in the presence of both an electric and magnetic field by substituting an approximate form for the first derivative of the four-velocity into the Lorentz-Dirac equation. The four-acceleration is then a function of the uniform fields and the four-velocity only. The differential equations are separable in terms of the square of the three-velocity transverse to the fields and the three-velocity parallel to the fields. The limitations of the approximation are not examined in detail. The result in the case of the absence of an electric field agrees with the results of this work.

II. EQUATIONS OF MOTION IN A UNIFORM MAGNETIC FIELD

The general covariant equation of motion of a Dirac particle (charge e and mass m) in an electromagnetic field with radiative damping was first derived by Dirac[1]:

$$\dot{u}_i = \frac{e}{mc} F_{ik} u^k + \frac{2e^2}{3mc^3}\left(\ddot{u}_i + \frac{1}{c^2} u_i \dot{u}_J \dot{u}^J\right) \quad (i=1,2,3,4)\,, \quad (1)$$

where the dot represents differentiation with respect to proper time τ, F_{ik} (called the Maxwell field tensor) contains the components of the electric and magnetic fields, and c is the speed of light. A repeated index means an implied sum, as in the usual Einstein summation convention. The components of u_i are

$$u_i = \gamma(v_1, v_2, v_3, c) \quad (2)$$

and

$$u^J u_J = c^2\,, \quad (3)$$

where γ is a total energy normalized to the rest energy, and v_1, v_2, and v_3 are the usual components of the three-velocity v in the x, y, and z directions, respectively.

For a uniform magnetic field in the z direction, H, Eq. (1) can be written as the following four coupled, nonlinear, second-order, ordinary differential equations in the four-velocity, u_i:

$$\dot{u}_1 = \omega_0 u_2 + \tau_0 \left(\ddot{u}_1 + \frac{1}{c^2} u_1 \dot{u}_J \dot{u}^J\right) , \tag{4}$$

$$\dot{u}_2 = -\omega_0 u_1 + \tau_0 \left(\ddot{u}_2 + \frac{1}{c^2} u_2 \dot{u}_J \dot{u}^J\right) , \tag{5}$$

$$\dot{u}_3 = \tau_0 \left(\ddot{u}_3 + \frac{1}{c^2} u_3 \dot{u}_J \dot{u}^J\right) , \tag{6}$$

$$\dot{u}_4 = \tau_0 \left(\ddot{u}_4 + \frac{1}{c^2} u_4 \dot{u}_J \dot{u}^J\right) , \tag{7}$$

where $\omega_0 = eH/mc$ is the Larmor frequency, and $\tau_0 = 2e^2/3mc^3$ is the light travel time across the classical electron radius.

However, no more than three of the four differential equations are independent, by virtue of Eq. (3). Similarly, it can be shown that the z component of the three-velocity, v_3, is a constant,[2] introducing another constraint on the system and reducing the number of independent differential equations to two. The two independent differential equations can be written

$$\dot{u} = -i\omega_0 u + \tau_0 \left(\ddot{u} + \frac{1}{c^2} u \dot{u}_J \dot{u}^J\right) \tag{8}$$

and

$$\dot{u}_4 = \tau_0 \left(\ddot{u}_4 + \frac{1}{c^2} u_4 \dot{u}_J \dot{u}^J\right) , \tag{7}$$

where

$$u \equiv u_1 + i u_2 \tag{9}$$

and $\dot{u}_J \dot{u}^J$ is a function of the remaining independent coordinates. The magnitude of u can be computed from the solution to (7) and the constraints given above, so that Eq. (8) contains only one independent variable. Finally, the Lorentz-invariant power $\dot{u}_J \dot{u}^J$ can be evaluated to zero order in τ_0, resulting in

$$\dot{u}_J \dot{u}^J = -\omega_0^2 \left[\left(1 - \frac{v_3^2}{c^2}\right) u_4^2 - c^2\right] . \tag{10}$$

The use of this approximation introduces restrictions on the limits of validity of the solution. These limits and details of the derivation are given in the Appendix. This approximate form of the Lorentz-invariant power decouples Eq. (7) from Eq. (8), so that

$$\dot{\gamma} = \tau_0 \left\{ \ddot{\gamma} - \gamma \omega_0^2 \left[\left(1 - \frac{v_3^2}{c^2}\right)\gamma^2 - 1\right]\right\} , \tag{11}$$

where γc has been substituted for u_4. This can be rewritten as

$$\ddot{\gamma} - \frac{\dot{\gamma}}{\tau_0} - \omega_0^2 \left(1 - \frac{v_3^2}{c^2}\right)\gamma^3 + \omega_0^2 \gamma = 0 . \tag{12}$$

The solution of this equation gives all the essential features of the motion.

III. THE SOLUTION OF THE ENERGY EQUATION

A. Linear approximation

Let γ be written as

$$\gamma = \gamma_3 + w ,$$

where $\gamma_3^2 \equiv (1 - v_3^2/c^2)^{-1}$. Then the derivatives can be written as

$$\dot{\gamma} = \dot{w}$$

and

$$\ddot{\gamma} = \ddot{w} .$$

Substitution of w and its derivatives into Eq. (12) gives

$$\ddot{w} - \frac{\dot{w}}{\tau_0} - \omega_0^2 w \left(2 + 3\frac{w}{\gamma_3} + \frac{w^2}{\gamma_3^2}\right) = 0 . \tag{13}$$

If we assume that

$$\left|\frac{w}{\gamma_3}\right| \ll 1 , \tag{14}$$

Eq. (13) becomes the following linear, homogeneous, second-order differential equation with constant coefficients:

$$\ddot{w} - \frac{\dot{w}}{\tau_0} - 2\omega_0^2 w = 0 . \tag{15}$$

The solution gives

$$w = A \exp\left\{\frac{\tau}{2\tau_0}\left[1 + (1 + 8\omega_0^2\tau_0^2)^{1/2}\right]\right\} + B \exp\left\{\frac{\tau}{2\tau_0}\left[1 - (1 + 8\omega_0^2\tau_0^2)^{1/2}\right]\right\} . \tag{16}$$

But for magnetic fields less than 10^{15} gauss, this can be written as

$$w = A e^{\tau/\tau_0} + B e^{-2\omega_0^2 \tau_0 \tau} . \tag{17}$$

So the solution for the energy is

$$\gamma = \gamma_3 + A e^{\tau/\tau_0} + (\gamma_0 - \gamma_3 - A) e^{-2\omega_0^2\tau_0\tau} , \tag{18}$$

where γ_0 is the value of γ when $\tau = 0$. Equation (18) is exact for $H = 0$, so that in this case the solution becomes

$$\gamma = \gamma_0 - A + A e^{\tau/\tau_0} . \tag{19}$$

This solution is valid for all τ. In this simple case a "runaway solution" is present completely analogous to that found in the uniform-electric-field case. One is tempted to identify *a priori* a solution of the form e^{τ/τ_0} as that solution. Such a solution has the intrinsic property that the energy gain or loss of the particle is independent of the applied fields. This is physically unacceptable. In particular, in Eq. (19) this reasoning leads to the result that $A=0$; then $\gamma=\gamma_0$ for all τ. A function of this type is present in Eq. (18), where the expansion of the exponent in Eq. (16) is valid for all magnetic fields less than 10^{15} gauss. By application of the same argument as above, it is assumed that $A=0$ for the general solution, where $H\neq 0$. However, condition (14) is not satisfied for all τ if A is finite.

Therefore, we cannot take the limit without examining the choice of A.[7] Let us choose an arbitrary value of $A>0$ in Eq. (18), for example, $A=A_1$. Let us then follow the motion until $\gamma(\tau_1)\geq\gamma_0$, where $\gamma(\tau_1)$ is the value of the energy at some future time $\tau_1>0$. Such a situation is physically impossible because the field does no work on the particle, yet the approximate solution to differential equation (4) is still valid since condition (14) is fulfilled (the limits of validity of the approximation are examined below in Sec. III B). We have obviously chosen too large a value of A. Therefore, let us choose a new value, $A=A_2<A_1$. If we follow the motion again until $\gamma(\tau_2)\geq\gamma_0$, we would find $\tau_2>\tau_1$ and $\gamma(\tau_2)=\gamma(\tau_1)$. By repeating the same argument, we obtain a physically meaningless result, so that A is still too large. In this way, we obtain a sequence $\{A(\tau_n)\}$ in which the last term is arbitrarily small, and a corresponding sequence $\{\tau_n\}$ in which the last term is arbitrarily large. The entire argument can be repeated for $A<0$ in Eq. (18) except that at some proper time $\tau>0$ for any nonzero A, $\gamma(\tau_n)\leq\gamma_3$. The result is physically impossible since the minimum energy of the particles is that associated with the motion along the z axis. Finally, we obtain

$$\gamma=\gamma_3+(\gamma_0-\gamma_3)e^{-2\omega_0^2\tau_0\tau}\ . \tag{20}$$

Using Eq. (20), we obtain the relation between laboratory time and proper time through a simple integration:

$$t=\gamma_3\tau+\frac{\gamma_0-\gamma_3}{2\tau_0\omega_0^2}[1-e^{-2\omega_0^2\tau_0\tau}]\ . \tag{21}$$

B. Limits imposed by the approximation $|w/\gamma_3|\ll 1$

By substitution

$$\frac{w}{\gamma_3}=\frac{\gamma-\gamma_3}{\gamma_3}\ . \tag{22}$$

Since the energy decreases for increasing proper time, the approximation at $\tau=0$ can be written as

$$1\gg\frac{\gamma_0-\gamma_3}{\gamma_3}=\left(1+\frac{v_{01}^2+v_{02}^2}{c^2-v_3^2-v_{01}^2-v_{02}^2}\right)^{1/2}-1\ , \tag{23}$$

where v_{01} and v_{02} are the initial velocities in the x and y directions, respectively. After a straightforward manipulation,

$$\gamma_3\left(\frac{v_{01}^2+v_{02}^2}{c^2}\right)^{1/2}\ll 1\ . \tag{24}$$

If condition (24) is met, we are assured of condition (14) for all τ. Thus, we have the condition that the product of the normalized energy in the direction of the field and the square root of the normalized energy in oscillations perpendicular to the field (normalized to the rest energy) cannot exceed unity. This severely limits the applicability of Eq. (20), but the solution leads to one other particularly useful relation,

$$\left|\frac{\tau_0\ddot{\gamma}}{\dot{\gamma}}\right|=2(\tau_0\omega_0)^2\ .$$

This quantity is much smaller than unity for magnetic fields less than 10^{15} gauss, so that

$$\left|\frac{\tau_0\ddot{\gamma}}{\dot{\gamma}}\right|\ll 1\ . \tag{25}$$

C. A more general solution

If

$$|\ddot{\gamma}|\ll\left|\frac{\dot{\gamma}}{\tau_0}\right|\ , \tag{26}$$

as in the limited solution in Sec. III A above, we can write for Eq. (12)

$$-\frac{\dot{\gamma}}{\tau_0}-\omega_0^2\frac{\gamma^3}{\gamma_3^2}+\omega_0^2\gamma=0\ . \tag{27}$$

This approximation will be examined at the conclusion of the analysis. Equation (27) is a nonlinear, first-order ordinary differential equation whose variables are separable. Its solution is

$$\gamma=\frac{\gamma_3}{\{1-[(\gamma_0^2-\gamma_3^2)/\gamma_0^2]e^{-2\omega_0^2\tau_0\tau}\}^{1/2}}\ . \tag{28}$$

Clearly, under condition (23), Eq. (28) reduces to Eq. (20). By ignoring the second derivative, only the exponentially increasing part of the solution has disappeared, and no essential properties of the motion have been lost. Since $\gamma=dt/d\tau$, a simple integration gives

$$t=\frac{\gamma_3}{\gamma_0^2\tau_0}\ln\frac{\gamma_0+[\gamma_0^2-(\gamma_0^2-\gamma_3^2)e^{-2\omega_0^2\tau_0\tau}]^{1/2}}{(\gamma_0+\gamma_3)e^{-\omega_0^2\tau_0\tau}}\ , \tag{29}$$

where it has been assumed that $t=0$ when $\tau=0$. An algebraically involved but calculationally simple rearrangement gives

$$\tau=\frac{1}{\omega_0\tau_0^2}\ln\left(\cosh\frac{\omega_0^2\tau_0 t}{\gamma_3}+\frac{\gamma_3}{\gamma_0}\sinh\frac{\omega_0^2\tau_0 t}{\gamma_3}\right)\ . \tag{30}$$

This can be substituted into Eq. (28) to give the complete solution to the total energy as a function of laboratory time.

D. Limits of the approximation $|\ddot{\gamma}\tau_0/\dot{\gamma}|$

Using Eq. (28), the ratio $|\ddot{\gamma}\tau_0/\dot{\gamma}|$ is easily evaluated to be

$$\left|\frac{\ddot{\gamma}\tau_0}{\dot{\gamma}}\right| = 2\omega_0{}^2\tau_0{}^2\frac{\gamma_0{}^2+\frac{1}{2}(\gamma_0{}^2-\gamma_3{}^2)e^{-2\omega_0{}^2\tau_0\tau}}{\gamma_0{}^2-(\gamma_0{}^2-\gamma_3{}^2)e^{-2\omega_0{}^2\tau_0\tau}} . \quad (31)$$

This ratio has its maximum value at $\tau=0$. This value is

$$\left|\frac{\ddot{\gamma}\tau_0}{\dot{\gamma}}\right| = 2\omega_0{}^2\tau_0{}^2\frac{\frac{3}{2}\gamma_0{}^2-\frac{1}{2}\gamma_3{}^2}{\gamma_3{}^2} . \quad (32)$$

Then, the approximation is valid if

$$2\omega_0{}^2\tau_0{}^2\frac{\frac{3}{2}\gamma_0{}^2-\frac{1}{2}\gamma_3{}^2}{\gamma_3{}^2} \ll 1 . \quad (33)$$

For particles in circular motion only, Eq. (28) would accurately describe the decay of the motion if

$$\gamma_0 H \ll 10^{15} , \quad (34)$$

where H is the magnetic field in gauss. If the fields surrounding pulsars are of the order of 10^{12} gauss, the limit becomes

$$\gamma_0 \ll 10^3\gamma_3 . \quad (35)$$

For the motion in the vicinity of pulsars, this solution is applicable as long as the total energy of the particles does not greatly exceed the energy the particles would have if all their motion were along the field line, the limit being given by condition (35).

In general, the limits above are more stringent than that imposed by condition (A17) governing the use of the approximate form of the Lorentz-invariant power. This guarantees that solutions consistent with condition (33) are consistent with all approximations imposed on the differential equations used to derive the solutions. Since the restrictions are most stringent for $\tau=0$, solutions consistent with condition (33) [or condition (A11)] for $\tau=0$ are valid for any other proper time.

IV. THE TRANSVERSE MOTION

The transverse motion—motion orthogonal to the field—is given by the solution to

$$\dot{u} = -i\omega_0 u + \tau_0\left(\ddot{u} + \frac{1}{c^2}u\dot{u}_J\dot{u}^J\right) . \quad (36)$$

This equation can be solved by substituting Eq. (10) for $\dot{u}_J\dot{u}^J$ to get

$$\dot{u} = -i\omega_0 u + \tau_0\left\{\ddot{u} - \frac{u\omega_0{}^2}{c^2}\left[\left(1-\frac{v_3{}^2}{c^2}\right)u_4{}^2 - c^2\right]\right\} . \quad (37)$$

But since $u_4=\gamma c$ and $\gamma_3{}^2=(1-v_3{}^2/c^2)^{-1}$, Eq. (37) becomes

$$\ddot{u} - \frac{\dot{u}}{\tau_0} - \left[i\frac{\omega_0}{\tau_0} + \frac{\omega_0{}^2}{\gamma_3{}^2}(\gamma^2-\gamma_3{}^2)\right]u = 0 . \quad (38)$$

γ is a known function of the proper time only, given by Eq. (28). Equation (38) is then a linear, homogeneous ordinary differential equation of second order. Assume a solution of the form

$$u = |u|e^{-i\alpha\tau} , \quad (39)$$

where arbitrary phase factors have been suppressed, and substitute this into Eq. (38). The real and imaginary parts give

$$2\alpha|\dot{u}| + \frac{\omega_0}{\tau_0}|u| - \frac{\alpha}{\tau_0}|u| = 0 , \quad (40)$$

$$|\ddot{u}| - \frac{|\dot{u}|}{\tau_0} - \alpha^2|u| - \frac{\omega_0{}^2}{\gamma_3{}^2}(\gamma^2-\gamma_3{}^2)|u| = 0 , \quad (41)$$

where $|\dot{u}|$ or $|\ddot{u}|$ represent first or second derivatives with respect to time of the magnitude of u rather than the magnitude of the derivatives.

To eliminate the second derivative in Eq. (41), differentiate Eq. (40) to obtain

$$2\alpha|\ddot{u}| + \frac{\omega_0}{\tau_0}|\dot{u}| - \frac{\alpha}{\tau_0}|\dot{u}| = 0 .$$

Solve this equation for $|\ddot{u}|$ and substitute the result into Eq. (41). This gives

$$-\left(\frac{\omega_0-\alpha}{2\alpha}+1\right)\frac{|\dot{u}|}{\tau_0} - \alpha^2|u| - \frac{\omega_0{}^2}{\gamma_3{}^2}(\gamma^2-\gamma_3{}^2)|u| = 0 . \quad (42)$$

Equation (42) can be put into a more useful form by noting that

$$|u| = \frac{(\gamma^2-\gamma_3{}^2)^{1/2}c}{\gamma_3} , \quad (43)$$

which follows from the definition of γ and the constancy of γ_3. This confirms that statement made earlier that Eq. (8) contains only one independent variable if the solution for the energy is known. This independent variable is the phase factor for the proper time in Eq. (39), and represents the reciprocal of the time constant for the decay of the transverse velocity. Then Eq. (43) becomes

$$\left(\frac{\omega_0+\alpha}{2\alpha}\right)\frac{|\dot{u}|}{\tau_0} + \alpha^2|u| + \frac{\omega_0{}^2}{c^2}|u|^3 = 0 . \quad (44)$$

Equation (44) is a nonlinear, first-order ordinary differential equation whose variables are separable. Its solution is

$$|u|^2 = \frac{\alpha^2(\gamma_0^2 - \gamma_3^2)c^2 e^{-2\alpha^2\tau_1\tau}}{\alpha^2\gamma_3^2 + \omega_0^2(\gamma_0^2 - \gamma_3^2) - \omega_0^2(\gamma_0^2 - \gamma_3^2)e^{-2\alpha^2\tau_1\tau}} , \quad (45)$$

where $\tau_0 = \tau_1[(\omega_0 + \alpha)/2\alpha]$, and $|u|_0 = c(\gamma_0^2 - \gamma_3^2)^{1/2}/\gamma_3$ is the initial transverse velocity at $\tau = 0$. By substituting the solution to γ given by Eq. (28) into Eq. (43), an independent expression is obtained for $|u|^2$:

$$|u|^2 = \frac{(\gamma_0^2 - \gamma_3^2)c^2 e^{-2\omega_0^2\tau_0\tau}}{\gamma_0^2 - (\gamma_0^2 - \gamma_3^2)e^{-2\omega_0^2\tau_0\tau}} . \quad (46)$$

However, since Eqs. (45) and (46) represent the same solution, α is determined. The result is

$$\alpha = \omega_0 . \quad (47)$$

Finally, the result for the transverse motion is

$$u = \frac{c(\gamma^2 - \gamma_3^2)^{1/2}}{\gamma_3} e^{-i(\omega_0\tau + \beta)} , \quad (48)$$

where β is some constant phase factor determined by the orientation of the transverse coordinate system with respect to the initial direction of the transverse velocity of the particle. For large τ the motion described by Eq. (48) is that of an underdamped harmonic oscillator in which the natural frequency of the system is ω_0.

V. SUMMARY AND CONCLUSIONS

We have presented here the solution to the equation of motion of Dirac classical particles, including radiation reaction, in a uniform magnetic field. The equations are complete solutions within the limitations of the approximations used to derive them. The solutions are complete in the sense that they contain the appropriate number of arbitrary constants. One of these constants is shown to be zero. This eliminated a possible solution similar to the "runaway solution" obtained in the uniform-electric-field case. Then the initial acceleration of the particle need not be specified. The other constant is determined by the initial velocity of the particle. The solution is valid for all τ, and the motion can be followed over the entire range of proper time. This can be seen from Eq. (31) and Eq. (A15), where the approximation actually improves as the proper time increases. The most stringent test of the approximation is for $\tau = 0$, and the result is a restriction on the choice of the magnitude of the magnetic field. For a given magnetic field, Eq. (33) gives additional restrictions on the choice of the total initial energy of the particle and the initial velocity along the field line.

The three-velocity in the direction of the field remains constant, while the three-velocity transverse to the field decays nearly exponentially as the particle oscillates harmonically around the field line with the Larmor frequency—that is, the transverse particle motion is nearly that of a damped harmonic oscillator. The solution of the equation of motion for the component of the four-velocity orthogonal to the field line is given in Eq. (48). The solution for the total energy given in Eq. (28) shows that the energy of the particle decays exponentially until all that remains is the energy associated with the motion along the field line.

ACKNOWLEDGMENTS

The author wishes to thank Professor W. J. Cocke of Steward Observatory, Professor J. M. Cohen of the University of Pennsylvania, and Dr. K. Wolf of Argonne National Laboratory for helpful discussions and comments. The author also wishes to thank Dr. A. V. Ferris-Prabhu for a comment.

APPENDIX: DERIVATION OF THE APPROXIMATE FORM FOR THE INVARIANT RADIATED POWER IN THE REST FRAME AND RESULTING LIMITS ON THE RANGE OF VALIDITY OF THE SOLUTION

The evaluation of the Lorentz-invariant power $\dot{u}_J\dot{u}^J$ is necessary to solve the radiation-reaction problem. As suggested by Shen,[4] a useful approximate form can most easily be derived in the rest frame of the charged particle. We have used the superscript R for quantities evaluated in this frame. Equation (1) can then be written

$$(\dot{u}_\alpha)^R = \frac{e}{m} E_\alpha^R + \tau_0(\ddot{u}_\alpha)^R \quad (\alpha = 1, 2, 3) , \quad (A1)$$

where E_α^R is the electric field seen by the charged particle in its rest frame. The inner product is given approximately by

$$\sum_{\alpha=1}^{3} (\dot{u}_\alpha)^R(\dot{u}^\alpha)^R = -\left(\frac{e}{m}\right)^2 \sum_{\alpha=1}^{3} (E_\alpha^R)^2 , \quad (A2)$$

where terms of the order τ_0 or higher have been ignored. However, from the orthogonality of u and $\dot{u}$, $(\dot{u}_4)^R = 0$. Then

$$(\dot{u}_J)^R(\dot{u}^J)^R = \sum_{\alpha=1}^{3} (\dot{u}_\alpha)^R(\dot{u}^\alpha)^R . \quad (A3)$$

Since

$$\dot{u}_J\dot{u}^J = (\dot{u}_J)^R(\dot{u}^J)^R , \quad (A4)$$

then

$$\dot{u}_J\dot{u}^J = -\left(\frac{e}{m}\right)^2 \sum_{\alpha=1}^{3} (E_\alpha^R)^2 . \quad (A5)$$

The electric field in the rest frame can be determined by a Lorentz transformation from the rest frame (called the original frame) to the laboratory frame (called the transformed frame). The transformation velocity is the negative of the particle three-velocity v. The components of the electric field are then

$$E_{\perp}^{R} = \pm \frac{H}{c^2} u_4 v \sin\Theta \text{ and } E_{\parallel}^{R} = 0 , \tag{A6}$$

where $\parallel$ and $\perp$ mean parallel and perpendicular to the transformation velocity, Θ is the angle between that velocity and the direction of the field, and the sign is determined by the choice of the coordinate axes relative to the magnetic field and velocity vectors. Finally, since the velocity along the field is a constant, Θ can be represented by

$$\cos\Theta = \frac{v_3}{v} , \tag{A7}$$

which can easily be written as

$$\sin\Theta = \left[\frac{(1 - v_3^2/c^2)(u_4^2/c^2) - 1}{u_4^2/c^2 - 1}\right]^{1/2} . \tag{A8}$$

Then

$$\sum_{\alpha=1}^{3} (E_{\alpha}^{R})^2 = H^2\left[\left(1 - \frac{v_3^2}{c^2}\right)\frac{u_4^2}{c^2} - 1\right] . \tag{A9}$$

Finally

$$\dot{u}_J \dot{u}^J = -\omega_0^2\left[\left(1 - \frac{v_3^2}{c^2}\right) u_4^2 - c^2\right] . \tag{A10}$$

The use of this approximation imposes limits on the validity of the solution. The relative difference between the approximate form of the Lorentz-invariant power[8] given by Eq. (A10) (called L_A) and that generated by the solution itself (called L_S) can be made small by certain restrictions on the physical parameters. We demand that

$$|L_S - L_A| \ll |L_A| . \tag{A11}$$

L_S can be derived from Eq. (48), the definition of u_4, and the condition that v_3 is a constant. The result is

$$L_S = -\frac{c^2\omega_0^2}{\gamma_3^2}(\gamma^2 - \gamma_3^2) - \frac{c^2}{\gamma_3^2}\frac{\gamma^2\dot{\gamma}^2}{\gamma^2 - \gamma_3^2} + \dot{\gamma}^2(c^2 - v_3^2) , \tag{A12}$$

where $\gamma_3^2 \equiv (1 - v_3^2/c^2)^{-1}$ and $\gamma \equiv u_4/c$. When Eqs. (A10) and (A12) are substituted into Eq. (A11), the result is

$$\frac{c^2}{\gamma_3^2}\frac{\gamma^2\dot{\gamma}^2}{\gamma^2 - \gamma_3^2} - \dot{\gamma}^2(c^2 - v_3^2) \ll \frac{c^2\omega_0^2}{\gamma_3^2}(\gamma^2 - \gamma_3^2) . \tag{A13}$$

Condition (A13) can be written after algebraic manipulation as

$$\left(\frac{\dot{\gamma}}{\gamma_3}\right)^2 \ll \omega_0^2\left(\frac{\gamma^2 - \gamma_3^2}{\gamma_3^2}\right)^2 . \tag{A14}$$

Equation (28) can be used to relate condition (A14) to the physical parameters of the problem. We get

$$\omega_0^2\tau_0^2 \ll \frac{\gamma_0^2 - (\gamma_0^2 - \gamma_3^2)e^{-2\omega_0^2\tau_0\tau}}{\gamma_0^2} \tag{A15}$$

for any τ. Condition (A15) is most restrictive for $\tau = 0$ since

$$\frac{\gamma_3^2}{\gamma_0^2} \leq \frac{\gamma_0^2 - (\gamma_0^2 - \gamma_3^2)e^{-2\omega_0^2\tau_0\tau}}{\gamma_0^2} \tag{A16}$$

for all proper time $\tau \geq 0$. Finally the restrictions on the range of validity of the solution are found from

$$\omega_0^2\tau_0^2\frac{\gamma_0^2}{\gamma_3^2} \ll 1 . \tag{A17}$$

Since

$$\omega_0^2\tau_0^2\frac{\gamma_0^2}{\gamma_3^2} \leq \omega_0^2\tau_0^2\left(\frac{3\gamma_0^2 - \gamma_3^2}{\gamma_3^2}\right) , \tag{A18}$$

we conclude that if condition (33) is satisfied, condition (A17) is guaranteed to be satisfied. Clearly, we could derive Eq. (A10) by ignoring the change in total energy of the particle associated with the radiated power so that the total energy is treated as a constant for periods of time of the order τ_0. Then, the only force on the particle arises from the Lorentz force associated with the velocity transverse to the magnetic field.

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[7]An alternate proof to that given below is to let the interaction constant τ_0 approach zero. Then Eq. (16) is not finite unless $A = 0$. This leads to $w = B$ and $\gamma = \gamma_3 + B$. B represents the portion of the total energy associated with the transverse motion. This argument is consistent with that given by Herrera (Ref. 3) and Sen Gupta (Ref. 2).

[8]The invariant referred to here and throughout this work as the Lorentz-invariant power is proportional to that quantity rather than equal to it.