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Odd-parity stability of a Reissner-Nordström black hole*

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The odd-parity perturbations of a Reissner-Nordström black hole are studied through the use of a variational principle for the perturbation equations. The Hamiltonian for the perturbations is expressed in terms of a natural set of canonical variables and shown to be (weakly) positive-definite. This result is used to prove the nonexistence of unstable normal-mode solutions of the perturbation equations. The two wave equations governing gravitational and electromagnetic perturbations are decoupled by a simple transformation.

I. INTRODUCTION

In this paper we establish the stability of the Reissner-Nordström family of black holes against small, odd-parity perturbations. We first establish the positive-definiteness of the Hamiltonian for the coupled gravitational and electromagnetic perturbations. The assumption of an unstable normal-mode solution of the perturbation equations (obeying suitable boundary conditions) is then shown to lead to a contradiction. The methods used here were developed in two recent papers[1,2] and applied there to Schwarzschild perturbations[1] and to the perturbations of isentropic, perfect-fluid stellar models[2] (for which a stability criterion was derived).

A variational principle for the perturbations is derived by taking the second variation of the appropriate, exact variational integral.[1,3] The perturbation functions are first expanded in Regge-Wheeler tensor harmonics.[4] A canonical transformation is then performed from the original, Regge-Wheeler variables to a new set which is more naturally adapted to the gauge symmetry of the perturbed Einstein-Maxwell equations. In terms of the new variables the Hamiltonian becomes manifestly gauge-invariant and positive-definite as soon as the initial-value constraints are taken into account. Finally, a simple transformation is given which decouples the wave equations for the gravitational and electromagnetic perturbations.

Other studies of the Reissner-Nordström perturbations have recently been made by Zerilli[5] and by Chitre, Price, and Sandberg.[6]

II. PERTURBATION FORMALISM

The variational integral for the perturbations is derived by taking the second variation of the exact expression[7]

$$16\pi I = \int d^4x \Big\{ \pi^{ij} \frac{\partial g_{ij}}{\partial t} + A_i \frac{\partial \mathcal{E}^i}{\partial t} - N[g^{-1/2}(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2) - g^{1/2}R + \tfrac{1}{2}g^{-1/2}g_{ij}(\mathcal{E}^i\mathcal{E}^j + \mathcal{B}^i\mathcal{B}^j)]$$
$$- N_i[-2\pi^{ij}{}_{|j} - g^{ik}\epsilon_{klm}\mathcal{E}^l\mathcal{B}^m] - A_0\mathcal{E}^i{}_{,i} \Big\} , \tag{1}$$

with

$$\mathcal{B}^i = \tfrac{1}{2}\epsilon^{ijk}(A_{k,j} - A_{j,k}) . \tag{2}$$

The notation used here is the same as that of Ref. 7 except that we have absorbed a factor of 2 into the definitions of $\mathcal{E}^i$ and A_μ. After taking the second variation of Eq. (1) we substitute, for the unperturbed metric, the Reissner-Nordström solution

$$ds^2 = -N^2dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) , \tag{3}$$

where

$$N^2 = e^{-2\lambda} = 1 - 2m/r + e^2/r^2 , \tag{4}$$

which, for $|e| \leq m$, represents the exterior of a black hole with charge e and mass m.

The perturbation quantities,

$$\delta g_{ij} \equiv h_{ij},\ \ \delta\pi^{ij} \equiv p^{ij},\ \ \delta A_\mu \equiv A'_\mu ,$$
$$\delta\mathcal{E}^i \equiv \mathcal{E}^{i\prime},\ \ \delta N \equiv N',\ \ \delta N_i \equiv N'_i, \tag{5}$$

are expanded in Regge-Wheeler harmonics,[4] an example of which is

$$(A_i') = A(r,t)\left(0, -\left(\frac{1}{\sin\theta}\right)\frac{\partial Y_{LM}}{\partial\varphi}, \sin\theta\frac{\partial Y_{LM}}{\partial\theta}\right),$$
$$(g^{-1/2}g_{ij}\mathcal{E}^{j\prime}) = \frac{e^{-\lambda}E(r,t)}{L(L+1)} \times\left(0, -\left(\frac{1}{\sin\theta}\right)\frac{\partial Y_{LM}}{\partial\varphi}, \sin\theta\frac{\partial Y_{LM}}{\partial\theta}\right). \tag{6}$$

For the remaining, odd-parity perturbations, we have the two Regge-Wheeler functions $h_1(r,t)$ and $h_2(r,t)$ (which determine h_{ij}), their conjugate momenta $p_1(r,t)$ and $p_2(r,t)$, and the single perturbed shift vector function $h_0(r,t)$. Since all the perturbation equations are independent of the azimuthal harmonic number M,[4] we may set $M=0$ and take all the perturbation functions ($A, E, h_1, h_2, p_1, p_2, h_0$) to be real-valued. Solutions for arbitrary M can be obtained from those with $M=0$ by a rotation.

For $L \geq 2$ both gravitational and electromagnetic waves can occur. In this case it is convenient to perform the canonical transformation to new variables $k_1, k_2, f_1, \pi_1, \pi_2, \pi_f$ defined by

$$\begin{aligned} k_1 &= h_1 + \tfrac{1}{2}h_{2,r} - (1/r)h_2, \\ \pi_1 &= p_1, \\ k_2 &= h_2, \\ \pi_2 &= p_2 + \tfrac{1}{2}p_{1,r} + \left(\frac{1}{r}\right)p_1 - \frac{eL(L+1)}{r^2}A_{,r}, \\ f_1 &= E - L(L+1)e(h_2/r^2)_{,r}, \\ \pi_f &= A. \end{aligned} \tag{7}$$

All of the new variables are gauge-invariant except for k_2, whose conjugate momentum π_2 is the single, odd-parity initial-value constraint (it is a general feature of gravitational perturbation theory that constraints and gauge-dependent variables occur as canonically conjugate pairs). In terms of the new variables the variational integral for perturbations outside the event horizon at $r=r_+=m+(m^2-e^2)^{1/2}$ is given by

$$16\pi I_{\text{pert}} = \int dt\int_{r_+}^{\infty} dr(\pi_1 k_{1,t} + \pi_2 k_{2,t} + \pi_f f_{1,t} - \mathcal{H}), \tag{8}$$

where

$$\begin{aligned} H &\equiv \int_{r_+}^{\infty} dr\,\mathcal{H} \\ &= \int_{r_+}^{\infty} dr[N/L(L+1)](\tfrac{1}{2}e^{\lambda}(\pi_1)^2 + [2r^2e^{-\lambda}/(L-1)(L+2)]\{\pi_2 - (1/2r^2)[r^2\pi_1 - 2eL(L+1)\pi_f]_{,r}\}^2) \\ &\quad + \tfrac{1}{2}L(L+1)\int_{r_+}^{\infty} dr N[L(L+1)(e^{\lambda}/r^2)\pi_f{}^2 + e^{-\lambda}(\pi_{f,r})^2] + \tfrac{1}{2}L(L+1)\int_{r_+}^{\infty} dr Ne^{-\lambda}[f_1/L(L+1) + (2e/r^2)k_1]^2 \\ &\quad + \tfrac{1}{2}L(L+1)\int_{r_+}^{\infty} dr(Ne^{-\lambda}/r^2)(L-1)(L+2)(k_1)^2 - 2\int_{r_+}^{\infty} dr\,h_0\pi_2. \end{aligned} \tag{9}$$

Variation of the shift function h_0 gives the initial-value constraint $\pi_2=0$, which is conserved in time since k_2 is cyclic. Hamilton's equations for k_1, f_1, π_1, and π_f may be combined to yield two coupled second-order equations which have a gauge-invariant significance. For the quantities π_f and

$$\pi_g \equiv r\pi_1 - (2e/r)L(L+1)\pi_f \tag{10}$$

we find

$$-\pi_{f,tt} + \pi_{f,r^*r^*} - (N^2/r^2)[L(L+1) + (4e^2/r^2)]\pi_f = \{2N^2e/[r^3L(L+1)]\}\pi_g \tag{11}$$

and

$$-\pi_{g,tt} + \pi_{g,r^*r^*} - (N^2/r^2)[L(L+1) - (6m/r) + (4e^2/r^2)]\pi_g = [2N^2e/r^3]L(L+1)(L-1)(L+2)\pi_f, \tag{12}$$

where r^* is defined by

$$\frac{dr}{dr^*} = N^2. \tag{13}$$

Owing to the vanishing of N^2/r^2 as $r\to\infty$ and as $r\to r_+$ ($r^*\to-\infty$), the normal-mode solutions [with time dependence $\exp(i\omega t)$] of Eqs. (11) and (12) have the asymptotic behavior

$$\pi_{f,g} = C_{f,g}\exp(i\omega t - i\omega r^*) \quad \text{as } r\to\infty, \tag{14}$$

$$\pi_{f,g} = D_{f,g}\exp(i\omega t + i\omega r^*) \quad \text{as } r\to r_+ \ (r^*\to-\infty), \tag{15}$$

where $C_{f,g}$ and $D_{f,g}$ are certain constants and where we have imposed the boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves at the event horizon.

The conservation equation obeyed by the Hamiltonian H is easily found to be

$$H_{,t}=\{L(L+1)\pi_{f,t}\,\pi_{f,r^*}+[(L-1)(L+2)L(L+1)]^{-1}(1/r)\pi_{g,t}(r\pi_g)_{,r^*}\}_{r=r_+}^{r=\infty}\ , \tag{16}$$

which expresses the time derivative of H in terms of a flux through the boundaries at spatial infinity and at the event horizon. For solutions with asymptotic behavior given by (the real parts of) expressions (14) and (15) this flux is nonpositive, so that $dH/dt \leqslant 0$. Thus H either remains constant or diminishes with time, as one would expect for waves obeying the assumed boundary conditions.

III. STABILITY ANALYSIS

An unstable normal-mode solution is one with time dependence $\exp(i\omega t)$ and for which the frequency ω has a negative imaginary part. Such solutions grow exponentially in time. From the asymptotic forms (14) and (15) we see that such solutions decay exponentially in r^* (at constant t) as $|r^*|\to\infty$. Consequently, they have vanishing flux at $r=\infty$ and $r=r_+$ and thus give

$$H_{,t}=0 \quad \text{or} \quad H=h=\text{constant}\ . \tag{17}$$

The constant h will be finite owing to the exponential decay of an unstable solution for large $|r^*|$. However, the Hamiltonian is a non-negative function of the field variables (since $\pi_2=0$ for any actual solution) which can vanish only if the gauge-invariant perturbation functions vanish. Evaluated for an unstable solution, however, H would consist of a sum of positive terms growing exponentially in time and thus could not satisfy Eq. (17). This contradiction proves the nonexistence of unstable normal-mode solutions obeying the assumed boundary conditions.

Defining $\hat{\pi}_g=\pi_g/[L(L+1)]$ and $\hat{\pi}_f=(L-1)^{1/2}\times(L+2)^{1/2}\pi_f$, we may write Eqs. (11) and (12) as

$$\Box\begin{pmatrix}\hat{\pi}_f\\ \hat{\pi}_g\end{pmatrix}=(N^2/r^3)T\begin{pmatrix}\hat{\pi}_f\\ \hat{\pi}_g\end{pmatrix}\quad , \tag{18}$$

where

$$\Box=-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^{*2}}-(N^2/r^2)[L(L+1)-3m/r+4e^2/r^2] \tag{19}$$

and

$$T=\begin{pmatrix}3m & 2e(L-1)^{1/2}(L+2)^{1/2}\\ 2e(L-1)^{1/2}(L+2)^{1/2} & -3m\end{pmatrix}. \tag{20}$$

Evidently, the orthogonal transformation which diagonalizes T decouples the two wave equations.

A similar analysis can be made for the $L=1$ modes in which only electromagnetic radiation can occur. We shall discuss the stability problem for the even-parity perturbations in a subsequent paper.

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APPENDIX

To compare our equations with those of Zerilli[5] we must relate our functions π_f and π_g to Zerilli's $f_{LM}^{(m)}$ and $R_{LM}^{(m)}$. By a direct comparison of the harmonic expansions used here and those used by Zerilli, one finds that

$$\pi_f=-2f_{LM}^{(m)}\ . \tag{A1}$$

Hamilton's equations give

$$\pi_{g,t}=-(Ne^{-\lambda}/r)(L-1)(L+2)L(L+1)\times[h_1+\tfrac{1}{2}h_{2,r}-(1/r)h_2]\ . \tag{A2}$$

We evaluate this equation in the Regge-Wheeler gauge (signified by an asterisk) to obtain

$$\pi_{g,t}=-[Ne^{-\lambda}/r](L-1)(L+2)L(L+1)h_1^*\ , \tag{A3}$$

since $h_2^*=0$. Thus, using Zerilli's convention for a normal-mode solution ($\partial/\partial t\to -i\omega$) we get

$$\pi_g=[Ne^{-\lambda}/(i\omega r)]\,h_1^*L(L+1)(L-1)(L+2) \tag{A4}$$

or

$$\pi_g=(1/i\omega)L(L+1)(L-1)(L+2)R_{LM}^{(m)}\ . \tag{A5}$$

The connection between corresponding field equations is now immediate.

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