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Is the Williams modification of one-pion-exchange amplitudes equivalent to the addition of crossed-channel exchanges?

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It is shown that for the processes $np \to pn$ and $\gamma p \to \pi^- \Delta^{++}$, the Williams method of modifying the one-pion-exchange amplitude is *not* equivalent to the addition of crossed-channel particle exchanges.

I. INTRODUCTION

It is well known that the forward-direction behavior of the processes $\gamma p \to \pi^+ n$, $np \to pn$, and $\gamma p \to \pi^- \Delta^{++}$ at high energies *cannot* be explained by simple one-pion-exchange (OPE) Born models. Specifically, these models do not account for the experimentally observed forward-direction cross sections. To remedy this deficiency it was found[1-3] sufficient to include crossed-channel exchanges in conjunction with the original OPE contributions within the framework of a simple Born model. A different approach was to modify the OPE amplitude by absorptive corrections.[4] This also accounts fairly well for the observed small-t behavior *as well as* the behavior at higher momentum transfers.

Recently a very simple and parameter-free absorption prescription was suggested by Williams.[5] This method entails the removal of the exceptional Kronecker δ terms which appear in the s-channel partial-wave expansion. These terms are eliminated without recourse to a full partial-wave decomposition simply by evaluating the residue of the pion pole at $t = m_\pi^2$ and retaining this value for all t values. One writes the OPE Born amplitude in the s-channel c.m. system as[6]

$$M_{\lambda_4,\lambda_3;\lambda_2,\lambda_1} = [\sin(\tfrac{1}{2}\theta)]^{|\lambda-\mu|}[\cos(\tfrac{1}{2}\theta)]^{|\lambda+\mu|} \times \frac{P(\lambda,\mu,s,t)}{t-m_\pi^2}, \tag{1.1}$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_3 - \lambda_4$, and where $P(\lambda, \mu, s, t)$ is a polynomial in t whose order depends on the spins of the external particles. Absorption of the Kronecker δ terms in the partial waves is obtained by replacing $P(\lambda, \mu, s, t)$ by $P(\lambda, \mu, s, m_\pi^2)$. The resulting amplitude was named the "OPE-δ amplitude" by Williams. In addition to the δ absorption one corrects the wrong high-$|t|$ behavior by a further application of an exponential form factor[7] or a similar Regge factor.[8]

It has been recently stated[8,9] that the Williams method is equivalent to the inclusion of crossed-channel contributions, i.e., to the "full" Born model, for $\gamma p \to \pi^+ n$ and the np charge-exchange reaction (CEX). It is the purpose of this note to check this statement explicitly for the above-mentioned reactions as well as for the photoproduction process $\gamma p \to \pi^- \Delta^{++}$. The statement can be confirmed by an explicit calculation for the process $\gamma p \to \pi^+ n$; however, for $np \to pn$ and $\gamma p \to \pi^- \Delta^{++}$ we shall demonstrate in Secs. II and III the inequivalence of the OPE-δ model and the full Born model The conclusions to be drawn from this result will be discussed in Sec. IV.

II. THE PROCESS $np \to pn$

The Born amplitude for the π^+ exchange in the t channel for $p(1) + n(2) \to n(3) + p(4)$ is given by

$$M^{\pi^+} = \frac{2g^2}{t-\mu^2}[\bar{u}(p_3)\gamma_5 u(p_1)][\bar{u}(p_4)\gamma_5 u(p_2)], \tag{2.1}$$

where $g^2/4\pi = 14.7$ and $\mu \equiv m_\pi$. The corresponding s-channel helicity amplitudes can be directly obtained and are

$$M^{\pi^+}_{\lambda_4,\lambda_3;\lambda_2,\lambda_1} = \delta_{\lambda_3,-\lambda_1}\delta_{\lambda_4,-\lambda_2}\lambda_4 \frac{4g^2}{t-\mu^2}\frac{p^2}{m^2} \times \sin^2(\tfrac{1}{2}\theta), \tag{2.2}$$

where p and θ denote the nucleon momentum and scattering angle in the c.m. system and where m is the nucleon mass. On the other hand one has for $\lambda_3 = -\lambda_1$, $\lambda_4 = -\lambda_2$

$$|\lambda-\mu| = \begin{cases} 2 & \text{for } \lambda_1 = -\lambda_2 \\ 0 & \text{for } \lambda_1 = \lambda_2 \end{cases}$$

and

$$|\lambda+\mu| = 0. \tag{2.3}$$

The OPE-δ amplitudes, henceforth denoted by $\tilde{M}$, are therefore

$$\tilde{M}^{\pi^+}_{\lambda_4,\lambda_3;\lambda_1,\lambda_1} = \delta_{\lambda_3,-\lambda_1}\delta_{\lambda_4,-\lambda_1}\lambda_4 \frac{4g^2}{t-\mu^2}\frac{p^2}{m^2}\sin^2(\tfrac{1}{2}\hat{\theta}), \tag{2.4}$$

$$\tilde{M}^{\pi^+}_{\lambda_4,\lambda_3;-\lambda_1,\lambda_1} = \delta_{\lambda_3,-\lambda_1}\delta_{\lambda_4,\lambda_1}\lambda_4 \frac{4g^2}{t-\mu^2}\frac{p^2}{m^2}\sin^2(\tfrac{1}{2}\theta),$$

where $\sin^2(\frac{1}{2}\hat{\theta}) \equiv \sin^2(\frac{1}{2}\theta)|_{t=\mu^2}$. The contribution of the π^0 exchange in the u channel is

$$M^{\pi^0} = \frac{g^2}{u-\mu^2}[\bar{u}(p_3)\gamma_5 u(p_2)][\bar{u}(p_4)\gamma_5 u(p_1)], \quad (2.5)$$

which yields the following helicity amplitudes:

$$M^{\pi^0}_{\{\lambda\}} = -\delta_{\lambda_4,-\lambda_1}\delta_{\lambda_3,-\lambda_2}\lambda_3 \frac{2g^2}{u-\mu^2}\frac{p^2}{m^2}\cos^2(\tfrac{1}{2}\theta). \quad (2.6)$$

It is obvious that, for arbitrary s, $(M^{\pi^0}+M^{\pi^+})_{\{\lambda\}} \neq \tilde{M}^{\pi^+}_{\{\lambda\}}$. However, this is a trivial observation and holds even for $\pi^+ n$ photoproduction. Only for $s \gg m^2$ do the Williams and the full Born amplitude coincide in $\pi^+ n$ photoproduction. Therefore one would expect the corresponding equality in np CEX only in the same limit, i.e., as $s \to \infty$. We therefore look for the asymptotic values of (2.2), (2.4), and (2.6). Recalling $p^2 \approx s/4$, $\sin^2(\frac{1}{2}\theta) \approx -t/s$, $\cos^2(\frac{1}{2}\theta) \approx 1$, one obtains

$$M^{\pi^+}_{\{\lambda\}} \approx -\delta_{\lambda_3,-\lambda_1}\delta_{\lambda_4,-\lambda_2}\lambda_4 \frac{g^2}{m^2}\frac{t}{t-\mu^2}, \quad (2.2')$$

$$\tilde{M}^{\pi^+}_{\{\lambda\}} \approx -\delta_{\lambda_3,-\lambda_1}\delta_{\lambda_4,-\lambda_2}\lambda_4 \frac{g^2}{m^2}\frac{1}{t-\mu^2} \times[\mu^2\delta_{\lambda_1,\lambda_2} + t\delta_{\lambda_1,-\lambda_2}], \quad (2.4')$$

$$M^{\pi^0}_{\{\lambda\}} \approx \delta_{\lambda_4,-\lambda_1}\delta_{\lambda_3,-\lambda_2}\lambda_3 \frac{g^2}{2m^2}. \quad (2.6')$$

From this one easily sees that asymptotically $\tilde{M}^{\pi^+}_{\{\lambda\}}$ equals $(M^{\pi^0}+M^{\pi^+})_{\{\lambda\}}$ for $\lambda_1 = -\lambda_2 = -\lambda_3 = \lambda_4$ but *differs* from it for all other helicity configurations. Let us also compare the cross sections predicted by the two models. The cross section is related to the helicity amplitudes by[10]

$$\frac{d\sigma}{dt} \approx \frac{m^4}{\pi s^2}\frac{1}{4}\sum_{\{\lambda\}}|M_{\{\lambda\}}|^2. \quad (2.7)$$

Hence

$$\frac{d\tilde{\sigma}}{dt} \approx \frac{g^4}{8\pi s^2}\frac{t^2+\mu^4}{(t-\mu^2)^2}, \quad (2.8)$$

while

$$\frac{d\sigma}{dt} \approx \frac{g^4}{16\pi s^2}\frac{3t^2+\mu^4}{(t-\mu^2)^2}, \quad (2.9)$$

i.e., the Williams cross section is twice the one due to the full Born model for $|t| \ll \mu^2$ and has a quite different shape. Comparison with the discussion in Ref. 2 shows that $d\tilde{\sigma}/dt$ might be in better agreement with the data, although the experimental situation is not completely clear due to possible systematic errors.

It is interesting to note here that had we taken the π^0 exchange in the Born model with an appropriate form factor, its contribution would have been negligible due to its being far off the mass shell at $s \gg m^2$. It is in fact due to their *inclusion* of form factors in the Born model that Islam and Preist[11] did not succeed in reproducing the forward peak by pion exchanges alone. In our introduction to Ref. 2 we have failed to state this clearly.[12,13] It is also interesting to note[13] that the full Born amplitude violates unitarity. However, this is not too alarming, since both the Born and Williams models need further absorption corrections to their wrong high-$|t|$ behavior.

III. THE PROCESS $\gamma p \to \pi^- \Delta^{++}$

The Born amplitude for π^+ exchange in the t channel for $\gamma(1) + p(2) \to \pi^-(3) + \Delta^{++}(4)$ is

$$M = eG2\epsilon_\nu(p_1)p_3^\nu \frac{1}{t-\mu^2}\bar{u}_\mu(p_4)u(p_2)p_2^\mu, \quad (3.1)$$

with $G^2/4\pi = 18.9\ \text{GeV}^{-2}$ and $e^2/4\pi = 1/137$. The s-channel helicity amplitudes implied by (3.1) are

$$\begin{aligned}
M_{3/2;\,1/2,\,\lambda_1} &= \frac{eG}{t-\mu^2}\lambda_1 p'pf_-\sin^2\theta\cos(\tfrac{1}{2}\theta),\\
M_{3/2;\,-1/2,\,\lambda_1} &= -\frac{eG}{t-\mu^2}\lambda_1 p'pf_+\sin^2\theta\sin(\tfrac{1}{2}\theta),\\
M_{1/2;\,1/2,\,\lambda_1} &= -\frac{eG}{t-\mu^2}\frac{\lambda_1}{\sqrt{3}}p'\sin\theta\left[pf_+\sin\theta\sin(\tfrac{1}{2}\theta) + 2\frac{p'p_0 - pp_0'\cos\theta}{M}f_-\cos(\tfrac{1}{2}\theta)\right],\\
M_{1/2;\,-1/2,\,\lambda_1} &= -\frac{eG}{t-\mu^2}\frac{\lambda_1}{\sqrt{3}}p'\sin\theta\left[pf_-\sin\theta\cos(\tfrac{1}{2}\theta) - 2\frac{p'p_0 - pp_0'\cos\theta}{M}f_+\sin(\tfrac{1}{2}\theta)\right].
\end{aligned} \quad (3.2)$$

This implies the following Williams amplitudes:

$$\tilde{M}_{3/2;\,1/2,1}=\frac{eG}{t-\mu^2}4p'pf_-\cos^2(\tfrac{1}{2}\hat{\theta})\sin^2(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)\,,$$

$$\tilde{M}_{3/2;\,1/2,-1}=-\frac{eG}{t-\mu^2}4p'pf_-\sin^2(\tfrac{1}{2}\hat{\theta})\cos^3(\tfrac{1}{2}\theta)\,,$$

$$\tilde{M}_{3/2;\,-1/2,1}=-\frac{eG}{t-\mu^2}4p'pf_+\cos^2(\tfrac{1}{2}\hat{\theta})\sin^3(\tfrac{1}{2}\theta)\,,$$

$$\tilde{M}_{3/2;\,-1/2,-1}=\frac{eG}{t-\mu^2}4p'pf_+\sin^2(\tfrac{1}{2}\hat{\theta})\sin(\tfrac{1}{2}\theta)\cos^2(\tfrac{1}{2}\theta)\,,$$

$$\tilde{M}_{1/2;\,1/2,1}=\frac{-2eG}{t-\mu^2}\frac{1}{\sqrt{3}}p'\cos(\tfrac{1}{2}\hat{\theta})\left[pf_+\sin\hat{\theta}\,\sin(\tfrac{1}{2}\theta)+2\frac{p'p_0-pp_0'\cos\hat{\theta}}{M}f_-\cos(\tfrac{1}{2}\hat{\theta})\right]\sin(\tfrac{1}{2}\theta)\,,\tag{3.3}$$

$$\tilde{M}_{1/2;\,1/2,-1}=\frac{4eG}{t-\mu^2}\frac{p'}{\sqrt{3}}\left[pf_+\sin^2(\tfrac{1}{2}\hat{\theta})+\frac{p'p_0-pp_0'\cos\hat{\theta})}{M}f_-\right]\sin(\tfrac{1}{2}\theta)\cos^2(\tfrac{1}{2}\theta)\,,$$

$$\tilde{M}_{1/2;\,-1/2,1}=-\frac{4eG}{t-\mu^2}\frac{p'}{\sqrt{3}}\left[pf_-\cos^2(\tfrac{1}{2}\hat{\theta})-\frac{p'p_0-pp_0'\cos\hat{\theta}}{M}f_+\right]\sin^2(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)\,,$$

$$\tilde{M}_{1/2;\,-1/2,-1}=\frac{2eG}{t-\mu^2}\frac{p'}{\sqrt{3}}\sin(\tfrac{1}{2}\hat{\theta})\left[pf_-\sin\hat{\theta}\,\cos(\tfrac{1}{2}\theta)-2\frac{p'p_0-pp_0'\cos\hat{\theta}}{M}f_+\sin(\tfrac{1}{2}\hat{\theta})\right]\cos(\tfrac{1}{2}\theta)\,.$$

In these formulas $p'=|\vec{p}_3|=|\vec{p}_4|$, $p=|\vec{p}_1|=|\vec{p}_2|$, $p_0\equiv p_2^0\equiv E_2$, $p_0'\equiv p_4^0\equiv E_4$, $\cos\theta=\vec{p}_1\cdot\vec{p}_3/p'p$, $\cos\hat{\theta}=\cos\theta|_{t=\mu^2}$, M is the mass of Δ^{++}, and

$$f_\pm=\tfrac{1}{2}\left[\frac{(p_0'+M)(p_0+m)}{mM}\right]^{1/2}\left[1\pm\frac{p'p}{(p_0'+M)(p_0+m)}\right].\tag{3.4}$$

For $s\gg M^2$ one has $\cos\theta\approx 1+(2t/s)$, $\sin^2(\frac{1}{2}\theta)\approx -t/s$, $2(p'p_0-pp_0'\cos\hat{\theta})\approx m^2-(M^2+\mu^2)$, $2(mM)^{1/2}f_+\approx\sqrt{s}$, and $2(mM)^{1/2}f_-\approx m+M$. This gives to leading order in s

$$\tilde{M}_{3/2;\,1/2,\lambda_1}=-\frac{eG}{t-\mu^2}\frac{m+M}{2(mM)^{1/2}}(t\delta_{\lambda_1,1}-\mu^2\delta_{\lambda_1,-1})\,,$$

$$\tilde{M}_{3/2;\,-1/2,\lambda_1}=\frac{eG}{t-\mu^2}\frac{(-t)^{1/2}}{2(mM)^{1/2}}(t\delta_{\lambda_1,1}-\mu^2\delta_{\lambda_1,-1})\,,$$

$$\tilde{M}_{1/2;\,1/2,\lambda_1}=\lambda_1\frac{eG}{t-\mu^2}\frac{(-t)^{1/2}}{2(mM)^{1/2}\sqrt{3}}\times\left[\mu^2+(m+M)\frac{M^2+\mu^2-m^2}{M}\right],\tag{3.5}$$

$$\tilde{M}_{1/2;\,-1/2,\lambda_1}=\frac{eG}{t-\mu^2}\frac{1}{2(mM)^{1/2}\sqrt{3}}\times\left[m+M+\frac{M^2+\mu^2-m^2}{M}\right][t\delta_{\lambda_1,1}-\mu^2\delta_{\lambda_1,-1}].$$

Therefore

$$\frac{d\tilde{\sigma}}{dt}=\frac{1}{4\pi}\frac{mM}{(s-m^2)^2}\tfrac{1}{4}\sum_{\{\lambda\}}|\tilde{M}_{\{\lambda\}}|^2$$
$$\approx\frac{1}{32\pi}\frac{e^2G^2}{(t-\mu^2)^2}\left\{(t^2+\mu^4)\left[(m+M)^2+\tfrac{1}{3}\left(m+M+\frac{M^2+\mu^2-m^2}{M}\right)^2-t\right]-\tfrac{2}{3}t\left[\mu^2+\frac{m+M}{M}(M^2+\mu^2-m^2)\right]^2\right\}.\tag{3.6}$$

Numerical calculation shows that (3.6) coincides, within experimental errors, with the data for $0<\sqrt{-t}<0.08$ GeV. It lies above the data for all higher values of $|t|$. This is a slightly poorer achievement than that of the Born model and is even worse than the "low-t theorem" prediction of Campbell *et al.*[3]

To compare the three *theoretical* predictions let us choose characteristic values of $|t|$; $|t|=0$, $t=m_\pi^2$, and $t=0.16$ GeV^2. One obtains for $(s-m^2)^2d\sigma/dt$ (in μb GeV^2) from Eq. (3.6) the values 600, 1330, and 890, respectively; from the Born model one obtains 530, 1000, 550; and from the "low-t theorem" 530, 1200, and 820, respectively. The Williams curve has, as we see, a steeper increase from $|t|=0$ to $|t|=m_\pi^2$, i.e., it is also of a different shape than the other two curves besides yielding higher values of $d\sigma/dt$.

IV. DISCUSSION

We have shown that the *theoretical* results for a full Born amplitude and a Williams model are *different*. In particular, for $np \to pn$ the forward differential cross section in the Williams model is twice that in the Born model and has moreover a different shape for small values of $|t|$. The same holds for $\pi^-\Delta^{++}$ photoproduction, with the sole difference that now the Williams model yields a cross section which is only 1.2–1.3 times the Born cross section.

We have also seen that experiment cannot yet discriminate between the two models. The best place to look at is np charge exchange. Unfortunately present-day experiments have normalization uncertainties[2] just of the magnitude (i.e., a factor of 2) we are looking for.

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