Their proofs are omitted.

We also tabulate, in Table V, the result of a numerical computation of $\pi^0(l, I, 0)$ for $l=24$. For small $I$, $\pi^0(24, I, 0)$ is approximately $8=\frac{1}{3}l$, confirming (B11). Notice that the value of $\pi^0(l, 0)$ without isospin conservation, as given by (A7), is bigger by $\sim\frac{1}{6}$ than this value. The meaning of this observation is as follows: For $q=0$, charge conservation gives a slight edge to $\pi^0$ over $\frac{1}{2}(\pi^+ + \pi^-)$ if one does not consider isospin conservation, as (A7) shows explicitly. This favoritism disappears when isospin conservation is considered, as (B11) and (B12) show explicitly, because of the greater symmetry between $\pi^+$, $\pi^-$ and $\pi^0$.

If one now keeps $q=0$, but considers a value of $I$ comparable to $l$, then the favoritism for $\pi^0$ must reappear, because the average of $\pi^0(l, I, 0)$ over $I$ must give the same result as (A7). Indeed this is so, since (B15) leads to

$$\pi^0(l, l, 0) = l^2(2l-1)^{-1} > \tfrac{1}{3}l \ .$$

The numerical value of $\pi^0(24, 24, 0)$ in Table V agrees with this formula, as expected.

---

*Work supported in part by the National Science Foundation under Grant No. GP32998X.

[1]B. Alper, H. Bøggild, G. Jarlskog, G. Lynch, J. M. Weiss, P. Booth, L. J. Carroll, J. N. Jackson, M. Prentice, G. Von Dardel, L. Jönsson, G. Damgaard, K. H. Hansen, E. Lohse, F. Bulos, L. Leistam, A. Klovning, E. Lillethun, B. Duff, F. Heymann, and D. Quarrie, Phys. Lett. 44B, 521 (1973); B. Alper, H. Bøggild, G. Jarlskog, J. M. Weiss, P. Booth, L. J. Carroll, J. N. Jackson, M. Prentice, G. Von Dardel, G. Damgaard, F. Bulos, W. Lee, G. Manning, P. Sharp, L. Leistam, A. Klovning, E. Lillethun, B. Duff, K. Potter, D. Quarrie, and S. Sharrock, *ibid.* 44B, 527 (1973); M. Banner, J. L. Hamel, J. P. Pansart, A. V. Stirling, T. Teiger, H. Zaccone, J. Zsembery, G. Bassompierre, M. Croissiaux, J. Gresser, R. Morand, M. Riedinger, and M. Schneegans, *ibid.* 44B, 537 (1973); F. W. Büsser, L. Camilleri, L. DiLella, G. Gladding, A. Placci, B. G. Pope, A. M. Smith, J. K. Yoh, E. Zavattini, B. J. Blumenfeld, L. M. Lederman, R. L. Cool, L. Litt, and S. L. Segler, *ibid.* 46B, 471 (1973); J. W. Cronin, H. J. Frisch, M. J. Shochet, J. P. Boymond, P. A. Piroué, and R. L. Sumner, Phys. Rev. Lett. 31, 1426 (1973).

[2]There have been considerations in the literature related to the present work. See especially C. H. Llewellyn Smith and A. Pais, Phys. Rev. D 6, 2625 (1972), and earlier works referred to in that paper. See also A. DiGiacomo, Phys. Lett. 40B, 569 (1972).

---

PHYSICAL REVIEW D VOLUME 9, NUMBER 9 1 MAY 1974

# Single-cluster formation in the statistical bootstrap model*

C. J. Hamer
*Brookhaven National Laboratory, Upton, New York 11973*
(Received 24 January 1974)

Single-cluster formation in hadronic reactions is discussed, within the context of the statistical bootstrap model. This process is analogous to compound nucleus scattering in nuclear physics, and similar formulas hold for the formation cross section. If the average resonance width should rise indefinitely with energy, the model will eventually run into conflict with unitarity; the trouble is traced to a breakdown of the "narrow-resonance approximation." The effects of angular momentum conservation on the cluster decay are considered, and formulas are presented for the multiplicity and single-particle momentum distribution as a function of the cluster's spin. Brief discussions are given of possible experimental tests of the model, including the annihilation reactions $e^+e^- \to$ hadrons and $N\bar{N} \to$ mesons, which are particularly favorable cases. In an appendix it is shown how to estimate asymptotic parameters in a "realistic" model by analytic means.

## I. INTRODUCTION

It is a familiar fact that low-energy nuclear interactions are well described by the "compound nucleus" model of Bohr,[1] in which reactions are assumed to proceed via an incoherent sum over long-lived direct-channel resonances. The average behavior of the system (e.g., momentum distributions, branching ratios, etc.) can then be described by statistical means, that is, by computing ratios of the phase space available in the various final states. In order to do this, one needs to know the level density of excited nuclear states (resonances) into which the compound nucleus may decay. Bethe's model,[2] which is employed for this purpose, is again statistical in nature. Comparisons of the resulting predictions with experiment provide the most important tests of both models.

It is natural to expect analogous phenomena to occur in hadron physics. That is, a certain fraction of the reaction cross section in any "nonexotic" channel may be attributed to incoherent direct-channel resonance fluctuations, whose average behavior may be described statistically. We shall refer to such processes as "single-cluster formation," to distinguish them from the exchange-type (coherent) processes which are apparently predominant at high energies, and which will generally lead to the production of *several* "clusters." By analogy with the nuclear case, it is likely that experimental studies of single-cluster formation will provide the most stringent tests available of any statistical model of hadrons.

Suggestions of this sort were already made several years ago by Ericson.[3] More recent discussions have been given by several authors,[4-6] within the context of the statistical bootstrap model of hadrons developed by Hagedorn[7] and Frautschi.[8] It has been pointed out by Frautschi,[4] in particular, that the relevance of a statistical approach in the hadron case has been less obvious than in the nuclear case, simply because the widths of the hadron resonances are much greater in comparison with the spacing between them. This makes it harder to disentangle individual resonances with the naked eye, and increases the relative importance of the *coherent* terms built up by the overlapping resonance states. Nevertheless, it is clear in principle that analogous methods should be applicable in both cases.

Our purpose in this paper is to carry these discussions somewhat further, and to derive predictions from the statistical bootstrap model which can be compared with experiment in some specific situations. In Sec. II formulas are set down for the single-cluster formation cross section in terms of the resonance partial widths. For the most part, these are precisely the same as those of the old Bohr model.[1,3] However, they do show the interesting fact that the usual assumptions of the statistical bootstrap may come into conflict with unitarity if the average resonance widths increase with their mass. This point is discussed further in Sec. III. A better understanding of these widths and their effects is necessary for further development of the model.

In Sec. IV the effects of angular momentum conservation on the cluster decay characteristics are dealt with. The multiplicity of the pions emitted, and the single-particle momentum distribution, are derived. Section V discusses experimental tests of the model, with emphasis on the two most important situations, namely $e^+e^-$ annihilation into hadrons via the one-photon process and $N\bar{N}$ annihilation into mesons. In the first case angular momentum conservation can be essentially ignored, while in the second case polarization effects are important.

In Sec. V our results and conclusions are summarized.

## II. SINGLE-CLUSTER FORMATION CROSS SECTION

Expressions for the single-cluster formation cross section can be taken straight from the compound-nucleus model.[1,3,6] We shall ignore all inessential complications, such as the spins of the initial and final particles, and terms in the scattering amplitude other than the incoherent resonance fluctuation terms. These complications have been treated by Ericson,[3] who showed how the particle spins can be taken care of by adding angular momenta, while coherent terms in the amplitude simply give rise to additional terms in the cross section which can be considered quite separately. The coherent terms naturally tend to become more predominant as the energy increases.

The reaction cross section for spinless particles can be expanded in partial waves:

$$\sigma_{fi}(E)=\pi \lambda\!\!\!^{-}_i{}^2 \sum_{L=0}^{\infty}(2L+1)|S_{fi}^L|^2, \tag{1}$$

where $S_{fi}^L$ is the partial-wave amplitude for $i\to f$ corresponding to total angular momentum $L$. The partial-wave amplitude at energy $E$ can be written

$$S_{fi}^L(E)=i\sum_n \frac{a_n^L}{E-E_n^L}, \tag{2}$$

if we assume it to be dominated by a number of narrow resonances, labeled by the index $n$. The residues $a_n^L$ are equal to products of the couplings to initial and final channels

$$a_n^L=\gamma_{fn}^L\gamma_{ni}^L. \tag{3}$$

Now let us average over an energy interval $\Delta E$ which is large compared with the average resonance widths and spacings. The average cross section will be

$$\langle\sigma_{fi}\rangle=\pi \lambda\!\!\!^{-}_i{}^2\sum_L(2L+1) \times\left\langle \sum_{m,n}\frac{a_m^L a_n^{L*}}{(E-E_m^L)(E-E_n^{L*})}\right\rangle. \tag{4}$$

By integrating in the complex energy plane[3] and applying the theorem of residues, this can be rewritten

$$\langle \sigma_{fi} \rangle = \pi \lambda\!\!\!^{-}_i{}^2 \sum_L (2L+1) \frac{2\pi i}{\Delta E} \sum_{m,n} \frac{a_m^L a_n^{L*}}{E_n^{L*} - E_m^L}. \quad (5)$$

We now assume only random correlations between terms with $m \neq n$, which thus do not contribute to the average; that is, we consider only the incoherent piece of the cross section due to resonance fluctuations, which we shall henceforth label $\sigma^F$. The remaining diagonal terms in Eq. (5) then give

$$\langle \sigma_{fi}^F \rangle = \pi \lambda\!\!\!^{-}_i{}^2 \sum_L (2L+1) \frac{2\pi}{D^L} \frac{\langle |a^L|^2 \rangle}{\Gamma^L}, \quad (6)$$

where $D^L$ is the average spacing between resonances of angular momentum $L$ in the energy range considered, $\Gamma^L$ is their average width, and $\langle |a^L|^2 \rangle$ is their mean square residue. This last quantity can be reexpressed as follows:

$$\begin{aligned} \langle |a^L|^2 \rangle &= \langle |\gamma_{fn}^L \gamma_{ni}^L|^2 \rangle \\ &= \langle |\gamma_{fn}^L|^2 \rangle \langle |\gamma_{ni}^L|^2 \rangle \\ &= \langle \Gamma_f^L \rangle \langle \Gamma_i^L \rangle . \end{aligned} \quad (7)$$

Here it has been assumed that the correlations between the couplings to exit and entrance channels are also random, and an expression in terms of the average partial widths $\langle \Gamma_f^L \rangle, \langle \Gamma_i^L \rangle$ has been obtained.

If one now sums over all exit channels $f$, the resulting total cross section is

$$\langle \sigma_i^F \rangle_{\text{tot}} = \pi \lambda\!\!\!^{-}_i{}^2 \sum_L (2L+1) \frac{2\pi}{D^L} \langle \Gamma_i^L \rangle . \quad (8)$$

This is the quantity which we have called the cross section for single-cluster formation. It is the analog of the "compound nucleus" cross section in nuclear physics, and Eq. (8) may be found in any elementary nuclear physics text.

Note that the ratio $\langle \Gamma_i^L \rangle / D^L$ must obey a restriction if Eq. (8) is to be valid; it cannot increase indefinitely with energy, or partial-wave unitarity would eventually be violated. We shall return to this point in Sec. III, but let us ignore it for the moment.

Now

$$\frac{1}{D^L} = \rho^L(E; L_z = 0), \quad (9)$$

where $\rho^L(E; L_z=0)$ is the density of resonance states with spin $L$ and $L_z = 0$ at energy $E$. Furthermore, in a statistical model one may assume that

$$\langle \Gamma_i^L \rangle = \frac{\phi_i^L(E; L_z=0)}{\phi_{\text{tot}}^L(E; L_z=0)} \Gamma^L, \quad (10)$$

where $\phi_i^L(E; L_z=0)$ is the phase space available in channel $i$ for energy $E$, angular momentum $L$ and $L_z = 0$, and $\phi_{\text{tot}}$ is the phase space summed over all channels. Finally, the assumption specific to the statistical bootstrap model is that

$$\rho^L(E; L_z=0) = \phi_{\text{tot}}^L(E; L_z=0) \quad (E \text{ large}), \quad (11)$$

i.e., the density of resonance states is equal to the total available phase space,[7,8] if the latter is computed within a box of volume $V$ equal to the average resonance volume. This assumption is the analog of Bethe's model[2,8] in the nuclear case.

Substituting Eqs. (9)–(11) in Eq. (8), it follows that

$$\langle \sigma_i^F \rangle_{\text{tot}} = 2\pi^2 \lambda\!\!\!^{-}_i{}^2 \sum_L (2L+1) \phi_i^L(E; L_z=0) \Gamma^L . \quad (12)$$

This is the final result, unless one is willing to make an assumption about how the resonance widths depend on $L$. The simplest assumption is that they are independent of $L$, in which case

$$\begin{aligned} \langle \sigma_i^F \rangle_{\text{tot}} &= 2\pi^2 \lambda\!\!\!^{-}_i{}^2 \Gamma \sum_L (2L+1) \phi_i^L(E; L_z=0) \\ &= 2\pi^2 \lambda\!\!\!^{-}_i{}^2 \Gamma \phi_i^{\text{tot}}(E). \end{aligned} \quad (13)$$

As $E \to \infty$, $\lambda\!\!\!^{-}_i{}^2 \phi_i^{\text{tot}}(E) \to$ constant; therefore the energy dependence of the single-cluster cross section is the same as that of the average width $\Gamma$. The widths must again obey certain restrictions if these equations are to be valid. Unitarity will not allow either $\Gamma^L$ in Eq. (12) or $\Gamma$ in Eq. (13) to increase indefinitely with energy. This point will be discussed further in the next section.

## III. RESONANCE WIDTHS AND THE STATISTICAL BOOTSTRAP

In this section we would like to return and consider the implications of Eqs. (8) and (13) of Sec. II, and their possible conflict with unitarity. Nahm[9] has suggested that the trouble lies in the assumption of Eq. (10) describing the decay of the resonances, the so-called principle of reciprocity.[10] We shall employ an argument similar to his, but arrive at a somewhat different conclusion.

Consider a quantity of hadronic material with total energy $E$ at equilibrium within a certain volume, which for simplicity we shall suppose to be the standard resonance volume $V$ (although this assumption may be changed without affecting the argument). Then the principle of detailed balance states that

$$\rho_A(E) P_{A\to B} = \rho_B(E) P_{B\to A}, \quad (14)$$

where $A$ and $B$ are any two states, $P_{A\to B}$ is the probability per unit time that state $A$ will make the transition to state $B$ within the volume $V$, and $\rho_A(E)$, $\rho_B(E)$ are the phase-space densities of states $A$ and $B$ within $V$. Now suppose that $A$ denotes the average one-particle resonance state of mass $E$, and we take the sum over all states $B$

consisting of 2, 3 ⋯ ∞ particles into which the resonance $A$ may decay. Then Eq. (14) leads to

$$\rho_A(E)\,\frac{\langle\Gamma_A\rangle T}{\hbar}=\sum_B \rho_B(E)\int_0^T dt\,P_{B\to A}(t)\le\sum_B \rho_B(E)\,, \tag{15}$$

where we have integrated over a time interval $T$, and the final inequality follows from unitarity. Now the statistical bootstrap assumption[7,8] [Eq. (11)] for the density of states is just

$$\rho_A(E)\underset{E\to\infty}{\sim}\sum_B \rho_B(E)\,. \tag{16}$$

That is, the density of resonances at a given energy is just equal to the density of available decay channels, counted in the volume $V$. It is clear that this will eventually *conflict* with unitarity [Eq. (15)] if the average resonance width $\langle\Gamma_A\rangle\equiv\Gamma$ rises indefinitely with energy. This is a parallel conflict to the one arising in Eqs. (8) and (13) of Sec. II.

The alert reader will have noticed, however, that Eq. (15) contains an approximation, in that we have set

$$\sum_B \int_0^T dt\,P_{A\to B}(t)=\frac{\Gamma_A T}{\hbar}\,.$$

The correct expression is

$$\sum_B \int_0^T dt\,P_{A\to B}(t)=1-e^{-\Gamma_A T/\hbar}\,. \tag{17}$$

One can use the first-order approximation to this expression only if the resonances $A$ are *narrow*, so that "absorption" of the initial state is negligible within the given time interval. It seems clear that our treatment of Sec. II implicitly involves a similar assumption, and is only valid in a "narrow-resonance approximation."

Now in fact the statistical bootstrap model for the level density itself, Eq. (16), also relies on the narrow-resonance approximation.[7,8] The model treats both resonances and stable particles on an equal footing, as independent particles, when computing phase-space densities. For this procedure to be valid, the resonances must be narrow.[11,12] Dashen, Ma, and Bernstein,[12] for instance, give as a requirement that

$$\Gamma<kT\underset{E\to\infty}{\sim}\text{constant in the statistical bootstrap}\,. \tag{18}$$

This requirement is clearly broken if $\Gamma$ rises indefinitely with energy. It is no longer possible to count each resonance as an independent state.

How should one change the assumptions of the model if the average widths do indeed rise with energy? One possibility is Frautschi's suggestion[8] that the resonance density times the average width should be proportional to the density of decay channels, i.e.,

$$\rho_A(E)\langle\Gamma_A\rangle\propto\sum_B \rho_B(E)\,. \tag{19}$$

Sertorio, Toller, and Bassetto[13] have discussed this sort of proposal. However, the correct procedure is still unclear to the present author.

The question is clearly not an academic one, because the widths of the particles listed in the Particle Data Group tables[14] show a general tendency to rise linearly with their mass. If this tendency were to persist indefinitely, the statistical bootstrap model as presently formulated would necessarily fail. Eventually one would run into a theoretical conflict with unitarity, as outlined above, and Eq. (13) would predict a large and increasing single-cluster cross section, in contradiction with experiment. On the other hand, a recent analysis of fluctuations in $\pi N$ scattering at 5 GeV/$c$ by Schmidt *et al.*[15] has suggested that the average resonance width has become relatively small at this energy. Further analyses of this sort are urgently required.

For the remainder of this paper we shall assume that the present formulation of the statistical bootstrap is valid.

## IV. EFFECTS OF ANGULAR MOMENTUM CONSERVATION IN CLUSTER DECAY

Having discussed the formation of the cluster, we must now consider its decay into the observed final states. The average behavior of the system can be predicted by statistical means, once the resonance spectrum is known. Within the context of the statistical bootstrap model, several treatments of this process have recently been given.[5,9,10,16-18] So far, angular momentum conservation has been neglected in these treatments, yet in a high-energy two-body collision very large angular momenta are involved and are likely to produce important effects.[19] In this section, we set out to remedy this neglect.

The decay of the cluster is determined by phase space, and follows the same pattern as the decay of an individual resonance of equal mass, in this model. Methods have now been developed for deriving the asymptotic behavior of the resonance spectrum,[20,21] and of the moments of various distributions over this spectrum.[17,18] A useful picture is to regard each heavy resonance as being composed of a large number of the "input" states or constituents (such as pions) into which it ultimately decays, all crammed into a volume $V$. The distribution of input states inside the box is a thermodynamic one, controlled by the "limiting temperature" $T_0$,[7] and therefore it is described by a partition function $Z_{in}(\beta_0)$, $\beta_0 \equiv 1/kT_0$, where

$$Z_{in}(\beta_0) = \int dE\, e^{-\beta_0 E} \int dm\, \rho_{in}(m) \frac{V}{h^3} \times \int d^3p\, \delta(E - (m^2 + \vec{p}^2)^{1/2}), \tag{20}$$

and $\rho_{in}(m)$ is the density of input states at mass $m$; $h$ is Planck's constant. Hence one may find the average value of any dynamical variable $F$, averaged over these input states within the box, by the formula[22]

$$\langle F \rangle = \frac{1}{Z_{in}(\beta_0)} \int dE\, e^{-\beta_0 E} \int dm\, \rho_{in}(m) \frac{V}{h^3} \times \int d^3p\, F \delta(E - (m^2 + \vec{p}^2)^{1/2}). \tag{21}$$

These averages then describe the distributions of the ultimate decay products.

For instance, the average energy of these constituents, $\langle E \rangle$, can be found from Eq. (21): It is fixed, and of the order $\frac{3}{2}kT_0$. Therefore a heavy resonance of mass $m$ eventually decays into $m/\langle E \rangle$ input states, on the average, and finally gives rise to a mean pion multiplicity equal to[18]

$$n(m) \underset{m\to\infty}{\sim} \frac{m}{\langle E \rangle} \langle N \rangle, \tag{22}$$

where $\langle N \rangle$ is the partition average number of pions produced by the input states themselves. Higher cumulants of the multiplicity distribution also turn out to be proportional to $m/\langle E \rangle$ at large $m$, which results from the random-walk process by which the heavy resonances are built out of the input states.[17,18] Similar results apply for distributions over other quantum numbers (e.g., spin) connected with the resonance spectrum, provided that the quantity concerned is additive, and may vary over an unrestricted range.[18]

The result (22) is only valid, however, when one averages over resonances of all spin. Generally, it is to be expected that the mean multiplicity will depend on the spin $J$; in fact, one finds that

$$n(m, J) \underset{m\to\infty}{\sim} am \exp\left(-d \frac{J^2}{m^2}\right), \tag{23}$$

where the constants $a$ and $d$ are given by

$$a = \frac{\langle N \rangle}{\langle E \rangle},$$

$$d = \frac{1}{2} \frac{\langle E \rangle^2}{\langle J_z^2 \rangle} \left[ \frac{\langle EN \rangle}{\langle E \rangle \langle N \rangle} + \frac{\langle EJ_z^2 \rangle}{\langle E \rangle \langle J_z^2 \rangle} - \frac{\langle J_z^2 N \rangle}{\langle N \rangle \langle J_z^2 \rangle} - \frac{\langle E^2 \rangle}{\langle E \rangle^2} \right]. \tag{24}$$

We relegate the derivation of this result to Appendix A, in order to avoid a clutter of mathematical formulas.

*A priori* one might have expected the Gaussian width of the multiplicity distribution (23) to be proportional to $(m/\langle E \rangle)\, \langle J_z^2 \rangle$, the mean square spin of the resonance states.[18,23] Instead, Eq. (23) shows it to be proportional to $m^2$, so that the multiplicity is essentially *independent* of $J$ for the bulk of the resonance states at large $m$, and only cuts off when $J/m$ becomes comparable to the radius of the box. This can be physically understood as follows. The bootstrap states are built out of a large number of "input" constituents, each of which carries a fixed average energy and angular momentum. Thus the angular momentum $J$ of the bootstrap states is built up in a random walk, and follows a Gaussian distribution[23,18] whose width is proportional to the number of constituents, $m/\langle E \rangle$, by virtue of the central limit theorem. The number of constituents itself, on the other hand, should indeed be largely independent of $J$. Hence the decay multiplicity also is asymptotically independent of $J$.

We have endeavored to check the result (23) by computing the actual multiplicities in a simple numerical model (case 1a of Ref. 8). They are shown in Fig. 1, and it can be seen that a Gaussian in $J$ fits very well out to quite large $J$. The Gaussian coefficient, however, does not settle down to the asymptotic form $d/m^2$ given by Eq. (23) within the range of masses considered. In fact, if one calculates the expected value of $d$ from Eq. (24) for this particular case, it turns out to be *negative*, so that the multiplicity will eventually *increase* with $J$ (for $J \ll kR$). This may be interpreted as implying that the mean angular momen-

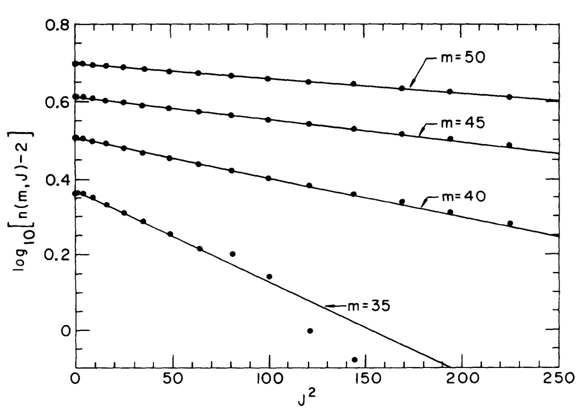


FIG. 1. Decay multiplicity as a function of angular momentum for a simple numerical example of the statistical bootstrap. Closed circles: computed multiplicities. Solid lines: Gaussian fits to the results.

tum of the constituents is not sensitive to their energy, so that the larger the number of constituents, the larger the angular momentum of the asymptotic state. In such a case, the asymptotic form (23) is not very useful. We shall continue to treat $d$ as a positive quantity for the moment.

Next, let us consider the multiplicity of pions resulting from clusters formed in a two-body collision. We shall again make the simple assumption that the resonance widths are independent of $J$, so that Eq. (13) may be used. Then the probability of forming a cluster with a given angular momentum $J$ is given by

$$P^J(E) \propto (2J+1)\phi_i^J(E; J_z=0). \qquad (25)$$

For the purposes of calculation, we shall also assume that the resonance volume $V$ is Gaussian in shape,[23] i.e., that integrals over $V$ take the form

$$\int_V d^3r \to \int d^3r\, e^{-r^2/R^2}, \qquad (26)$$

so

$$V = [(\pi)^{1/2}R]^3. \qquad (27)$$

Then the phase space for given angular momentum can be computed[23,24]:

$$\phi_i^J(E; J_z=0) = 4\pi^3R^2\,\frac{E_1E_2}{E}\exp\left(-\frac{p^2R^2}{2}\right)I_{J+1/2}\left(\frac{p^2R^2}{2}\right), \qquad (28)$$

where $E_1$, $E_2$, and $p$ are the c.m. energies and momentum of the initial particles. Using (23), (25), and (28), the mean multiplicity of pions resulting from single-cluster formation can now be computed:

$$\bar{n}(E) = \frac{\sum_J P^J(E)n(E,J)}{\sum_J P^J(E)} = \frac{\pi^{1/2}}{pR}\,e^{-p^2R^2/2}\sum_J (2J+1)I_{J+1/2}\left(\frac{p^2R^2}{2}\right)aE \times e^{-dJ^2/E^2}. \qquad (29)$$

We can get an approximate expression for this sum by setting

$$I_{J+1/2}\left(\frac{p^2R^2}{2}\right) \approx \begin{cases} \dfrac{1}{pR(\pi)^{1/2}}\exp\left(\dfrac{p^2R^2}{2}\right), & J<pR \\ 0, \text{ otherwise} \end{cases}$$

and converting the sum into an integral, giving

$$\bar{n}(E) \underset{E\to\infty}{\approx} \frac{4aE}{dR^2}\left[1-\exp\left(-\frac{dR^2}{4}\right)\right]. \qquad (30)$$

Thus the multiplicity is less than the result $aE$ one would have obtained by ignoring the conservation of angular momentum, but still rises linearly with energy. We shall endeavor to make realistic estimates of the size of this effect in Sec. V.

Let us now go on to consider the momentum distribution of pions emitted from the cluster. For the purposes of this exercise, we shall ignore the question of particle statistics, and consider only the simple case where the input spectrum consists of a single neutral "pion." Then if angular momentum conservation is ignored, one expects the single-particle inclusive distribution to take a simple Maxwell-Boltzmann form[16-18]:

$$\rho_1(m;\vec{p}_1) \equiv \frac{1}{\sigma^F}\,2E_1\,\frac{d^3\sigma^F}{d^3p_1} \propto 2E_1\,e^{-\beta_0E_1} \text{ in the limit } m\to\infty, \qquad (31)$$

where $m$ is the mass of the cluster; $\vec{p}_1$ is the momentum of the observed pion in the center of mass of the cluster, and $E_1$ its energy. If angular momentum is included, and one specifies another additive quantum number $J_z$ as well as the energy, one might expect (semiclassically) that the corresponding form should be[12]

$$\rho_1(m, J_z; \vec{p}_1, M) \propto 2E_1\,e^{-\beta E_1+\mu M} \text{ as } m\to\infty, \qquad (32)$$

where $J_z$ is the $z$ component of the spin of the initial cluster and $M$ is the angular momentum component in the $z$ direction of the emitted pion. In Appendix A 2 it is shown that this is indeed the case. The single-particle inclusive distribution function is (summing over $M$)

$$\rho_1(m, J_z; \vec{p}_1) \underset{\substack{m\to\infty \\ J_z/m \text{ small}}}{\sim} \frac{m}{\langle E\rangle}\left[\frac{2E_1}{h^3Z_{\text{in}}}\int d^3r\, e^{-r^2/R^2}\, e^{-\beta_0(h_0)E_1}\sum_{M=-\infty}^{\infty} e^{ih_0M}P_M\right]. \qquad (33)$$

Here $P_M$ is an operator which projects out states with angular momentum component $M$, and the resonance volume has been given a Gaussian shape as in Eq. (26). The constants involved are (from Appendix A and Ref. 18)

$$Z_{\text{in}} \equiv Z_{\text{in}}(\beta_0(h_0), h_0) \equiv 2\ln 2 - 1,$$

$$\beta_0(h_0) = \beta_0 - \frac{1}{2}\,\frac{\langle J_z^2\rangle}{\langle E\rangle}h_0^2 + \cdots, \qquad (34)$$

$$h_0 = -iJ_z\langle E\rangle/m\langle J_\pi^2\rangle.$$

The distribution in square brackets in Eq. (33) is normalized to give unity after an invariant integration over momenta $\vec{p}_1$, so the over-all multiplicity of produced pions is $m/\langle E\rangle$, as expected.

Now, following Cerulus,[24] the projection operator $P_M$ can be written

$$P_M \to \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-iM\phi}\, e^{i\vec{K}(\phi)\cdot\vec{r}}, \tag{35}$$

where

$$\vec{K}(\phi) = (\vec{p}\times\hat{z})\sin\phi + (\vec{p}\times\hat{z})\times\hat{z}(1-\cos\phi). \tag{36}$$

Therefore

$$\rho_1(m, J_z; \vec{p}_1) \underset{\substack{m\to\infty \\ J_z/m \text{ small}}}{\sim} \frac{2mE_1 e^{-\beta_0(h_0)E_1}}{\langle E\rangle Z_{\text{in}} h^3}\int d^3r\, e^{-r^2/R^2}\sum_M e^{ih_0 M}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-iM\phi}\, e^{i\vec{K}(\phi)\cdot\vec{r}}$$

$$= \frac{m}{\langle E\rangle}\,\frac{2E_1 V e^{-\beta_0(h_0)E_1}}{Z_{\text{in}} h^3}\,\frac{1}{2\pi}\sum_M e^{ih_0 M}\int_{-\pi}^{\pi} d\phi\, e^{-iM\phi}\exp[-\tfrac{1}{4}R^2(p_{1T}^2\sin^2\phi + p_{1z}^2(1-\cos\phi)^2],$$

$$\tag{37}$$

where $p_{1T}$ and $p_{1z}$ are the transverse and longitudinal projections of $\vec{p}_1$. For $h_0$ very small, one can expand in powers of $h_0$, make the replacement

$$\sum_M e^{-iM\phi} \to 2\pi\delta(\phi),$$

and eventually obtain

$$\rho_1(m, J_z; \vec{p}_1) \underset{\substack{m\to\infty \\ J_z/m \text{ small}}}{\sim} \frac{m}{\langle E\rangle}\,\frac{2E_1 V e^{-\beta_0 E_1}}{Z_{\text{in}} h^3}\left(1 - \tfrac{1}{4}h_0^2 R^2 p_{1T}^2 + \tfrac{1}{2}h_0^2\frac{\langle J_z^2\rangle}{\langle E\rangle}E_1\right). \tag{38}$$

Using Eq. (34) and the fact that[18]

$$\langle J_z^2\rangle = \tfrac{1}{3}R^2\langle p^2\rangle, \tag{39}$$

the result can be rewritten

$$\rho_1(m, J_z; \vec{p}_1) \underset{\substack{m\to\infty \\ J_z/m \text{ small}}}{\sim} \frac{m}{\langle E\rangle}\,\frac{2E_1 V e^{-\beta_0 E_1}}{Z_{\text{in}} h^3}\left[1 + \frac{9}{2R^2\langle p^2\rangle}\left(\frac{J_z\langle E\rangle}{m}\right)^2\left(\frac{1}{2}\,\frac{p_{1T}^2}{\langle p^2\rangle} - \frac{1}{3}\,\frac{E_1}{\langle E\rangle}\right)\right]. \tag{40}$$

Note that this quantity is averaged over states of all total spins $J \geq J_z$.

Next, consider the limit $m\to\infty$, $J_z/m$ finite but very small. In this situation,

$$\rho_1(m, J, J_z = J; \vec{p}_1) \sim \rho_1(m, J_z; \vec{p}_1)$$

$$\underset{\substack{m\to\infty \\ J/m \text{ small}}}{\sim} \frac{m}{\langle E\rangle}\,\frac{2E_1 V e^{-\beta_0 E_1}}{Z_{\text{in}} h^3}\left[1 + \frac{9}{2R^2\langle p^2\rangle}\left(\frac{J\langle E\rangle}{m}\right)^2\left(\frac{1}{2}\,\frac{p_{1T}^2}{\langle p^2\rangle} - \frac{1}{3}\,\frac{E_1}{\langle E\rangle}\right)\right]. \tag{41}$$

That is, $J \simeq J_z$, because $(\langle J_x^2\rangle + \langle J_y^2\rangle)^{1/2} \sim (m)^{1/2}$ only,[18] and can be ignored. Then one can estimate (semiclassically) that

$$\rho_1(m, J, J_z = 0; \vec{p}_1) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \rho_1(m, J, J_\phi = J; \vec{p}_1) \tag{42}$$

$$\underset{\substack{m\to\infty \\ J/m \text{ small}}}{\sim} \frac{m}{\langle E\rangle}\,\frac{2E_1 V e^{-\beta_0 E_1}}{Z_{\text{in}} h^3}\left\{1 + \frac{9}{2R^2\langle p^2\rangle}\left(\frac{J\langle E\rangle}{m}\right)^2\left[\frac{p_1^2}{\langle p^2\rangle}\left(\frac{1+\cos^2\theta}{4}\right) - \frac{1}{3}\,\frac{E_1}{\langle E\rangle}\right]\right\}. \tag{43}$$

Finally, the single-particle distribution resulting from single-cluster formation can be estimated in similar fashion to Eq. (30):

$$\bar{\rho}_1(E; p_1) = \frac{\sum_J P^J(E)\rho_1(E, J, J_z = 0; \vec{p}_1)}{\sum_J P^J(E)}$$

$$\approx \frac{E}{\langle E\rangle}\,\frac{2E_1 V e^{-\beta_0 E_1}}{Z_{\text{in}} h^3}\left\{1 + \frac{9}{16}\,\frac{\langle E\rangle^2}{\langle p^2\rangle}\left[\frac{p_1^2}{\langle p^2\rangle}\left(\frac{1+\cos^2\theta}{4}\right) - \frac{1}{3}\,\frac{E_1}{\langle E\rangle}\right]\right\}. \tag{44}$$

This equation comprises our final result. Some of the steps between Eqs. (40) and (44) are of slightly dubious validity, but we believe that the result makes good sense physically. The distribution peaks in the forward and backward directions, and is symmetric about $\theta = 90°$, as one would expect.[19,24] It approaches a *limiting shape*, which can also be understood easily: The average angular momentum in the two-body collision increases linearly with the energy, but so does the multiplicity of decay products, so the angular momentum carried off by each final-state pion approaches a constant. This implies a constant limiting shape for its momentum distribution. Finally, if one integrates over the distribution, the multiplicity turns out to be $m/\langle E\rangle$ as we expect [note that the calculation of the momentum distribution has not been taken to the same order of accuracy as the multiplicity calculation, Eq. (23); no account has been taken of the variation of multiplicity with $J$].

## V. EXPERIMENTAL TESTS

### A. $e^+e^-$ annihilation into hadrons

Assuming that this process goes predominantly via the one-photon process at presently accessible energies, $e^+e^-$ annihilation is the first place to look for single-cluster formation. Since only a single angular momentum ($J=1$) contributes, the "incoherent" piece of the cross section due to resonance fluctuations should be a larger proportion of the total than in other processes.[3] Also, since the total angular momentum remains fixed and small as the energy increases, polarization effects will be negligible and the cluster decay distribution should rapidly become isotropic in the center-of-mass system.[25] Thus the energy distribution in the center-of-mass can be directly compared with a simple Boltzmann or Bose-Einstein-type distribution.

The annihilation cross section due to noninterfering vector-meson states has, in fact, already been written down by Sakurai and others,[26] within the context of a generalized vector-dominance model. For a given final state $f$, the cross section is

$$\sigma^F_{e^+e^-\to f}(s^{1/2}) = \frac{12\pi}{s}\sum_V m_V{}^2 \frac{\Gamma_{fV}\Gamma(V\to e^+e^-)}{(s-m_V{}^2)^2+m_V{}^2\Gamma_V{}^2}\,. \tag{45}$$

When averaged over a suitable energy interval, this becomes

$$\begin{aligned}\langle\sigma^F_{e^+e^-\to f}(s^{1/2})\rangle &= \frac{12\pi}{s}\frac{2\pi}{D^V}\frac{\langle\Gamma_{fV}\rangle\langle\Gamma(V\to e^+e^-)\rangle}{4\Gamma^V}\\ &= \frac{6\pi^2}{s}\rho_V(s^{1/2})\frac{\langle\Gamma_{fV}\rangle\langle\Gamma(V\to e^+e^-)\rangle}{\Gamma^V}\,,\end{aligned} \tag{46}$$

where $\rho_V(s^{1/2})$ is the density of vector-meson states at energy $s^{1/2}$. This formula is the same as Eqs. (6) and (7) of Sec. II, except for an extra factor of $\frac{1}{4}$ accounting for the spins of the initial electron-positron pair. Now one can also write

$$\Gamma(V\to e^+e^-) = \frac{4\pi\alpha^2 m_V}{3f_V{}^2}\,, \tag{47}$$

where $em_V{}^2/f_V$ is the usual $\gamma$-$V$ coupling constant. Thus

$$\langle\sigma^F_{e^+e^-\to f}(s^{1/2})\rangle = \sigma_{\mu\,\text{pair}}(s^{1/2})\times 6\pi^2 s^{1/2}\rho_V(s^{1/2}) \times\left\langle\frac{1}{f_V{}^2}\right\rangle\frac{\langle\Gamma_{fV}\rangle}{\Gamma_V}\,, \tag{48}$$

where $\sigma_{\mu\,\text{pair}}(s^{1/2}) = 4\pi\alpha^2/3s$ is the usual reference cross section. Summing over final states $f$, one obtains

$$\langle\sigma^F_{e^+e^-}(s^{1/2})\rangle = \sigma_{\mu\,\text{pair}}(s^{1/2})\times 6\pi^2 s^{1/2}\,\rho_V(s^{1/2})\left\langle\frac{1}{f_V{}^2}\right\rangle, \tag{49}$$

which is the total cross section for single-cluster formation.

The decay of a cluster formed in $e^+e^-$ annihilation, as predicted by statistical models, was first discussed by Bjorken and Brodsky.[27] The statistical-bootstrap-model predictions have recently been treated in detail by Engels, Schilling, and Satz[25] (see also Frautschi and Hamer[16]), and we have little to add to that treatment. It is worth pointing out, however, that the fluctuation or single-cluster cross section $\sigma^F$ is only expected to be a *part* of the total cross section. In general, one expects dynamical and coherent resonance terms to appear in the reaction amplitudes, giving rise to another piece of the cross section $\sigma^c$ (e.g., production of "jets"). This coherent term may be presumed (by analogy with the nuclear physics case) to dominate when the center-of-mass energy gets large, or when the momentum of a produced particle gets big.

The question then arises, how can one isolate the cluster term $\sigma^F$ from the coherent term $\sigma^c$? It is doubtful whether they could be separated by looking at the multiplicity distribution, especially if neutral pions are allowed to go undetected. Instead, one should look at the single-particle inclusive momentum distribution in the center-of-mass system. The term $\sigma^F$ should give rise to an exponential rise at low momentum,

$$\frac{d\sigma}{d^3p} \propto \exp\left(-\frac{E}{kT}\right), \tag{50}$$

where $T$ is the effective temperature. One should look for this effect at energies high enough that a statistical treatment should be roughly applicable (say $s^{1/2} \simeq 2$ GeV and up), and out to momenta of order 1 GeV/$c$, beyond which the coherent term $\sigma^c$ might be expected to take over. At some still higher energy, $\sigma^c$ may be expected to take over entirely, and the exponential peak should disappear.

If such exponential behavior is observed, the next question is: How does the temperature $T$ vary with $s^{1/2}$? In the statistical bootstrap model it is predicted[16] to approach a constant of order 160 MeV[7]:

$$T_{\text{eff}}(s^{1/2}) \underset{s\to\infty}{\sim} T_0 - \frac{\text{const.}}{s^{1/2}}. \tag{51}$$

In the old Fermi model,[28] by way of contrast, it would be predicted to behave as

$$T_{\text{eff}}(s^{1/2}) \underset{s\to\infty}{\sim} \text{const.} \times s^{1/6}. \tag{52}$$

Analyses of the above sort will provide us with the most direct and important tests of the statistical bootstrap model that one can presently foresee. They should be supplemented by fluctuation analyses[4,15] in various individual annihilation channels (see, for instance, the recent predictions for $e^+e^- \to \pi^+\pi^-$ of Margolis, Meggs, and Rudaz[29]). The coherence width of these fluctuations should also provide us with one of the best available estimates of the average behavior of the resonance widths, albeit for only a single value of angular momentum.

We note in passing that there is no reason to expect the single-cluster cross section to "scale," and it should be subtracted out from the total annihilation cross section before looking for scaling behavior.

### B. $N\bar{N}$ annihilation

The next most obvious place to look for single-cluster formation is in nucleon-antinucleon annihilation, where again the energy released is high, the angular momentum is relatively low, and resonance fluctuations should play an important role. Annihilation at rest has already been treated from the viewpoint of the statistical bootstrap by the present author,[5] with reasonable success. The model provides a natural explanation for the high multiplicity of produced pions, and the low branching ratios into individual final states. The predicted relative rates of resonance production were of about the right order of magnitude.

Let us now consider briefly the extension of these ideas to annihilation in flight. The data on cross sections for individual annihilation channels show a rapid exponential-type decrease with energy,[30] inconsistent with coherent Regge behavior, up to quite high energies (several GeV). Such behavior may easily be explained in a statistical model, as being due to competition among a rapidly increasing number of final states. This suggests that single-cluster formation may remain dominant over this whole range of energies.[31]

As the incident antinucleon momentum rises, however, the average angular momentum of the cluster will increase rather rapidly. The effects of this may be estimated using the results of Sec. IV. The average multiplicity, for instance, can be computed from Eq. (29). To determine the parameters involved, we set the volume $V$ to the value found previously[5] to fit $N\bar{N}$ annihilation at rest. The corresponding values of $a$ and $d$ are computed in Appendix B for our "realistic" statistical bootstrap model,[5] and are found to be given by

$$n(m,0) = 0.29\left(\frac{m}{m_\pi}\right) + 0.7 \sim am \tag{53}$$

(for nonstrange annihilations), and

$$d = -0.39 \tag{54}$$

(units $\hbar = c = m_\pi = 1$). Hence the multiplicity for annihilation in flight can be predicted.

Unfortunately, though, the parameter $d$ has again turned out to be negative, so that judging by the example of Sec. IV our asymptotic formulas (23) and (29) are useless. At energies of interest, the multiplicity is likely to decrease with $J$ as in Fig. 1, rather than increase as the asymptotic formula would predict. Thus we expect that the effect of angular momentum conservation will be to lower the predicted cluster multiplicity by amounts of the order 20%, but we are unable to compute the effect without a long and detailed numerical program along the lines of Ref. 5, which we do not propose to carry out here.

It was shown previously[5] that if one neglects angular momentum conservation, the multiplicity is expected to increase linearly with energy, with a slope of about 2.1 GeV$^{-1}$. Experimentally, a slope of 1.3 GeV$^{-1}$ has been measured for nonstrange annihilations in flight by Fields *et al.*,[32] while Oh *et al.*[33] measure a slope of 1.8 GeV$^{-1}$ for pions produced in association with a $K\bar{K}$ pair. We feel that these results are in qualitative agreement with the statistical bootstrap model, once one allows for some reduction in slope due to angular momentum conservation.

Finally, we may predict the single-particle inclusive momentum distribution, using the results of Sec. IV. Since this has not been calculated in such detail as the multiplicity, it is perhaps worth reemphasizing the qualitative conclusions first. The distribution is expected to approach a limiting shape, with forward-backward symmetry, of roughly a Maxwell-Boltzmann type (this seems to be in agreement with current experimental evidence[33]). The detailed form we obtained was

$$\bar{\rho}_1(E;\vec{p}_1)\propto E_1\, e^{-\beta_0 E_1}\left\{1+c\left[\frac{p_1^2}{\langle p^2\rangle}\left(\frac{1+\cos^2\theta}{4}\right)-\frac{1}{3}\frac{E_1}{\langle E\rangle}\right]\right\}, \qquad (55)$$

where $\vec{p}_1$, $E_1$ are the momentum and energy of the observed particle in the center-of-mass system, $\langle p^2\rangle, \langle E\rangle$ are the single-particle averages of the momentum squared and energy, and the constants $\beta_0$ and $c$ can be estimated from the results of Appendix B to be

$$\beta_0 \simeq 1.2\,, \quad c\simeq 0.5 \qquad (56)$$

(units $\hbar = c = m_\pi = 1$). It would be interesting to know if Eq. (55) can provide at least a rough description of the data.

### C. Other reactions

In general, one expects to see single-cluster formation in any reaction where resonances may be formed in the direct channel. Annihilation reactions are the best places to look, because coherence between different partial waves is less important there; but similar effects should also be seen in $\pi^\pm p$ reactions, etc. When the direct channel is "exotic" as in $pp$ or $K^+p$ reactions, on the other hand, these effects should be absent.

The characteristic feature of single-cluster formation will again be the presence of an approximately isotropic, exponentially peaked single-particle distribution at low center-of-mass energies. Erwin *et al.*[34] have already reported seeing just such behavior in $K^+p$ scattering at 11.8 GeV/$c$. But this is an exotic channel, where we have just said that single-cluster formation would not be expected. One can interpret the experimental result in two possible ways:

(i) At these energies ($s^{1/2}\cong 4.8$ GeV) a spectrum of exotic baryon resonances exists, giving rise to true single-cluster formation.

(ii) The reaction is actually giving rise to one or two fast particles together with a heavy cluster moving slowly in the center-of-mass system, so that we are seeing cluster "*production*" rather than "formation." Only a detailed experimental analysis can decide between these two possibilities, but we feel that the second is the more likely explanation.

## VI. SUMMARY AND CONCLUSIONS

This paper has been devoted to a study of single-cluster formation in hadronic reactions, a process analogous to "compound nucleus" scattering[1,3] in nuclear physics. The statistical bootstrap model[7,8] was used to describe this process. It was pointed out that experimental studies of single-cluster formation are likely to provide the most stringent tests available of the statistical approach.

In Sec. II the single-cluster formation cross section was derived, and its energy dependence was found to be the same as that of the average resonance width Γ. This immediately raises a difficulty, in that if the resonance widths should rise indefinitely with the energy, unitarity would eventually be violated. The source of this trouble was analyzed in Sec. III, and it was concluded that if the widths increase with energy then the "narrow-resonance approximation,"[11,12] which is implicit in the statistical bootstrap model, breaks down. It is no longer possible to count each resonance as an independent particle. What the correct procedure should be in such a situation is still unclear.

The experimental information available on average resonance widths was briefly discussed. Very little exists at intermediate energies, and further fluctuation analyses[3,4] of the sort carried out by Schmidt *et al.*[15] would be very welcome for this purpose.

Next, the effects of angular momentum conservation on the cluster (or resonance) decay were considered. They are basically rather unimportant. The mean multiplicity of pions produced by a resonance of mass $m$ and spin $J$, for instance, was found to be

$$n(m,J)\underset{m\to\infty}{\sim} am\,\exp\left(-d\frac{J^2}{m^2}\right), \qquad (23)$$

where $a$ and $d$ are constants. Since the average resonance spin is only proportional to $m^{1/2}$, it follows that the multiplicity becomes essentially *independent* of $J$ for very large $m$. This can be quite simply understood. In this model, the massive resonances are built out of a large number of constituents, each of which carries a fixed average energy and angular momentum, after the fashion of a random walk.[8,17,18] Thus, as a first approximation, one expects the density of states as a function of spin to follow a Gaussian distribution, while the multiplicity of constituents is independent of the spin.

For clusters formed in a two-body collision, however, the average spin increases like $m$ (that is, if one makes the natural assumption that the average impact parameter for these collisions approaches a constant—which is equivalent to our assumption that the resonance widths are independent of $J$). Then the angular momentum dependence does have non-negligible effects. One expects that the cluster decay multiplicity still rises linearly with its mass, but at a slower rate than one would obtain by neglecting angular momentum conservation. The magnitude of this effect was roughly estimated to be of the order 20% for $N\bar{N}$ annihilation in flight.

The single-particle inclusive momentum distribution of the decay products was also considered. The distribution peaks in the forward and backward directions, and is symmetric[19] about 90°. It is expected to approach a *limiting shape,* because both the average spin of the cluster and its decay multiplicity increase linearly with the energy: Therefore the average angular momentum carried off by each final-state particle approaches a constant. The specific form obtained is given in Eq. (55).

Finally, some experimental tests were discussed. The prime example is $e^+e^-$ annihilation into hadrons via the one-photon process.[27,25] Since the total angular momentum remains fixed and small in this reaction as the energy increases, the piece of the cross section due to direct-channel resonance fluctuations (i.e., single-cluster formation) ought to be relatively large; and furthermore, angular momentum conservation should have negligible effect. Comparison of the single-particle inclusive distribution at low momenta with a thermodynamic form should provide a crucial test of the statistical bootstrap model.

The other main example touched upon was $N\bar{N}$ annihilation into hadrons.[5,31] In this case the effects of angular momentum conservation should be included as outlined above. Present experimental results seem to be in qualitative agreement with the predictions of the statistical bootstrap, but further comparisons remain to be done.

As a final comment, it is perhaps worth remarking that the resonance fluctuations and cluster formation phenomena which we have discussed might alternatively be described by a dual resonance model.[35] Several approaches of this sort have appeared already.[36] It will be of great interest in the future to compare and contrast the predictions of the two models,[37] and to see which has the greater success in fitting experiment.

ACKNOWLEDGMENTS

I would like to thank Jim Vary for supplying me with a least-squares fitting program, and the theory group at Brookhaven National Laboratory for discussions.

## APPENDIX A

In this appendix formulas for the angular momentum dependence of the resonance decay multiplicities and momentum distributions are derived, within the statistical bootstrap model. The methods to be used were described in a previous paper,[18] and employ techniques due in particular to Nahm[20] and Montvay.[17]

### 1. Decay multiplicity

First of all, let $\bar{\sigma}(E)$ represent the total density of single-resonance states of energy $E$ within the volume $V$, given by[38]

$$\bar{\sigma}(E)=\frac{V}{h^3}\int d^3p\int dm\,\rho(m)\,\delta(E-(m^2+\vec{p}^{\,2})^{1/2}), \tag{A1}$$

where $\rho(m)$ is the density of states with *mass* $m$. One can then project out[24] the density of states $\bar{\sigma}_{J_z}(E)$ with a specific spin component $J_z$ in some arbitrary direction $\hat{z}$—we will not need to know its specific form for the present.

Next, let $p(E, J_z, N)$ be the average probability that a state with energy $E$ and spin $J_z$ will ultimately decay into $N$ pions. The prototype statistical bootstrap equation for this quantity reads

$$\bar{\sigma}_{J_z}(E)p(E,J_z,N)=[\bar{\sigma}_{J_z}(E)p(E,J_z,N)]_{\text{in}} +\sum_{n=2}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int dE_i\sum_{J_z=-\infty}^{\infty}\sum_{N_i=0}^{\infty}\bar{\sigma}_{J_{zi}}(E_i)p(E_i,J_{zi},N_i)\delta\Big(E-\sum_i E_i\Big)\delta\Big(J_z-\sum_i J_{zi}\Big)\delta\Big(N-\sum_i N_i\Big), \tag{A2}$$

where the subscript "in" denotes an integral over input states, and the right-hand side includes a sum over $n=2,3,\ldots,\infty$ constituents of the bootstrap states. This equation can be solved by taking integral transforms with respect to $E$, $J_z$, and $N$. Define

$$\bar{\sigma}(E,h_J,h_N)=\sum_{J_z=-\infty}^{\infty}\sum_{N=0}^{\infty}\bar{\sigma}_{J_z}(E)\,p(E,J_z,N)\times e^{ih_JJ_z}e^{ih_NN}, \tag{A3}$$

take the Laplace transform

$$Z(\beta,h_J,h_N)=\int dE\, e^{-\beta E}\tilde{\sigma}(E,h_J,h_N), \qquad (A4)$$

and solve Eq. (A2); one then finds[18] that asymptotically

$$\tilde{\sigma}(E,h_J,h_N) \underset{E\to\infty}{\sim} c(h_J,h_N)E^{-3/2}\exp[\beta_0(h_J,h_N)E], \qquad (A5)$$

where the function $\beta_0(h_J,h_N)$ is determined by the sum rule

$$Z_{\rm in}(\beta_0(h_J,h_N),h_J,h_N)=\int dE\, e^{-\beta_0(h_J,h_N)E}\tilde{\sigma}_{\rm in}(E,h_J,h_N) \equiv 2\ln 2-1. \qquad (A6)$$

Now define a characteristic function for the associated spin and multiplicity distribution:

$$\psi(E,h_J,h_N)=\tilde{\sigma}(E,h_J,h_N)/\bar{\sigma}(E). \qquad (A7)$$

This can be expanded in the usual way[39] in terms of absolute moments $F_{l,m}(E)$ of the distribution,

$$\psi(E,h_J,h_N)=\sum_{l,m=0}^{\infty}F_{l,m}(E)\frac{(ih_J)^l}{l!}\frac{(ih_N)^m}{m!}, \qquad (A8)$$

and in terms of "cumulants" $f_{l,m}(E)$,

$$\psi(E,h_J,h_N)=\exp\left[\sum_{l,m=0}^{\infty}f_{l,m}(E)\frac{(ih_J)^l}{l!}\frac{(ih_N)^m}{m!}\right] \qquad [f_{0,0}(E)\equiv 0]. \qquad (A9)$$

It follows from Eq. (A5) that the asymptotic form of this function is

$$\psi(E,h_J,h_N) \underset{E\to\infty}{\sim} \exp\{[\beta_0(h_J,h_N)-\beta_0(0,0)]E\}. \qquad (A10)$$

Therefore

$$i^{-l-m}\frac{\partial^{l+m}\beta_0(h_J,h_N)}{\partial h_J{}^l\partial h_N{}^m}\bigg|_{h_J=h_N=0}=\lim_{E\to\infty}\left[\frac{f_{l,m}(E)}{E}\right] \equiv C_{l,m}. \qquad (A11)$$

Using Eqs. (A7)–(A11), one can substitute in Eqs. (A6) and expand in powers of $h_J$ and $h_N$. Comparing the two sides of (A6) term by term, one then obtains relationships between the *asymptotic* moments and partition averages[18] over the *input* spectrum, denoted by $\langle F_{l,m}\rangle$ as in Eq. (21). Finally, note that $E/m\to 1$ as $E\to\infty$ for the average state in the asymptotic spectrum. Hence we arrive at the final results:

$$C_{0,1}=\lim_{m\to\infty}\left[\frac{f_{0,1}(m)}{m}\right]=\frac{\langle N\rangle}{\langle E\rangle},$$

$$C_{1,0}=0,$$

$$C_{0,2}=\frac{1}{\langle E\rangle}\left[\langle N^2\rangle+\langle E^2\rangle\left(\frac{\langle N\rangle}{\langle E\rangle}\right)^2-\frac{2\langle NE\rangle\langle N\rangle}{\langle E\rangle}\right],$$

$$C_{2,0}=\frac{\langle J_z{}^2\rangle}{\langle E\rangle}, \qquad (A12)$$

$$C_{2,1}=\frac{1}{\langle E\rangle}\left[\langle J_z{}^2N\rangle-\frac{\langle EN\rangle\langle J_z{}^2\rangle}{\langle E\rangle}-\frac{\langle EJ_z{}^2\rangle\langle N\rangle}{\langle E\rangle}+\frac{\langle E^2\rangle\langle J_z{}^2\rangle\langle N\rangle}{\langle E\rangle^2}\right],$$

etc.

Neglecting all moments with $l>2$ and $m>1$, it follows that the spin distributions are Gaussian, so that one can write down the following forms: The number of states with spin projection $J_z$ is

$$\rho_{J_z}(m) \underset{m\to\infty}{\sim} \rho_{\rm tot}(m)\left[\frac{d_1(m)}{\pi}\right]^{1/2}\exp(-d_1(m)J_z{}^2), \qquad (A13)$$

correspondingly the number of states with *total* spin $J$ is[23,18]

$$\rho_J(m) \underset{m\to\infty}{\sim} \rho_{\rm tot}(m)\left[\frac{d_1{}^3(m)}{\pi}\right]^{1/2}(2J+1)^2 \times\exp(-d_1(m)J^2), \qquad (A14)$$

and the average decay multiplicity can be written

$$n(m,J) \underset{m\to\infty}{\sim} a(m)\exp(-d_2(m)J^2). \qquad (A15)$$

The cumulants of the associated spin and multiplicity distribution can be calculated from Eqs. (A13)–(A15), and one obtains

$$\begin{aligned} f_{0,1}(m) &\underset{m\to\infty}{\sim} a(m)\left[\frac{d_1(m)}{d_1(m)+d_2(m)}\right]^{3/2},\\ f_{2,0}(m) &\underset{m\to\infty}{\sim} \frac{1}{2d_1(m)},\\ f_{2,1}(m) &\underset{m\to\infty}{\sim} \tfrac{1}{2}a(m)\left[\frac{d_1(m)}{d_1(m)+d_2(m)}\right]^{3/2} \times\left[\frac{1}{d_1(m)+d_2(m)}-\frac{1}{d_1(m)}\right]. \end{aligned} \qquad (A16)$$

Comparing these equations with (A12), it follows that

$$d_1(m) \underset{m\to\infty}{\sim} \frac{1}{2C_{2,0}m}=\frac{1}{2m}\frac{\langle E\rangle}{\langle J_z{}^2\rangle},$$

$$a(m) \underset{m\to\infty}{\sim} C_{0,1}m=\frac{m}{\langle E\rangle}\langle N\rangle,$$

and

$$d_2(m) \underset{m\to\infty}{\sim} -\frac{1}{2m^2}\frac{C_{2,1}}{C_{0,1}(C_{2,0})^2} =\frac{1}{2m^2}\frac{\langle E\rangle^2}{\langle J_z{}^2\rangle}\left[\frac{\langle EN\rangle}{\langle E\rangle\langle N\rangle}+\frac{\langle EJ_z{}^2\rangle}{\langle E\rangle\langle J_z{}^2\rangle}-\frac{\langle J_z{}^2N\rangle}{\langle N\rangle\langle J_z{}^2\rangle}-\frac{\langle E^2\rangle}{\langle E\rangle^2}\right]. \qquad (A17)$$

This completes the required derivation of Eq. (23).

### 2. Momentum distribution

Next, let us consider the momentum distribution of decay products concomitant with a given angular momentum. Again, we use the same methods and notation used previously.[18] Let

$$\rho_N(E, J_z; \vec{p}_1, \ldots, \vec{p}_N)$$

be the average *inclusive* probability density that a single-resonance state with energy $E$ and spin projection $J_z$ will decay into a final state including $N$ pions with three-momenta $\vec{p}_1, \ldots, \vec{p}_N$ *plus* anything else. Write down a generating functional[40] for the momentum distribution:

$$\psi(E, J_z; [\phi]) = \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} d\vec{p}_i \, \rho_N(E, J_z; \vec{p}_1, \ldots, \vec{p}_N) \phi(\vec{p}_1) \cdots \phi(\vec{p}_N)$$
$$= \exp\left( \sum_{N=1}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} d\vec{p}_i \, \rho_N^c(E, J_z; \vec{p}_1, \ldots, \vec{p}_N) \phi(\vec{p}_1) \cdots \phi(\vec{p}_N) \right) , \tag{A18}$$

where the $\rho_N^c$ are correlation functions, and $\int d\vec{p}_i$ stands for the invariant momentum integral $\int d^3p_i / 2E_i$. Now, define

$$\bar{\sigma}(E, J_z; [\phi]) = \sigma_{J_z}(E) \psi(E, J_z; [\phi]), \tag{A19}$$

and carry out a Fourier transform with respect to $J_z$ and a Laplace transform with respect to $E$ as in Appendix A1:

$$Z(\beta, h; [\phi]) = \int dE \, e^{-\beta E} \sum_{J_z} e^{ihJ_z} \bar{\sigma}(E, J_z; [\phi]), \tag{A20}$$

where $h$ is the parameter previously designated $h_J$ and should not be confused with Planck's constant. The statistical bootstrap equation for this quantity will have exactly the same form as previously,[18] and gives rise to a similar solution:

$$\tilde{\sigma}(E, h; [\phi]) \underset{E \to \infty}{\sim} c(h, [\phi]) E^{-3/2} \exp(\beta_0(h, [\phi]) E), \tag{A21}$$

where the functional $\beta_0(h, [\phi])$ is determined by the sum rule

$$Z_{\text{in}}(\beta_0(h, [\phi]), h; [\phi]) = \int dE \, e^{-\beta_0(h, [\phi]) E} \tilde{\sigma}_{\text{in}}(E, h; [\phi])$$
$$\equiv 2 \ln 2 - 1 \tag{A22}$$

From Eq. (A21) it follows that the transformed density of states is asymptotically given by

$$\tilde{\sigma}(E, h; [\phi]) \underset{E \to \infty}{\sim} \bar{\sigma}(E) \exp\{(\beta_0(h, [\phi]) - \beta_0) E\} \tag{A23}$$

[where $\beta_0 \equiv \beta_0(0, 0)$], and performing the inverse Fourier transform with respect to $h$, we see that

$$\bar{\sigma}(E, J_z; [\phi]) \underset{E \to \infty}{\sim} \bar{\sigma}(E) \frac{1}{2\pi} \times \int_{-\pi}^{\pi} dh \, e^{-ihJ_z} \exp\{(\beta_0(h, [\phi]) - \beta_0) E\}. \tag{A24}$$

The single-particle inclusive distribution, by virtue of Eq. (A18), can now be found by taking a functional derivative with respect to $\phi(\vec{p}_1)$:

$$\rho_1(E, J_z; \vec{p}_1) \underset{E \to \infty}{\sim} \frac{\bar{\sigma}(E)}{\bar{\sigma}_{J_z}(E)} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} dh \, e^{-ihJ_z} E \frac{\delta \beta_0(h, [\phi])}{\delta \phi(\vec{p}_1)} \bigg|_{\phi=0} \times \exp((\beta_0(h) - \beta_0) E). \tag{A25}$$

Now for $E$ very large and $J_z$ finite, we need only consider very small values of $h$. The integrand in the above equation has a saddle point where

$$E \frac{\partial \beta_0(h)}{\partial h} - i J_z = 0 , \tag{A26}$$

i.e.,

$$h = \frac{i J_z}{E \partial^2 \beta_0 / \partial h^2 |_{h=0}} = \frac{-i J_z \langle E \rangle}{E \langle J_z^2 \rangle} \equiv h_0. \tag{A27}$$

Distorting the contour of integration, and performing a saddle-point integral, it follows that

$$\rho_1(E, J_z; \vec{p}_1) \underset{E \to \infty}{\sim} \frac{\bar{\sigma}(E)}{\bar{\sigma}_{J_z}(E)} \frac{E \delta \beta_0(h_0, [\phi])}{\delta \phi(\vec{p}_1)} \bigg|_{\phi=0} \left( \frac{\langle E \rangle}{2\pi \langle J_z^2 \rangle E} \right)^{1/2} \exp\left( - \frac{J_z^2 \langle E \rangle}{2E \langle J_z^2 \rangle} \right) \tag{A28}$$

$$= \frac{E \delta \beta_0(h_0, [\phi])}{\delta \phi(\vec{p}_1)} \bigg|_{\phi=0} . \tag{A29}$$

Now $\delta\beta_0(h_0, [\phi]) / \delta\phi(\vec{p}_1)$ can be found from the sum rule (A22).[18] Performing a functional differentiation, one obtains

$$\langle E\rangle \frac{\delta\beta_0(h_0,[\phi])}{\delta\phi(\vec{p}_1)}\bigg|_{\phi=0} = \langle \tilde{\rho}_1(p,h_0;\vec{p}_1)\rangle \,. \quad \text{(A30)}$$

In the simple case where the input spectrum consists of a single neutral "pion," we have[38]

$$\langle \tilde{\rho}_1(p,h_0;\vec{p}_1)\rangle$$

$$= \frac{1}{Z_{\rm in}(\beta_0(h_0),h_0)} \int dE\, e^{-\beta_0(h_0)E} \int dm\, \delta(m-m_\pi)\frac{V}{h^3}\int d^3p \sum_M e^{ih_0M} P_M[2E_1\delta^3(\vec{p}-\vec{p}_1)]\,\delta(E-(m_\pi{}^2+p^2)^{1/2}) \quad \text{(A31)}$$

$$\to \frac{1}{Z_{\rm in}(\beta_0(h_0),h_0)}\frac{1}{h^3}\int d^3r\, e^{-r^2/R^2}\, e^{-\beta_0(h_0)(m_\pi{}^2+\vec{p}_1{}^2)^{1/2}}\; 2(m_\pi{}^2+\vec{p}_1{}^2)^{1/2}\sum_M e^{ih_0M}P_M \,, \quad \text{(A32)}$$

where $P_M$ is an operator which projects out states of angular momentum $M$ in the $z$ direction, and we have assumed a Gaussian "shape" for the volume $V$ as in Sec. IV. A more realistic input spectrum will lead to a somewhat more complicated form for the single-particle distribution. We restrict ourselves to the simple form (A32) for the purposes of discussion.

## APPENDIX B

Here we shall try to estimate some asymptotic parameters for a "realistic" statistical bootstrap model (case 5 of Ref. 8). This introduces one important complication: One has to insert a restriction that forbids states with exotic SU(3) quantum numbers, which results in more involved bootstrap equations. To reduce the complication we shall here deal only with mesons, and ignore the existence of baryons. Following Hamer and Frautschi,[8] we shall take as input states the $J^P=0^-$ and $1^-$ meson nonets (namely $\pi$, $K$, $\eta$, $\eta'$, $\rho$, $K^*$, $\omega$, and $\phi$), and we shall allow only singlet and octet SU(3) states to be generated.

To treat this problem, we again follow the methods of Ref. 18. Whereas for the unrestricted case the statistical bootstrap equation for the partition function $Z(\beta)$ reads[20]

$$Z_{\rm in}(\beta) = 1 + 2Z(\beta) - \exp[Z(\beta)]\,, \quad \text{(B1)}$$

in the present case, the corresponding equation will be

$$Z^i_{\rm in}(\beta) = Z^i(\beta) - \tfrac{1}{2}c^{ijk}Z^j(\beta)Z^k(\beta) - \frac{1}{3!}c^{ijk}Z^j(\beta)c^{klm}Z^l(\beta)Z^m(\beta) - \cdots, \quad \text{(B2)}$$

where the indices $i,j,k,\ldots$ run from 1 to 4, denoting the four possible SU(3) isospin multiplets:

$[\{\underline{1}\}, I=0, Y=0]$,

$[\{\underline{8}\}, I=0, Y=0]$,

$[\{\underline{8}\}, I=\tfrac{1}{2}, Y=-1]$,

$[\{\underline{8}\}, I=1, Y=0]$.

The coefficients $c^{ijk}$ are the squares of the appropriate SU(3) isoscalar factors.[8] The partition functions $Z^i(\beta)$ will have a square-root singularity[20] at some $\beta\equiv\beta_0$, and in that vicinity power-series expansions may be made[18]:

$$Z^i_{\rm in}(\beta) = \sum_{n=0}^\infty b^i_n s^n\,,$$
$$Z^i(\beta) = \sum_{n=0}^\infty a^i_n s^n + s^{1/2}\sum_{n=0}^\infty c^i_n s^n\,, \quad \text{(B3)}$$

where

$$s=\beta-\beta_0\,.$$

Substituting into Eq. (B2) and solving term by term, one obtains

$$b^i_0 = a^i_0 - \tfrac{1}{2}c^{ijk}a^j_0a^k_0 - \frac{1}{3!}c^{ijk}a^j_0c^{klm}a^l_0a^m_0 - \cdots, \quad \text{(B4)}$$

$$0 = c^i_0 - c^j_0c^{ijk}a^k_0 - \frac{1}{3!}[c^j_0c^{ijk}c^{klm}a^l_0a^m_0 + 2c^l_0c^{ijk}a^j_0c^{klm}a^m_0] - \cdots, \quad \text{(B5)}$$

etc.

(we make use of the symmetry $c^{ijk}=c^{ikj}$). Now define the matrix

$$A^{ij} = \delta^{ij} - c^{ijk}a^k_0 - \frac{1}{3!}(c^{ijk}c^{klm}a^l_0a^m_0 + 2c^{ilk}a^l_0c^{kjm}a^m_0) - \cdots\,, \quad \text{(B6)}$$

Then the condition for Eq. (B5) to have a solution is that

$$\det A = 0\,. \quad \text{(B7)}$$

Furthermore, once the volume and the set of input states are specified, the coefficients $b^i_0 = Z_{\rm in}(\beta_0)$ are a function of $\beta_0$ alone. Therefore, Eqs. (B4) and (B7) form a set of five simultaneous equations in five unknowns, $a^i_0$ ($i=1,\ldots,4$) and $\beta_0$. These equations can be solved by a numerical least-squares fitting program. Then the coefficients $c^i_0$, and higher coefficients, can be found by ordinary matrix algebra. Thus a solution to the Eqs. (B2) can be obtained.

Next, one wants to know the asymptotic moments of various distributions over the resonance states. Take the multiplicity distribution, for example. Transforming with respect to both the multiplicity and energy as usual,[18] one forms the partition function

$$Z^i(\beta,h)=\int dE\,e^{-\beta E}\int dm\,\rho^i(m)\frac{V}{h^3}\int d^3p\sum_{N=0}^{\infty}p^i(m,N)e^{ihN}\delta(E-(m^2+\vec{p}^2)^{1/2}),\tag{B8}$$

where $p(m,N)$ is the probability that a resonance of mass $m$ will decay into $N$ pions.

The statistical bootstrap equation for this quantity is the same as (B2), and again leads to Eqs. (B4) and (B5), where the coefficients $a_n$, $b_n$, and $c_n$ are all now functions of $h$. Next, expand in powers of $h$.[18] The zeroth-order term gives the density of states, as above. The coefficients of the first power of $h$ can be equated in (B4) and (B5), giving

$$\left.\frac{\partial b_0^i}{\partial h}\right|_{h=0}=\left[\frac{\partial a_0^i}{\partial h}-\frac{\partial a_0^j}{\partial h}c^{ijk}a_0^k\right]_{h=0}=\left.A^{ij}\frac{\partial a_0^j}{\partial h}\right|_{h=0},\tag{B9}$$

$$\left.\frac{\partial}{\partial h}(\det A)\right|_{h=0}=0\,.\tag{B10}$$

Now

$$b_0^i(h)=\int dE\,e^{-\beta_0(h)E}\int dm\,\rho_{\rm in}^i(m)\frac{V}{h^3}\int d^3p\sum_{N=0}^{\infty}p_{\rm in}^i(m,N)e^{ihN}\delta(E-(m^2+\vec{p}^2)^{1/2}),\tag{B11}$$

so

$$\left.\frac{\partial b_0^i}{\partial h}\right|_{h=0}=\left[i\langle N\rangle^i-\frac{\partial\beta_0(h)}{\partial h}\langle E\rangle^i\right]b_0^i(0)\,.\tag{B12}$$

Therefore (B9) becomes

$$\langle N\rangle^i=\langle E\rangle^i\left.i^{-1}\frac{\partial\beta_0}{\partial h}\right|_{h=0}+\frac{A^{ij}}{b_0^i}\left.i^{-1}\frac{\partial a_0^j}{\partial h}\right|_{h=0}.\tag{B13}$$

After Eqs. (B4) and (B7) have been solved, Eqs. (B10) and (B13) form a set of five simultaneous *linear* equations in the five unknowns $i^{-1}\partial\beta_0/\partial h$, $i^{-1}\partial a_0^j/\partial h$ ($j=1,\ldots,4$), which can be solved by ordinary algebraic means. Thus one deduces immediately the first moment of the asymptotic multiplicity distribution, as in Appendix A:

$$C_{0,1}=i^{-1}\frac{\partial\beta_0}{\partial h}=\lim_{m\to\infty}\left[\frac{f_1^N(m)}{m}\right].\tag{B14}$$

Similarly, one can go back and look at second-order terms in $h$ to find the second cumulant of the multiplicity distribution, and other distributions can be treated in the same way. We shall not inflict any further details on the reader.

The results of these calculations are shown in Figs. 2–4. After we have chosen a definite set of input states, the only adjustable parameter remaining in the model is the resonance volume $V$, or the corresponding radius $R'$ ($V=\frac{4}{3}\pi R'^3$). Two additional choices were made in order to carry out the calculations. The first decision was to truncate the exponential series in Eq. (B2) at the third term; that is, we allow only 2- and 3-particle states in the bootstrap.[8] Secondly, the resonance volume was assumed to have a Gaussian "shape" as in Sec. IV. This affects the calculation of spin distributions; for instance, it leads to the result[18]

$$\langle J_z^2\rangle=\tfrac{1}{3}(\langle S_z^2\rangle+R^2\langle p^2\rangle)\tag{B15}$$

(note the distinction between $R$ and $R'$). Alteration of these two decisions is not likely to affect the results by more than a few percent.

In Fig. 2 the coefficients $a_0^i$ ($i=1,\ldots,4$) are plotted against $R'$. Figure 3 shows the inverse temperature $\beta_0$, and its derivatives as defined in Appendix A, as a function of $R'$—these derivatives

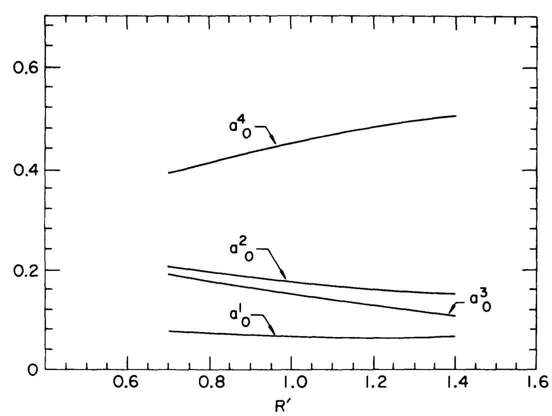


FIG. 2. Coefficients $a_0^i$ characterizing the density of states, as a function of the resonance radius $R'$ [cf. Eq. (B.3)]. Units $\hbar=c=m_\pi=1$.

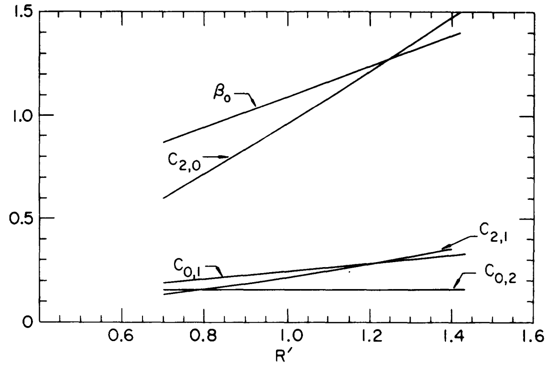


FIG. 3. The inverse temperature $\beta_0$, and its derivatives $C_{l,m}$ (as defined in Appendix A) characterizing the spin distribution, and the multiplicity distribution of pions produced in resonance decay. All quantities are plotted as functions of the resonance radius $R'$. Units $\hbar = c = m_\pi = 1$.

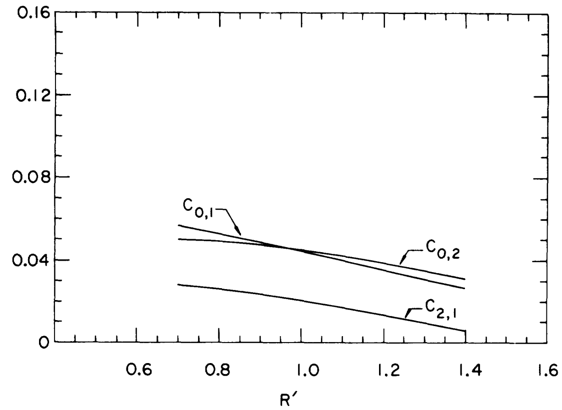


FIG. 4. The derivatives $C_{l,m}$ characterizing the multiplicity distribution of *kaons* produced in resonance decay, as a function of the radius $R'$. Units $\hbar = c = m_\pi = 1$.

characterize the spin distribution, and the distribution of final-state pions resulting from resonance decay. Figure 4 shows the same quantities for the final-state *kaon* distribution.

It is remarkable that whereas $C_{0,1}$ for pions (that is, the multiplicity divided by $m$) increases with the radius $R'$ as one would expect, the multiplicity for kaons actually *decreases*. This can be quite easily understood, however. As the radius increases, the "temperature" $\beta_0^{-1}$ decreases (Fig. 3), favoring states with low energy. The average energy of the emitted particles is already less than the kaon mass, and it becomes harder and harder to emit a kaon (or kaon pair) as opposed to a pion. Hence the multiplicity of kaons decreases.

*Work performed under the auspices of the U. S. Atomic Energy Commission.

[1]N. Bohr, Nature 137, 344 (1936); N. Bohr and F. Kalckar, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 14, 10 (1937). For an up-to-date review, see E. Vogt, Advan. Nucl. Phys. 1, 261 (1968).

[2]H. A. Bethe, Phys. Rev. 50, 332 (1936); Rev. Mod. Phys. 9, 69 (1937). For a modern review see T. Ericson, Adv. Phys. 9, 425 (1960).

[3]T. E. O. Ericson, Ann. Phys. (N.Y.) 23, 390 (1963); T. E. O. Ericson and T. Mayer-Kuckuk, Annu. Rev. Nucl. Sci. 16, 183 (1966).

[4]S. Frautschi, Nuovo Cimento 12A, 133 (1972).

[5]C. J. Hamer, Nuovo Cimento 12A, 162 (1972).

[6]B. Margolis, W. J. Meggs, and R. K. Logan, Phys. Rev. D 8, 199 (1973).

[7]R. Hagedorn, Nuovo Cimento Suppl. 3, 147 (1965); R. Hagedorn and J. Ranft, *ibid.* 6, 169 (1968).

[8]S. C. Frautschi, Phys. Rev. D 3, 2821 (1971); C. J. Hamer and S. C. Frautschi, *ibid.* 4, 2125 (1971).

[9]W. Nahm, Bonn University Report No. PI 2-142, 1973 (unpublished). See also G. Matthiae, Nucl. Phys. B7, 142 (1968).

[10]R. Hagedorn and I. Montvay, Nucl. Phys. B59, 45 (1973).

[11]E. Beth and G. E. Uhlenbeck, Physica 4, 915 (1937); S. Z. Belenky, Nucl. Phys. 2, 259 (1956).

[12]R. Dashen, S. Ma, and H. J. Bernstein, Phys. Rev. 187, 345 (1969).

[13]L. Sertorio and M. Toller, Nuovo Cimento 14A, 21 (1973); A. Bassetto and L. Sertorio, *ibid.* 14A, 548(1973).

[14]Particle Data Group, Rev. Mod. Phys. 45, S1 (1973).

[15]F. H. Schmidt, C. Baglin, P. J. Carlson, A. Eide, V. Gracco, E. Johansson, and A. Lundby, Phys. Lett. 45B, 157 (1973).

[16]C. B. Chiu, Nucl. Phys. B54, 170 (1973); C. B. Chiu and J. Johnsen, Phys. Lett. 42B, 475 (1972); S. C. Frautschi and C. J. Hamer, Nuovo Cimento 13A, 645 (1973); E. M. Ilgenfritz and J. Kripfganz, Nucl. Phys. B56, 241 (1973); *ibid.* B62, 141 (1973).

[17]I. Montvay, Phys. Lett. 42B, 466 (1972); Nucl. Phys. B53, 521 (1973).

[18]C. J. Hamer, Phys. Rev. D 8, 3558 (1973).

[19]This was already recognized by Fermi twenty years ago. See E. Fermi, Phys. Rev. 81, 683 (1951).

[20]W. Nahm, Nucl. Phys. B45, 525 (1972).

[21]J. Yellin, Nucl. Phys. B52, 583 (1973).

[22]We apologize for using angular brackets with two different meanings within this paper, and trust that no confusion will arise thereby.

[23]C. B. Chiu and R. L. Heimann, Phys. Rev. D 4, 3184 (1971).

[24]F. Cerulus, Nuovo Cimento 22, 958 (1961); H. Satz, Fortschr. Phys. 11, 445 (1963).

[25]J. Engels, K. Schilling, and H. Satz, Nuovo Cimento 17A, 535 (1973).

[26]A. Bramòn, E. Etim, and M. Greco, Phys. Lett. 41B, 609 (1972); M. Greco, Nucl. Phys. B63, 398 (1973); J. J. Sakurai, Phys. Lett. 46B, 207 (1973).

[27]J. D. Bjorken and S. Brodsky, Phys. Rev. D 1, 1416 (1970).

[28]E. Fermi, Prog. Theor. Phys. 1, 570 (1950).
[29]B. Margolis, W. J. Meggs, and S. Rudaz, Phys. Rev. D 8, 3944 (1973).
[30]See, for instance, the review by J. Vandermeulen, in *Symposium on Nucleon-Antinucleon Annihilations*, edited by L. Montanet (CERN, Geneva, 1972), p. 113.
[31]Treatments which are similar in spirit to ours have already been given by M. Jacob and S. Nussinov, Nuovo Cimento 14A, 335 (1973); S. J. Orfanidis and V. Rittenberg, Nucl. Phys. B59, 570 (1973).
[32]T. Fields *et al.*, Argonne Report No. ANL/HEP 7223, 1972 (unpublished).
[33]B. Y. Oh, P. S. Eastman, Z. Ming Ma, D. L. Parker, G. A. Smith, R. J. Sprafka, Nucl. Phys. B63, 1 (1973).
[34]J. Erwin, W. Ko, R. L. Lander, D. E. Pellett, and P. M. Yager, Phys. Rev. Lett. 27, 1534 (1971).
[35]G. Veneziano, Nuovo Cimento 57A, 190 (1968). For a recent review, see J. H. Schwarz, Phys. Rep. 8C, No. 4 (1973).
[36]E.g., T. Eguchi and K. Igi, Phys. Rev. D 8, 1537 (1973); L. Masperi and C. Rebbi, Nuovo Cimento 13A, 689 (1973).
[37]V. A. Miransky, V. P. Shelest, B. V. Struminsky, and G. M. Zinoviev, Phys. Lett. 43B, 73 (1973); H. Satz, Nuovo Cimento Lett. 4, 910 (1972).
[38]We use the normal "noncovariant" phase-space expression, rather than the "covariant" form used previously (Ref. 18). To *leading order* in $E$, similar methods and results apply in both cases (Ref. 20).
[39]E.g., B. V. Gnedenko, *Theory of Probability* (Chelsea, New York, 1962), Chap. VII.
[40]L. S. Brown, Phys. Rev. D 5, 748 (1972); S.-S. Shei and T.-M. Yan, *ibid.* 6, 1744 (1972); Z. Koba, H. B. Nielsen, and P. Olesen, Nucl. Phys. B43, 125 (1972); K. J. Biebl and J. Wolf, *ibid.* B44, 301 (1972).

PHYSICAL REVIEW D VOLUME 9, NUMBER 9 1 MAY 1974

# Nucleon-nucleon scattering near 50 MeV. II. Sensitivity of various $n$-$p$ observables to the phase parameters*

Judith Binstock and Ronald Bryan†
*Department of Physics and Cyclotron Institute, Texas A&M University, College Station, Texas 77843*
(Received 10 August 1972)

In the first paper in this series, we reported on a phase-shift analysis of existing $p$-$p$ and $n$-$p$ data in the energy range of 47.5 to 60.9 MeV. Two results were emphasized. The first is that the available $n$-$p$ data leave $\epsilon_1$ undetermined within the range $-10°$ to $+3°$, resulting in a range of phase-parameter solutions, rather than a single solution. The second result is that while $\epsilon_1$ is very poorly determined, $\delta(^1P_1)$ is rather well determined, but at a value which appears to conflict not only with values obtained at adjacent energies, but also with the value (or narrow range of values) predicted by meson-theoretical models. In that paper it is reported that the Harwell $n$-$p$ $d\sigma/d\Omega$ data are responsible for this value of $\delta(^1P_1)$. The remaining data, consisting only of $\sigma_{\text{tot}}$ data, polarization data, and other $d\sigma/d\Omega$ data, are consistent with the theoretical predictions. In this paper we look more closely at the sensitivity of experimental observables to variations in the partial-wave parameters. We extend the number of experimental observables under study to twenty, and consider the effect on these of varying seven different phase parameters: $\delta(^1S_0)_{np}$, $\delta(^3S_1)$, $\epsilon_1$, $\delta(^1P_1)$, $\delta(^3D_1)$, $\delta(^3D_2)$, and $\delta(^3D_3)$. We discover that the best observable to fix $\delta(^1P_1)$ is still the differential cross section, and recommend, as in the first paper, that it be measured both at extreme forward and extreme backward angles. We also discover that the reason $\epsilon_1$ is very poorly determined by the present data is that neither $\sigma_{\text{tot}}$, $d\sigma/d\Omega$, nor $P$ is sensitive to changes in $\epsilon_1$. We find that the experimental observables which are sensitive to $\epsilon_1$ and can fix this parameter are, in order of decreasing sensitivity, $A_{zz}$, $C_{pp}$, $A_t'$, $C_{KK}$, $A_t$, $D_t$, $C_{nn}$, and $A_{xx}$.

## I. INTRODUCTION

In a paper by Arndt, Binstock, and Bryan,[1] hereafter referred to as paper I, a phase-shift analysis of $n$-$p$ plus $p$-$p$ elastic-scattering data in the laboratory energy range 47.5–60.9 MeV was carried out. Charge independence was assumed for all but $\delta(^1S_0)$, and $F$ waves and higher partial waves were set to the OPEC (one-pion-exchange contribution) values. It was found that the available $n$-$p$ data leave $\epsilon_1$ undetermined within the range $-10°$ to $+3°$, resulting in a range of phase parameter solutions rather than a single solution. Furthermore, although $\epsilon_1$ was poorly determined, $\delta(^1P_1)$ was found to be rather well determined by the data, but at an anomalous value. In particular, for $\epsilon_1$ fixed at a reasonable 50-MeV value of $+2.78°$ (taken from Ref. 2), $\delta(^1P_1)$ searched to $-3.52 \pm 1.04°$ at 50 MeV, in conflict both with theoretical expectations of