PHYSICAL REVIEW D VOLUME 9, NUMBER 9 1 MAY 1974

Possible relationship between the ratio π^+/π^- and the average multiplicity*

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(Received 21 January 1974)

If charge distribution in a collision process becomes "statistical," then the ratio π^+/π^- could be used to measure the average multiplicity. A detailed mathematical analysis is given as well as a discussion of physical limitations.

I. MOTIVATION

Recent high-energy pp and p-nucleus collision experiments[1] at the CERN ISR and at NAL have shown that for large $p_\perp$ the ratio π^+/π^- is always close to but greater than unity. This result is not *a priori* surprising since the collision presumably produces several pions and a statistical distribution of charges would produce a π^+/π^- ratio close to unity. The positive charge excess in the collision then explains why π^+/π^- is greater than unity. We want in the present paper[2] to pursue this line of reasoning, defining what the heuristic concept of "statistical" distribution means *precisely* and raising the possibility of using the observed ratios such as π^+/π^-, p/n, and $2\pi^0/(\pi^+ + \pi^-)$ for possible information on the average multiplicity of pions.

Before proceeding, it must be emphasized that for some collision processes charge distribution cannot be statistical. In fact, in typical fragmentation processes, $\vec{\mathrm{I}}$ transfer (hence charge transfer) from one fragmenting particle to the other is strongly inhibited. In such processes, therefore, there cannot be an over-all "statistical" charge distribution. The consideration of this paper therefore only applies to a "violent" collision, and one hopes that for an inelastic process with a particle emitted with a $p_\perp$ greater than, say, 2 GeV/c the collision is violent and statistical distribution of charges and $\vec{\mathrm{I}}$ obtains.

II. STATISTICAL CHARGE DISTRIBUTION WITHOUT $\vec{\mathrm{I}}$ CONSERVATION

Consider the process

$$pp \xrightarrow{\text{violent}} NN(\pi)^l \ . \tag{1}$$

For a fixed final-momenta distribution of the nucleons and the pions, the possible charge states are listed in Table I for small values of l. Assuming (for fixed l) each of these states to have equal probability, the evaluation of the ratios π^+/π^-, p/n, etc. is straightforward. (Notice that the assumption of equal probability is the *only* reasonable precise formulation of the heuristic idea of statistical distribution.)

For larger values of l, these ratios are evaluated with the computer, and they are tabulated in Table II. It will be proved in Appendix A that

$$\frac{\pi^+}{\pi^-} = 1 + 3(l-1)^{-1} \ . \tag{2}$$

The asymptotic form of other ratios will be derived in Appendix A. They are

$$\frac{p}{n} = 1 + \frac{3}{2l} - \frac{9}{16l^2} + \cdots \ , \tag{3}$$

$$\frac{2\pi^0}{\pi^+ + \pi^-} = 1 + \frac{3}{4l} - \frac{3}{l^2} + \cdots \ . \tag{4}$$

III. STATISTICAL CHARGE DISTRIBUTION WITH $\vec{\mathrm{I}}$ CONSERVATION

For process (1) the total $\vec{\mathrm{I}}$ is unity and the total I_z is unity. That should be taken into account in the statistical consideration. There is a *unique* way of doing this. Let $O(I, I_z)$ be the projection operator for the state of the $NN(\pi)^l$ system, so that in the representation where the total I and total I_z are diagonal $O(J, J_z)$ is diagonal and is equal to 1 or 0 according to whether the equations

$$I = J \ ,$$
$$I_z = J_z$$

are valid or not.

The operator $O(1, 1)$ is then the density matrix we want for the statistical ensemble on the right-hand side of (1), so that the average of any operator A, such as the charge of each pion, is

$$[\mathrm{Tr} A O(1,1)][\mathrm{Tr}\, O(1,1)]^{-1} \ .$$

Notice that this quantity is independent of the representation we choose.

The evaluation of the averages π^+/π^-, etc. will be detailed in Appendix B. The result is tabulated in Table III and graphed in Fig. 1. Asymptotic forms for large l are derived there also, yielding

$$\frac{\pi^+}{\pi^-} = 1 + \frac{3}{l} + \frac{9}{4l^2} + \frac{81}{32l^3} + \cdots \ , \tag{5}$$

$$\frac{p}{n} = 1 + \frac{3}{2l} - \frac{9}{16l^2} + \cdots \ , \tag{6}$$

TABLE I. Calculation of mean numbers of π^+, π^0, and π^- for $l = 1$, 2, and 3, neglecting isospin conservation.

l	Possible final-state charge configurations	Number of possible charge configurations $N_{l,0}+2N_{l,1}+N_{l,2}$	Mean number of π^+'s $=\pi^+(l)$	Mean number of π^0's $=\pi^0(l)$	Mean number of π^-'s $=\pi^-(l)$
1	$pp0$ $pn+$ $np+$	3	$\frac{2}{3}$	$\frac{1}{3}$	0
2	$pp+-$ $pp-+$ $pp00$ $pn+0$ $pn0+$ $np+0$ $np0+$ $nn++$	8	1	$\frac{3}{4}$	$\frac{1}{4}$
3	$pp+-0$ $pp+0-$ $pp-+0$ $pp-0+$ $pp0+-$ $pp0-+$ $pp000$ $pn++-$ $pn+-+$ $pn+00$ $pn-++$ $pn0+0$ $pn00+$ $np++-$ $np+-+$ $np+00$ $np-++$ $np0+0$ $np00+$ $nn++0$ $nn+0+$ $nn0++$	22	$\frac{15}{11}$	$\frac{12}{11}$	$\frac{6}{11}$

TABLE II. Numerical calculation for various mean numbers and their ratios when isospin conservation is ignored.

l	π^+	π^0	π^-	p	n	π^+/π^-	p/n	$2\pi^0/(\pi^+ + \pi^-)$
0	0	0	0	2	0			
1	0.67	0.33	0	1.33	0.67		2.00	1.00
2	1.00	0.75	0.25	1.25	0.75	4.00	1.67	1.20
3	1.36	1.09	0.55	1.18	0.82	2.50	1.44	1.15
4	1.71	1.44	0.85	1.15	0.85	2.00	1.35	1.13
5	2.05	1.78	1.17	1.12	0.88	1.75	1.27	1.11
6	2.39	2.12	1.49	1.11	0.89	1.60	1.24	1.10
7	2.72	2.46	1.82	1.09	0.91	1.50	1.20	1.09
8	3.06	2.80	2.14	1.08	0.92	1.43	1.18	1.08
9	3.40	3.14	2.47	1.07	0.93	1.38	1.16	1.07
10	3.73	3.47	2.80	1.07	0.93	1.33	1.14	1.06

$$\frac{2\pi^0}{\pi^+ + \pi^-} = 1 - \frac{9}{16 l^2} + \cdots . \tag{7}$$

A comparison of these equations with (2)–(4) reveals that the most important change introduced by isospin conservation is the elimination of the l^{-1} term in the ratio $2\pi^0/(\pi^+ + \pi^-)$ in going from (4) to (7). This elimination is more explicitly traceable to the difference between the value of $\pi^0(l)$ in (A9) and (B14). We shall return to this topic at the end of Appendix B.

IV. DISCUSSION

Equations (5) and (7) [or, better, (B14)] can probably be used to yield information on the multiplicity l of pions. In particular one finds, after

TABLE III. Numerical calculation for various mean numbers and their ratios when isospin conservation is considered.

l	π^+	π^0	π^-	p	n	π^+/π^-	p/n	$2\pi^0/(\pi^+ + \pi^-)$
0	0	0	0	2	0			
1	0.75	0.25	0	1.25	0.75		1.67	0.667
2	1.05	0.65	0.30	1.25	0.75	3.50	1.67	0.963
3	1.43	0.97	0.60	1.17	0.83	2.38	1.40	0.951
4	1.77	1.31	0.91	1.14	0.86	1.94	1.34	0.979
5	2.12	1.65	1.24	1.12	0.88	1.73	1.26	0.982
6	2.46	1.98	1.56	1.10	0.90	1.58	1.23	0.988
7	2.80	2.32	1.88	1.09	0.91	1.48	1.20	0.991
8	3.13	2.65	2.21	1.08	0.92	1.42	1.18	0.993
9	3.47	2.99	2.54	1.07	0.93	1.38	1.16	0.994
10	3.81	3.32	2.87	1.07	0.93	1.33	1.14	0.995

averaging over all l both sides of (B14),

$$\frac{\langle\pi^+\rangle}{\langle\pi^-\rangle} = 1 + \frac{3}{\langle l\rangle} + O\left(\frac{1}{\langle l\rangle}\left\langle\frac{1}{l}\right\rangle\right), \tag{8}$$

where the left-hand side is the π^+/π^- ratio observed at a fixed large $p_\perp$, and where $\langle l\rangle$ = average number of pions in a collision in which one large $p_\perp$ pion is observed.

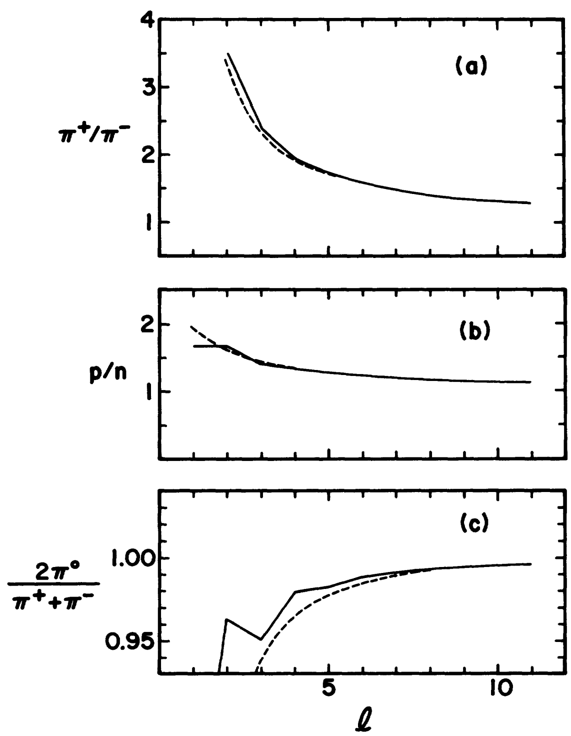

FIG. 1. (a) π^+/π^- vs l, (b) p/n vs l, (c) $2\pi^0/(\pi^+ + \pi^-)$ vs l. The solid curves are exact calculations found in Appendix B. The dashed curves are their asymptotic limits [i.e., Eqs. (5), (6), and (7)]. Isospin conservation is considered in all three cases. Note the scale in (c).

Notice that, to the accuracy indicated, (8) is also correct if one neglects isospin conservation, as is obvious from (A9). A simpler argument leading to (8), without detailed mathematical analysis, is as follows. For large l, the average charge of each nucleon is evidently $\frac{1}{2} + O(l^{-1})$. Thus by charge conservation *the charge of the l pions is on the average $1 + O(l^{-1})$*. It follows then from the obvious relations

$$\pi^+ + \pi^0 + \pi^- = l, \quad \pi^+ - \pi^- = 1 + O(l^{-1})$$

that

$$\pi^\pm = \tfrac{1}{2}(l - \pi^0) \pm \tfrac{1}{2} + O(l^{-1}) .$$

Further, it is clear that

$$\pi^0 = \tfrac{1}{3} l + O(1) .$$

Equation (8) then follows.

What happens if another nucleon-antinucleon pair is produced, i.e., for the process $pp \to NNN\bar{N}(\pi)^l$? The average net charge of the additional pair $(N\bar{N})$ is zero. Hence, by the argument in italic type above, Eq. (8) remains valid.

What happens if strangeness $\neq 0$ mesons and hyperons are produced, e.g., for the process $pp \to N\Lambda K(\pi)^l$? The average net charge of $N(\Lambda K)$ remains $1 + O(l^{-1})$, since N and K each contribute $\frac{1}{2}$ and Λ contributes 0. Thus the average net charge of the $(\pi)^l$ system remains $1 + O(l^{-1})$ and (8) still obtains.

In short, if the number of antinucleons and strange particles in the whole collision is much smaller than the number of pions, (8) remains valid. This seems a safe assumption, and (8) could be used to estimate the number $\langle l\rangle$ of pions emitted in the pp collision if it were known that one pion is emitted with a large $p_\perp$.

APPENDIX A

This is an appendix to Sec. II. We first define a quantity $N_{l,q}$ which is the number of possible ways to distribute total charge q over l pions. We obviously have

$$\pi^+(l, q) + \pi^0(l, q) + \pi^-(l, q) = l \tag{A1}$$

and

$$\pi^+(l, q) - \pi^-(l, q) = q , \tag{A2}$$

where $\pi^+(l, q)$ is defined to be the average number of π^+'s per state in these $N_{l,q}$ states. Similarly we define $\pi^{0,-}(l, q)$. The generating function for $N_{l,q}$ is

$$(x + 1 + x^{-1})^l = \sum_{q=-l}^{l} N_{l,q} x^q , \tag{A3}$$

which implies

$$\sum_{q=-l}^{l} N_{l,q} = 3^l .$$

To find $\pi^0(l, q)$, we start with the new generating function

$$\left(\frac{x}{3}+\frac{y}{3}+\frac{1}{3x}\right)^l \equiv \sum_{n,q} P_{l,n,q}\, y^n x^q ,$$

where $P_{l,n,q}$ is the probability of having n neutral pions under the conditions (A1) and (A2). We then have

$$\pi^0(l, q)=\sum_n nP_{l,n,q}\Big/\sum_n P_{l,n,q} .$$

The denominator is equal to the coefficient of x^q in the new generating function at $y=1$. Thus it is equal to $3^{-l}N_{l,q}$. The numerator is equal to the coefficient of x^q in

$$y\frac{\partial}{\partial y}\left(\frac{x}{3}+\frac{y}{3}+\frac{1}{3x}\right)^l\Big|_{y=1}=\frac{l}{3}\left(\frac{x}{3}+\frac{1}{3}+\frac{1}{3x}\right)^{l-1} .$$

Thus the numerator is equal to $l3^{-l}N_{l-1,q}$. Hence

$$\pi^0(l, q)=lN_{l-1,q}/N_{l,q} . \tag{A4}$$

Equation (A4) is easy to understand combinatorially. It merely states the fact that if we add one pion to $l-1$ pions, without changing the total charge q, the added pion must be neutral. The factor l is the number of possible choices for this "extra" pion.

We can now obtain $\pi^{+0-}(l, q)$ by solving (A1), (A2), and (A4) simultaneously. The only task that remains is to find $N_{l,q}$, which is straightforward:

$$N_{l,q}=\frac{1}{2\pi i}\oint\frac{dx}{x}\,\frac{(x+1+x^{-1})^l}{x^q}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta(1+2\cos\theta)^l\cos q\theta . \tag{A5}$$

It is useful to have an asymptotic expression for $N_{l,q}$ for $l\to\infty$, with q fixed. This is done by replacing the integrand of (A5) by

$$\left[1+2\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-\cdots\right)\right]^l\left(1-\frac{q^2\theta^2}{2}+\frac{q^4\theta^4}{24}\cdots\right)\to 3^l e^{-l\theta^2/3}\left(1-\frac{l\theta^4}{36}-\frac{13l\theta^6}{3240}+\frac{l^2\theta^8}{2592}+\cdots\right)\left(1-\frac{q^2\theta^2}{2}+\frac{q^4\theta^4}{24}-\cdots\right) .$$

The result is

$$N_{l,q}=3^l\left(\frac{3}{4\pi l}\right)^{1/2}\left(1-\frac{3+12q^2}{16l}+\frac{1+360q^2+144q^4}{512l^2}+\frac{\frac{135}{4}-651q^2-1260q^4-144q^6}{2048l^3}+\cdots\right). \tag{A6}$$

The asymptotic forms for $\pi^{+0-}(l, q)$ are then found to be

$$\pi^+(l, q)=\frac{l}{3}-\frac{1-6q}{12}-\frac{1-4q^2}{32l}+\cdots ,$$

$$\pi^0(l, q)=\frac{l}{3}+\frac{1}{6}+\frac{1-4q^2}{16l}+\cdots , \tag{A7}$$

$$\pi^-(l, q)=\frac{l}{3}-\frac{1+6q}{12}-\frac{1-4q^2}{32l}+\cdots .$$

We now have to average these quantities over physically allowed values of q. Table I will be helpful at this point. This is done as follows:

$$p(l)=\frac{2N_{l,0}+2N_{l,1}}{N_{l,0}+2N_{l,1}+N_{l,2}} ,$$

$$n(l)=\frac{2N_{l,1}+2N_{l,2}}{N_{l,0}+2N_{l,1}+N_{l,2}}=2-p(l), \tag{A8}$$

$$\pi^{+0-}(l)=\frac{N_{l,0}\pi^{+0-}(l,0)+2N_{l,1}\pi^{+0-}(l,1)+N_{l,2}\pi^{+0-}(l,2)}{N_{l,0}+2N_{l,1}+N_{l,2}},$$

where $p(l)$ is defined to be the expected number of protons, when there are l pions produced. Definitions of $n(l)$, $\pi^+(l)$, $\pi^0(l)$, and $\pi^-(l)$ are similar. They have the following asymptotic expressions:

$$p(l)=1+\frac{3}{4l}-\frac{27}{32l^2}+\cdots ,$$

$$n(l)=1-\frac{3}{4l}+\frac{27}{32l^2}+\cdots ,$$

$$\pi^+(l)=\frac{l}{3}+\frac{5}{12}-\frac{7}{32l}+\cdots . \tag{A9}$$

$$\pi^0(l)=\frac{l}{3}+\frac{1}{6}-\frac{5}{16l}+\cdots ,$$

$$\pi^-(l)=\frac{l}{3}-\frac{7}{12}+\frac{17}{32l}+\cdots .$$

There are a number of interesting identities among the various particle ratios. They are

$$\frac{\pi^+(l)-\pi^-(l)}{n(l)}=1 , \tag{A10}$$

$$\frac{\pi^+(l)+\pi^0(l)+\pi^-(l)}{n(l)+p(l)}=\tfrac{1}{2}l , \tag{A11}$$

$$\frac{\pi^+(l)}{\pi^-(l)}=\frac{l+2}{l-1} . \tag{A12}$$

Equation (A10) is nothing but the conservation of charge and the conservation of baryons. Equation (A11) is also trivial. It derives from the definition of l and the conservation of baryons. To prove (A12), we first note that an integration by parts for (A5) leads to

$$N_{l,q}=\frac{l}{q}(N_{l-1,q-1}-N_{l-1,q+1}) \tag{A13}$$

if $q \neq 0$. From the definition of $N_{l,q}$ it follows that

$$N_{l,q}=N_{l-1,q+1}+N_{l-1,q}+N_{l-1,q-1}\,. \tag{A14}$$

Equations (A8), (A13), and (A14) imply (A12). Since there are five kinds of particles, i.e., four kinds of particle ratios, Eqs. (A10), (A11), and (A12) fix the other three ratios whenever one ratio is known.

APPENDIX B

In Sec. II we separated the final states into four groups, according to the charges of the two nucleons, as is clear by looking at Table I. In Sec. III, to which this is an appendix, we include isospin conservation. The state of the pion system is then specified by its I and I_z, and the four states of the two nucleons can be linearly combined into states of total isospin 1 and 0. There are then seven ways to multiply the pion system with the nucleon system to give the total isospin 1 and the z component of isospin 1. These seven ways are listed in Table IV, which is valid for arbitrary l. To construct this table we first study the total I and I_z states of l pions.

Define a quantity $M_{l,I}$ which is the number of isopin multiplets with total isospin I for l pions. $M_{l,I}$ is equal to the difference between the number of possible states with $q=I$ and $I+1$, where q is the third component of isospin:

$$M_{l,I}=N_{l,I}-N_{l,I+1}\,, \tag{B1}$$

where $N_{l,I}$ is defined in Appendix A. We then have

$$M_{l,I}=\frac{1}{\pi}\int_{-\pi}^{\pi}d\theta\,(1+2\cos\theta)^l\sin\tfrac{1}{2}\theta\,\sin(I+\tfrac{1}{2})\theta$$

$$=3^l\left(\frac{3}{4\pi l}\right)^{1/2}\frac{3}{4l}(2I+1)\left(1-\frac{21+12I+12I^2}{16l}+\frac{685+1032I+1176I^2+288I^3+144I^4}{512l^2}+\cdots\right)\,. \tag{B2}$$

TABLE IV. Pion charge distribution in pp collisions.

Isospin state of NN	Nucleon state of NN	Isospin state of π^l	Weight for such a NN-π^l coupling	Mean number of π^0's in each state $=\pi^0(l,I,q)$	Number of isospin multiplets in such a π^l isospin state $=M_{l,I}$
$\binom{1}{1}$	pp	$\binom{0}{0}$	1	$\frac{1}{3}l$	$3^l\left(\frac{3}{4\pi l}\right)^{1/2}\frac{3}{4\pi}\left(1-\frac{21}{16l}+\frac{685}{512l^2}+\cdots\right)$
$\binom{1}{0}$	$\frac{1}{\sqrt{2}}(pn+np)$	$\binom{1}{1}$	$\frac{1}{2}$	$\frac{l}{3}-\frac{1}{8l}+\cdots$	$3^l\left(\frac{3}{4\pi l}\right)^{1/2}\frac{9}{4\pi}\left(1-\frac{45}{16l}+\frac{3325}{512l^2}+\cdots\right)$
$\binom{1}{1}$	pp	$\binom{1}{0}$	$\frac{1}{2}$	$\frac{l}{3}+\frac{1}{4l}+\cdots$	
$\binom{1}{-1}$	nn	$\binom{2}{2}$	$\frac{3}{5}$	$\frac{l}{3}-\frac{3}{4l}+\cdots$	
$\binom{1}{0}$	$\frac{1}{\sqrt{2}}(pn+np)$	$\binom{2}{1}$	$\frac{3}{10}$	$\frac{l}{3}+\frac{3}{8l}+\cdots$	$3^l\left(\frac{3}{4\pi l}\right)^{1/2}\frac{15}{4\pi}\left(1-\frac{93}{16l}+\frac{12061}{512l^2}+\cdots\right)$
$\binom{1}{1}$	pp	$\binom{2}{0}$	$\frac{1}{10}$	$\frac{l}{3}+\frac{3}{4l}+\cdots$	
$\binom{0}{0}$	$\frac{1}{\sqrt{2}}(pn-np)$	$\binom{1}{1}$	1	$\frac{l}{3}-\frac{1}{8l}+\cdots$	$3^l\left(\frac{3}{4\pi l}\right)^{1/2}\frac{9}{4l}\left(1-\frac{45}{16l}+\frac{3325}{512l^2}+\cdots\right)$

Numerically $M_{l,I}$ is computed by a recurrence scheme:

$$\begin{aligned} &M_{l,I}=M_{l-1,I+1}+M_{l-1,I}+M_{l-1,I-1} \quad \text{if } I\geqq 1, \\ &M_{l,I}=M_{l-1,I+1} \quad \text{if } I=0, \end{aligned} \tag{B3}$$

with the conditions

$$M_{l,I}=0 \quad \text{if } I>l,$$

$$M_{0,0}=1 .$$

In each isospin multiplet with total isospin I, there are $(2I+1)$ possible q's. We now want to find the mean number of neutral pions per state under the following conditions: (i) The total number of pions is l, (ii) the total isospin quantum number is I, and (iii) the third component of isospin is q. Let this quantity by $\pi^0(l,I,q)$. It can be obtained again by a recurrence formula as follows.

Suppose we have already solved the problem for $l-1$ pions. By adding one extra pion, we can generate the desired $\pi^0(l,I,q)$. It is given by

$$M_{l,I}\pi^0(l,I,q)=\sum_{i=-1}^{1}\sum_{j=-1}^{1}M_{l-1,I+i}\left[\begin{pmatrix}I+i & 1\\ q+j & -j\end{pmatrix}\middle|\begin{pmatrix}I\\ q\end{pmatrix}\right]^2[\pi^0(l-1,I+i,q+j)+\delta_{j0}], \tag{B4}$$

with $\pi^0(0,0,0)=0$.

Each of the nine terms on the right-hand side of (B4) consists of three factors. The first factor is the weight given to each multiplet. The second factor is the square of a suitable Clebsch-Gordan coefficient. The third factor is the mean number of neutral pions in the particular isospin configuration.

Equation (B4) can be greatly reduced because of the relation

$$\pi^0(l,I,q)=\tfrac{1}{3}l+[(I^2+I-3q^2)Z_{l,I}/M_{l,I}], \tag{B5}$$

where $Z_{l,I}$ depends only on l and I.

To prove (B5), we first consider the π^0-number operator for the ith pion, and split it into an isoscalar and an isotensor:

$$\begin{aligned} \hat{\pi}^{0(i)}&=1-[\hat{I}_z^{(i)}]^2 \\ &=\{1-\tfrac{1}{3}[\hat{\vec{I}}^{(i)}]^2\}+\tfrac{1}{3}\{[\hat{\vec{I}}^{(i)}]^2-3[\hat{I}_z^{(i)}]^2\}. \end{aligned} \tag{B6}$$

The first term on the right-hand of (B6) transforms under isospin rotations as an isoscalar, while the second term transforms as the zeroth component of an isospin tensor. In fact, the isoscalar term is simply $\frac{1}{3}$, because $[\vec{I}^{(i)}]^2=2$.

Since $\pi^0(l,I,q)$ is just the expectation value of $\sum_i\hat{\pi}^{0(i)}$ in the state $|I,q\rangle$, we take the expectation value of (B6). On the right-hand side the first term gives $\frac{1}{3}l$. According to the Wigner-Eckart theorem, the second term is proportional to I^2+I-3q^2 with a proportionality constant independent of q. Hence we have proved (B5).

Substitution of (B5) into (B4) results in a recurrence formula which allows for a computation of $Z_{l,I}$. A simpler recurrence formula for $Z_{l,I}$ can be obtained as follows. We first observe the sum rule

$$\sum_{I=q}^{l}M_{l,I}\pi^0(l,I,q)=lN_{l-1,q} . \tag{B7}$$

Equation (B7) is true because both sides are equal to $N_{l,q}$ times the mean number of π^0 [cf. (A4)] with fixed l and q. (This becomes apparent when one starts with the $N_{l,q}$ states of the l-pion system with total charge q and makes orthonormal transformations on them to obtain states with fixed total isospin. The number of states with isospin I is $M_{l,I}$.) We then substitute (B5) into (B7), making use of (B1), to get

$$\frac{1}{q^2}\sum_{I=q}^{l}(I^2+I)Z_{l,I}-3\sum_{I=q}^{l}Z_{l,I}=l(N_{l-1,q}-\tfrac{1}{3}N_{l,q})/q^2 . \tag{B8}$$

Subtract (B8) from its corresponding equation with $q\to q+1$. The result contains a term

$$\sum_{I=q+1}^{l}(I^2+I)Z_{l,I} .$$

Eliminate this term by using (B8) again. The result obtained is now subtracted from the same after the replacement $q\to q-1$. The final equation is

$$\begin{aligned} &-\frac{(I-1)(2I-3)Z_{l,I-1}}{2I-1}+\frac{(2I+3)(I+1)Z_{l,I}}{2I+1} \\ &\quad=\frac{l(M_{l-1,I-1}-\frac{1}{3}M_{l,I-1})}{2I-1}-\frac{l(M_{l-1,I}-\frac{1}{3}M_{l,I})}{2I+1} . \end{aligned} \tag{B9}$$

Equation (B9) is legitimate only for $I\geqslant 1$. Setting $I=1$, one obtains

$$Z_{l,1}=\tfrac{3}{10}l(M_{l-1,0}-\tfrac{1}{3}M_{l,0}-\tfrac{1}{3}M_{l-1,1}+\tfrac{1}{9}M_{l,1}) . \tag{B10}$$

A closed form for $Z_{l,I}$ can be obtained by solving Eqs. (B9) and (B10). The result is

$$\begin{aligned} Z_{l,I}=\frac{l}{3}\Bigg[&\frac{1}{I(2I-1)}M_{l-1,I-1}-\frac{1}{I(I+1)}M_{l-1,I} \\ &+\frac{1}{(2I+3)(I+1)}M_{l-1,I+1}\Bigg], \end{aligned}$$

with the asymptotic expression

$$Z_{l,I} = \frac{3^{l+1}}{32l^2}\left(\frac{3}{4\pi l}\right)^{1/2}(2I+1) \times\left(1-\frac{12I^2+12I+25}{16l}+\cdots\right),$$

or

$$Z_{l,I} \to \frac{1}{8l} M_{l,I}\left(1-\frac{1}{4l}\right)$$

for all fixed I. Substituting this into (B5), we find the asymptotic form for $\pi^0(l, I, q)$:

$$\pi^0(l, I, q) = \frac{l}{3} + \frac{1}{8l}(I^2+I-3q^2)+\cdots . \tag{B11}$$

Equation (B11) and the definitions of l and q then give

$$\pi^+(l, I, q) = \frac{l}{3} + \frac{q}{2} - \frac{I^2+I-3q^2}{16l}+\cdots ,$$
$$\pi^-(l, I, q) = \frac{l}{3} - \frac{q}{2} - \frac{I^2+I-3q^2}{16l}+\cdots . \tag{B12}$$

We now tabulate Table IV, which gives the seven possible isospin configurations, their individual weights, and the asymptotic forms of $M_{l,I}$ and $\pi^0(l, I, q)$. Only those multiplets with $2 \geq I \geq q$ concern us. The fourth column is the square of the Clebsch-Gordan coefficient which combines the isospin states of NN and π^l into the initial isospin state $|1, 1\rangle$ of two protons. The seven groups can be arranged into four classes. Different classes correspond to different sets of total isospin quantum numbers of the NN and π^l system. From this table, it is easy to show that

$$p(l) = \frac{2M_{l,0}+\frac{5}{2}M_{l,1}+\frac{1}{2}M_{l,2}}{M_{l,0}+2M_{l,1}+M_{l,2}} = 2-n(l) ,$$

$$\pi^{+0-}(l) = \frac{1}{M_{l,0}+2M_{l,1}+M_{l,2}} \times\{ M_{l,2}[\tfrac{3}{5}\pi^{+0-}(l, 2, 2)+\tfrac{3}{10}\pi^{+0-}(l, 2, 1) +\tfrac{1}{10}\pi^{+0-}(l, 2, 0)] +M_{l,1}[\tfrac{3}{2}\pi^{+0-}(l, 1, 1)+\tfrac{1}{2}\pi^{+0-}(l, 1, 0)] +M_{l,0}\pi^{+0-}(l, 0, 0)\} . \tag{B13}$$

Their asymptotic forms are

$$p(l) = 1+\frac{3}{4l}-\frac{33}{32l^2}+\cdots ,$$

$$n(l) = 1-\frac{3}{4l}+\frac{33}{32l^2}+\cdots ,$$

$$\pi^+(l) = \frac{l}{3}+\frac{1}{2}-\frac{5}{16l}+\cdots , \tag{B14}$$

$$\pi^0(l) = \frac{l}{3}-\frac{1}{8l}+\cdots ,$$

$$\pi^-(l) = \frac{l}{3}-\frac{1}{2}+\frac{7}{16l}+\cdots .$$

Equations (A10) and (A11) remain valid with isospin conservation because they are only dependent on nucleon conservation and charge conservation. (A12) is, however, no longer valid.

A number of interesting identities are listed below:

$$M_{l,l}=1 ,$$
$$M_{l,l-1}=l-1 \quad (l\geq 1) ,$$
$$\sum_{I=0}^{l}(2I+1)M_{l,I}=3^l ,$$
$$\frac{1}{2I+1}\sum_{q=-I}^{I}\pi^{+0-}(l, I, q)=\tfrac{1}{3}l ,$$
$$Z_{l,l}=[3(2l-1)]^{-1} , \tag{B15}$$
$$Z_{l,l-1}=(l-3)[3(2l-3)]^{-1} ,$$
$$\pi^0(l, 0, 0)=\tfrac{1}{3}l .$$

TABLE V. The average number of neutral pions per state at fixed I and $I_z=0$ for $l=24$.

I	$\pi^0(24,I,0)$
0	8.00
1	8.01
2	8.03
3	8.06
4	8.10
5	8.16
6	8.22
7	8.29
8	8.38
9	8.48
10	8.59
11	8.71
12	8.85
13	9.00
14	9.17
15	9.35
16	9.55
17	9.77
18	10.02
19	10.29
20	10.59
21	10.92
22	11.30
23	11.73
24	12.26

Their proofs are omitted.

We also tabulate, in Table V, the result of a numerical computation of $\pi^0(l, I, 0)$ for $l = 24$. For small I, $\pi^0(24, I, 0)$ is approximately $8 = \frac{1}{3}l$, confirming (B11). Notice that the value of $\pi^0(l, 0)$ without isospin conservation, as given by (A7), is bigger by $\sim\frac{1}{6}$ than this value. The meaning of this observation is as follows: For $q = 0$, charge conservation gives a slight edge to π^0 over $\frac{1}{2}(\pi^+ + \pi^-)$ if one does not consider isospin conservation, as (A7) shows explicitly. This favoritism disappears when isospin conservation is considered, as (B11) and (B12) show explicitly, because of the greater symmetry between π^+, π^- and π^0.

If one now keeps $q = 0$, but considers a value of I comparable to l, then the favoritism for π^0 must reappear, because the average of $\pi^0(l, I, 0)$ over I must give the same result as (A7). Indeed this is so, since (B15) leads to

$$\pi^0(l, l, 0) = l^2(2l-1)^{-1} > \tfrac{1}{3}l \ .$$

The numerical value of $\pi^0(24, 24, 0)$ in Table V agrees with this formula, as expected.

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*Work supported in part by the National Science Foundation under Grant No. GP32998X.

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PHYSICAL REVIEW D VOLUME 9, NUMBER 9 1 MAY 1974

Single-cluster formation in the statistical bootstrap model*

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(Received 24 January 1974)

Single-cluster formation in hadronic reactions is discussed, within the context of the statistical bootstrap model. This process is analogous to compound nucleus scattering in nuclear physics, and similar formulas hold for the formation cross section. If the average resonance width should rise indefinitely with energy, the model will eventually run into conflict with unitarity; the trouble is traced to a breakdown of the "narrow-resonance approximation." The effects of angular momentum conservation on the cluster decay are considered, and formulas are presented for the multiplicity and single-particle momentum distribution as a function of the cluster's spin. Brief discussions are given of possible experimental tests of the model, including the annihilation reactions $e^+e^- \to$ hadrons and $N\bar{N} \to$ mesons, which are particularly favorable cases. In an appendix it is shown how to estimate asymptotic parameters in a "realistic" model by analytic means.

I. INTRODUCTION

It is a familiar fact that low-energy nuclear interactions are well described by the "compound nucleus" model of Bohr,[1] in which reactions are assumed to proceed via an incoherent sum over long-lived direct-channel resonances. The average behavior of the system (e.g., momentum distributions, branching ratios, etc.) can then be described by statistical means, that is, by computing ratios of the phase space available in the various final states. In order to do this, one needs to know