Comment on the null-plane gauge

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Gauge-invariant Lagrangians are studied using the method of 't Hooft and Veltman with a particular type of noncovariant symmetry-breaking term. A limiting case corresponds to the formalism of null-plane or light-cone quantization. Ward identities and a feature of dimensional regularization are briefly discussed.

The null-plane (or light front or the so-called "infinite-momentum frame") quantization of fields has been studied at length by several authors.[1,2]

In a previous unpublished report, we have studied Yang-Mills fields in this formalism. This treatment has been extended to the case of the general gauge-invariant Lagrangian involving the Higgs-Kibble mechanism. A study of light-cone quantization and equivalence with functional formalism will be published elsewhere and presented in a thesis by one of us (C.D.). It is to be noted that in the light-cone quantization[1,2] a minimum number of independent fields survive, while unlike the unitary gauge[3] the term $k_\mu k_\nu/m^2$ (troublesome from the point of view of renormalization) is absent in the gauge-field propagator.

In this paper we propose to study the gauge fields from the following point of view: We will introduce a particular type of noncovariant gauge function (or symmetry-breaking term) in the Lagrangian and then follow the general technique of 't Hooft and Veltman.[4] Correspondence with the above-mentioned formalism[1,2] will be obtained as a limiting case.

I. THE NULL-PLANE GAUGE FUNCTION

Let us illustrate our method by an example, namely the Abelian Higgs model. We might as well have started with spinor electrodynamics, but it is interesting to display certain special features arising in the Higgs model. The non-Abelian case will be taken up later on.

The invariant Lagrangian can be written as

$$\mathcal{L}_{\text{inv}}(x) = -\tfrac{1}{4}[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)]^2 + (D_\mu\phi)^*(D^\mu\phi) + \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 \quad (\lambda v^2 = \mu^2 > 0), \qquad (1)$$

where $D_\mu \equiv \partial_\mu - ieA_\mu$ and $\phi = \phi_1 + i\phi_2 = (\tfrac{1}{2})^{1/2}(v + \psi + i\chi)$. The vacuum expectation values are given by zero values for all the fields except that

$$\langle \phi_1 \rangle = (\tfrac{1}{2})^{1/2} v .$$

$\mathcal{L}_{\text{inv}}$ can be separated into free and interaction parts in terms of A_μ, χ, and ψ.

To this we will add the symmetry-breaking term

$$-\frac{1}{2\alpha}[n\cdot A(x)]^2 , \qquad (2)$$

where α is an arbitrary gauge parameter and n is a *fixed* four-vector. We will be principally interested in the case[5] $n^2 = 0$, with $n_\mu = (\tfrac{1}{2})^{1/2}(1, 0, 0, 1)$, say (which is evidently indicated for correspondence with the null-plane formalism), but let us not exclude for the moment the cases $n^2 = \pm 1$, with, for example, $n = (1, 0, 0, 0)$ and $n = (0, 0, 0, 1)$, respectively.

The above choice replaces the usual covariant one[6]

$$-\frac{1}{2\alpha}[\partial\cdot A(x)]^2 . \qquad (3)$$

Corresponding to the generalization of (3) as[7]

$$-\tfrac{1}{2}\xi\left[\partial\cdot A(x) + \frac{\lambda}{\xi}\chi(x)\right]^2 , \qquad (4)$$

we might have chosen the gauge term

$$-\tfrac{1}{2}\xi\left[n\cdot A(x) + \frac{\lambda}{\xi}\chi(x)\right]^2 \qquad (5)$$

with a view to study equivalence properties.

In the following we will restrict ourselves to the case (2) ($\lambda = 0$).

Thus our starting point is the Lagrangian

$$\mathcal{L}(x) = \mathcal{L}_{\text{inv}}(x) - \frac{1}{2\alpha}[n\cdot A(x)]^2 \qquad (6)$$

and the generating functional for the Green's function

$$W = e^{iZ} = \int [dA_\mu][d\chi][d\psi] \times \exp\left\{ i \int d^4x [\mathcal{L}(x) + J_\psi(x)\psi(x) + J_\chi(x)\chi(x) + J_\mu(x)A^\mu(x)] \right\} . \qquad (7)$$

Z generates the connected Green's functions.

Our Green's functions will be defined as the variational derivatives of W (see the remarks in Appendix B of Ref. 8). For the model we are considering, the bilinear terms in $\int \mathcal{L}(x)\, d^4x$ can be arranged as

$$\tfrac{1}{2}A^\mu(x)\left[(\Box + m^2)g_{\mu\nu} - \partial_\mu\partial_\nu - \frac{1}{\alpha}\, n_\mu n_\nu\right]A^\nu(x)$$
$$-\tfrac{1}{2}\chi(x)\Box\chi(x)$$
$$-\tfrac{1}{2}m[A^\mu(x)\partial_\mu\chi(x) - \chi(x)\partial_\nu A^\nu(x)]$$
$$-\tfrac{1}{2}\psi(x)(\Box + 2\mu^2)\psi(x) \quad (m = ev)\,. \quad (8)$$

Let us first consider the case $n^2 = 0$. We obtain the following nonzero bare propagators:

$$\int e^{ik\cdot x}\langle (A_\mu(x)A_\nu(0))_+\rangle_{(0)}\, d^4x = \frac{-i}{k^2 - m^2}\left(g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{n\cdot k}\right) - i\,\alpha k_\mu k_\nu/(n\cdot k)^2, \quad (9)$$

$$\int e^{ik\cdot x}\langle (A_\mu(x)\chi(0))_+\rangle_{(0)}\, d^4x = \frac{mn_\mu}{(k^2-m^2)(n\cdot k)} - \frac{\alpha\, mk_\mu}{(n\cdot k)^2}, \quad (10)$$

$$\int e^{ik\cdot x}\langle (\chi(x)\chi(0))_+\rangle_{(0)}\, d^4x = \frac{i}{k^2-m^2} - \frac{i\alpha m^2}{(n\cdot k)^2}, \quad (11)$$

$$\int e^{ik\cdot x}\langle (\psi(x)\psi(0)_+\rangle_{(0)}\, d^4x = \frac{i}{k^2 - 2\mu^2}\,. \quad (12)$$

The subscript (0) denotes bare propagators. The subscript + should be taken to indicate ordering in the direction introduced in the asymptotic conditions defining the in and out fields, the corresponding vacua defining the functional integral of the action as $\langle \text{out}|\text{in}\rangle$. For the usual case the asymptotic directions are t or $x^0 \to \mp\infty$, and we have the usual T (or rather T^*) ordering. When we want to compare our results with those of quantization on the light cone[1,2] we should choose the directions $\tau \equiv n\cdot x \to \mp\infty$. This is more fully explained in the point (d).

The complete set of Feynman's rules are easily obtained. We will not, however, write them down.

Let us now note the following successive points.

(a) Comparison of the cases $n^2 > 0$, $n^2 = 0$, and $n^2 < 0$. For $n^2 \neq 0$, the results (obtained through straightforward calculation) become much less simple. To give but one example, in (11) we obtain an extra term

$$\frac{-i\,m^2 n^2}{(k^2-m^2)(n\cdot k)^2}\,.$$

For this reason we will henceforth consider exclusively the case $n^2 = 0$, instead of writing down complicated general formulas. It is interesting to compare the case $n^2 = 1$ (for $\alpha \to 0$) with those for the Coulomb gauge. (This point, however, does not concern directly the null-plane gauge and will not be studied here.) For $n^2 = -1$ (and $\alpha \to 0$) we obtain the "axial gauge" of Fradkin and Tyutin.[9]

(b) Ward identities. We have from (9) and (10),

$$\int e^{ik\cdot x}\langle (n\cdot A(x)A_\nu(0))_+\rangle_{(0)} = \frac{-i\alpha k_\nu}{(n\cdot k)} \quad (13)$$

and

$$\int e^{ik\cdot x}\langle (n\cdot A(x)\chi(0))_+\rangle_{(0)} = -\frac{\alpha m}{(n\cdot k)}\,. \quad (14)$$

These are bare propagators. But the Ward identities assure that these two are not renormalized.

The result corresponding to Eq. (4.1) of Ref. 6 is for our case

$$\frac{1}{\alpha}(n\cdot\partial)\frac{\delta Z}{\delta J^n(x)} - eJ_\psi(x)\frac{\delta Z}{\delta J_\chi(x)} + eJ_\chi(x)\left(v + \frac{\delta Z}{\delta J_\psi(x)}\right) - \partial_\mu J^\mu(x) = 0\,. \quad (15)$$

[In our notation

$$J_\mu(x)A^\mu(x) = J^n(x)A_n(x) + J^{\bar n}A_{\bar n}(x) + \sum_{i=1,2} J^i(x)A_i(x),$$

with $n^2 = 0$, $\bar n^2 = 0$, $n\cdot\bar n = 1$.]

From (15), exactly as in Ref. 6 we obtain that (13) and (14) hold for the respective full propagators.

Thus in the limit $\alpha \to 0$ the propagators involving $n\cdot A(x)$ all tend to zero. (We will not discuss here the effect of the singularity in the different propagators for $n\cdot k = 0$ and the mechanism of their cancellation. In the covariant gauge a corresponding problem arises for $k^2 = 0$.)

Starting from (15) we can easily obtain (as in Ref. 6) Ward identities satisfied by the generating functional for the irreducible vertices.

(c) Physical sources and fields. The criterion of 't Hooft and Veltman[4] gives (independently of the choice of the gauge term) the following constraint for the physical sources for our Lagrangian:

$$m\,J_\chi(x) - \partial_\mu J^\mu(x) = 0.$$

Hence the physical states should correspond to the fields

$$A_\mu(x) - \frac{1}{m}\,\partial_\mu\chi(x)$$

and $\psi(x)$. This is also brought out if in $\mathcal{L}(x)$ we make the substitution

$$A_\mu(x) = C_\mu(x) + \frac{1}{m}\,\partial_\mu\chi(x).$$

Then the free propagators for $C_\mu(x)$ and $\chi(x)$ are calculated from

$$-\tfrac{1}{4}[\partial_\mu C_\nu(x)-\partial_\nu C_\mu(x)]^2+\tfrac{1}{2}m^2 C_\mu(x)C^\mu(x) -\frac{1}{2\alpha}\left[n\cdot C(x)+\frac{1}{m}\,n\cdot\partial\chi(x)\right]^2. \quad (16)$$

We obtain (considering for brevity only the limit $\alpha\to 0$)

$$\int e^{ik\cdot x}\,d^4x\langle\langle C_\mu(x)C_\nu(0)\rangle_+\rangle_{(0)}=-\frac{i}{k^2-m^2}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{m^2}\right) \quad (17)$$

and

$$\int e^{ik\cdot x}\,d^4x\langle\langle C_\mu(x)\chi(0)\rangle_+\rangle_{(0)}=-\frac{1}{k^2-m^2}\left(\frac{k_\mu}{m}-\frac{mn_\mu}{n\cdot k}\right). \quad (18)$$

The other propagators are the same as before. Thus we see that $C_\mu(x)$ has already the Feynman propagator (17). It is interesting to compare the role played by $\chi(x)$ in our scheme with that in the covariant gauge, particularly for $\alpha\to 0$ [compare our Eqs. (10) and (11) with Eqs. (3.23) and (3.24) of Ref. 6].

(d) Comparison with null-plane quantization.[1,2] As is emphasized in Ref. 8, in defining the Green's functions through the functional derivatives one avoids from the beginning mutually cancelling non-covariant terms in the formalism. Since in our case the Lagrangian (6) is made explicitly non-covariant, we do not avoid terms linear in n_μ in the propagators. What we do avoid, however, is a term proportional to $n_\mu n_\nu$ in (9) [though this would leave (13) unaltered]. This fact can be traced more precisely to the absence of constraint equations in our formalism as compared to the usual treatment of null-plane quantization.[1,2] Since we never formally set $n\cdot A(x)=0$, the Heisenberg fields do not satisfy any constraint equation, though in the limit $\alpha\to 0$, $n\cdot A(x)$ can be shown to decouple in a consistent fashion.

Let us examine this aspect more explicitly. To start with, let us note that in the free part of $\mathcal{L}_{\rm inv}$ [Eq. (1)] the terms bilinear in $A_\mu(x)$ and $\chi(x)$ have the same structure as the boson part of the gluon model of Soper[1] [his field $B(x)$ corresponds to $\chi(x)$], and that in Yan's formalism[2] the corresponding fields are $\overline{B}_\mu(x)$ and $\Lambda(x)$. Thus certain features can be compared directly.

Now suppose that instead of introducing our gauge term (2) we put directly in (1), following Soper,[1]

$$n\cdot A(x)=0 \quad (19)$$

and evaluate $\bar{n}\cdot A(x)$ (with $\bar{n}^2=0$, $n\cdot\bar{n}=1$) through the consequent constraint equation[1] in terms of the independent fields $A_1(x), A_2(x), \chi(x), \psi(x)$, on which are imposed equal-τ $(=n\cdot x)$ commutation relations. The bare propagators of these latter fields can be calculated trivially as τ-ordered vacuum expectation values. Noting that, symbolically,

$$[\bar{n}\cdot A(x)]_{\text{free part}}=-\frac{1}{n\cdot\partial}\,[\partial_i A^i(x)+m\chi(x)]_{\text{free part}} \quad (20)$$

and that not only ∂_i but also $n\cdot\partial$ can be taken out of the τ-ordering (since $g_{nn}=0$), we can write finally [consistently with (19) and (20)]

$$\int e^{ik\cdot x}\langle\langle A_\mu(x)A_\nu(0)\rangle_+\rangle_{(0)}\,d^4x =\frac{-i}{k^2-m^2}\left(g_{\mu\nu}-\frac{k_\mu n_\nu+k_\nu n_\mu}{n\cdot k}\right)+\frac{in_\mu n_\nu}{(n\cdot k)^2}. \quad (21)$$

other propagators coincide with (10)–(12) for $\alpha=0$. Thus, as might have been expected, using the constraint equation in this fashion we obtain exactly the boson propagators of Yan,[2] who discusses the effect of the presence of a constraint equation on the variational derivatives.

Had we considered the vector-gluon model starting with the Lagrangian of Soper [Eq. (1) of Ref. 1], but introducing our gauge term (2), we would have obtained apart from results like (9), (10), and (11), the *usual* spinor propagator $iS_F(x)$ for the spinor field. At this point it is convenient to compare with the corresponding results of Yan [Eqs. (3.1), (3.38), and (5.1) of Ref. 2(b)]. Two extra terms in his propagators (one for the boson, one for the fermion) serve only to cancel the effect of two terms in the Hamiltonian, which again are consequences of the constraint equations.

The absence of constraint equations and the need for such cancellations is a general feature of our formalism.

A more complete study of the equivalence of the two approaches by comparing the different vertices and graphs will be presented in a thesis by one of us (C. D.).

II. THE NON-ABELIAN CASE

Here let us note very briefly certain features of the non-Abelian generalization of the Higgs model[4,7] using our gauge term

$$-\frac{1}{2\alpha}[n\cdot A^a(x)]^2 \quad (a=1,2,3).$$

Apart from the evident generalization due to the internal indices, the propagators (9)–(12) remain the same. But now there is a Faddeev-Popov term,

$$d\Phi \exp\left[i\Phi^{*a}(x)\left(\frac{1}{\sqrt{\alpha}}\, n\cdot\partial\Phi^a(x) - g\epsilon^{abc}\frac{1}{\sqrt{\alpha}}[n\cdot A(x)]^b\Phi^c(x)\right)\right]$$

(g is the coupling constant), (22)

and a ghost propagator,

$$\int e^{ik\cdot x}\langle(\Phi^{*a}(x)\Phi^b(0))_+\rangle_{(0)}\, d^4x = \frac{\alpha^{1/2}}{n\cdot k}\,\delta_{ab}\,. \qquad (23)$$

But the ghost is coupled only to $n\cdot A(x)$ and in the limit $\alpha\to 0$ both $n\cdot A(x)$ and the ghost may be shown to decouple in a consistent fashion. This introduces an element of simplicity in the formalism.

In the covariant gauge our coupling term $n_\mu\Phi^*A^\mu\Phi$ is replaced by $(\partial_\mu\Phi^*)A^\mu\Phi$ and the ghost is *not* coupled only to $\partial_\mu A^\mu$ (Refs. 4 and 7). Thus even in the limit of Landau gauge the F-P ghost is present.

Following the method of Ref. 7 the Ward identitites can be compactly displayed in the equation

$$\left\{-F_a\left(\frac{1}{i}\frac{\delta}{\delta J}\right)+J_i\left(\Gamma^b_{ij}\frac{1}{i}\frac{\delta}{\delta J_j}+\Lambda^b_i\right)\times\left[M^{-1}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]_{ba}\right\}W(J)=0\,, \qquad (24)$$

where Γ_{ij} and Λ_i are defined through the gauge-transformation properties

$$\delta\phi_i = (\Gamma^a_{ij}\phi_j+\Lambda^a_i)\,\omega_a \quad (\phi_i = A^a_\mu,\ \chi^a,\ \psi),$$

and in our case

$$F_a = (\alpha)^{-1/2}\, n\cdot A_a(x) \qquad (25)$$

and

$$M = (\alpha)^{-1/2}\,(n\cdot\partial + g n\cdot\vec{A}\times). \qquad (26)$$

III. DIMENSIONAL REGULARIZATION

As is known, dimensional regularization[10] is a powerful and elegant tool particularly suited for studying renormalizability of gauge theories.

Let us briefly note only one point concerning this aspect. To see whether this technique might be conveniently applicable with our particular choice of gauge function (with, for example, the Feynman rules in the limit $\alpha=0$), we should first consider integrals of the type

$$I=\int\frac{d^lk}{(k^2+2p\cdot k+\Lambda+i\epsilon)^\alpha[n\cdot(k+p')]^\beta} \qquad (27)$$

(we denote the dimension by l, and n is supposed to be redefined in such a way as to conserve the property $n^2=0$).

It can be shown that the lightlike nature of n permits a relatively simple evaluation of this integral.

Using Feynman parametrization and the 't Hooft-Veltman formula [for an integrand of the term $(k^2+2p\cdot k+\Lambda+i\epsilon)^{-\lambda}$], *and* $n^2=0$, we obtain

$$I = i\pi^{l/2}e^{-i\pi(\alpha+\beta)}\frac{\Gamma(\alpha+\beta-\frac{1}{2}l)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dz\, z^{\alpha-l/2-1}(1-z)^{\beta-1}[z(p^2-\Lambda)+p''\cdot n(1-z)]^{l/2-(\alpha+\beta)} \quad (p''=p-p')$$
$$= i\pi^{l/2}e^{-i\pi(\alpha+\beta)}\frac{\Gamma(\alpha-\frac{1}{2}l)}{\Gamma(\alpha)}(p^2-\Lambda)^{-(\alpha-l/2)}(n\cdot p'')^{-\beta}. \qquad (28)$$

It is quite interesting to derive this result by a method which does not involve Wick rotation and exploits directly the particular structure of I [Eq. (27)] concerning the components k_n and $k_{\bar{n}}$. Let

$$n = (\tfrac{1}{2})^{1/2}(1, 0, \ldots, 0, 1)$$

and

$$\bar{n} = (\tfrac{1}{2})^{1/2}(1, 0, \ldots, 0, -1),$$

with $(l-2)$ transverse components and

$$k = (k_0, k_1, \ldots, k_{l-1}) \qquad (29)$$

so that $d^lk = dk_n\, dk_{\bar{n}}\, d^{l-2}\underline{k}$, where $d^{l-2}\underline{k}=\prod_{r=1}^{l-2}dk_r$ (we have a similar definition of $\underline{p}$). Now shifting the variable (the values of l, α, β being supposed to permit it) and integrating over the Euclidean transverse vector $\underline{k}$ we obtain from (27)

$$I = e^{-i\pi(l-2)/2}\pi^{(l-2)/2}\frac{\Gamma(\alpha-\frac{1}{2}(l-2))}{\Gamma(\alpha)}\times\int\frac{dk_n\,dk_{\bar{n}}}{[2k_nk_{\bar{n}}-(p^2-\Lambda)+i\epsilon]^{\alpha+1-l/2}(n\cdot k-n\cdot p'')^\beta}\,. \qquad (30)$$

At this point, in order to carry out the $k_{\bar{n}}$ integration we note the result of Yan,[2(b)] namely,

$$\int_{-\infty}^{\infty}\frac{dy}{(xy-C+i\epsilon)^2}=\frac{i2\pi}{C}\,\delta(x). \qquad (31)$$

If we formally differentiate both sides with respect to C a suitable number of times and then use it to evaluate (30), we get exactly the result (28).

We will not study here the consequences of the possibility $p''\cdot n=0$. (See, however, Lee's remarks[6] concerning the $k^2=0$ singularities in the covariant gauge.)

We hope to discuss elsewhere more fully many points discussed briefly or touched upon in this paper.

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