Particle in an electromagnetic field: The Lorentz-Dirac equation

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A new method to solve the Lorentz-Dirac equation in the presence of an external electromagnetic field is presented. The validity of the approximation is discussed, and the method is applied to a particle in the presence of a constant magnetic field.

Recently there has been some interest[1] in the study of the behavior of a charged particle in the presence of a constant magnetic field when the radiation effects are taken into account, since it seems possible that, with the new generation of accelerators and very high magnetic fields, there are some possibilities to prove the validity of the Lorentz-Dirac equation (we use the metric $+---$)

$$\dot{u} = \omega u + \omega_0^{-1}[\ddot{u} + (\dot{u}^T \dot{u})\, u] \ . \tag{1}$$

In this equation the dots denote derivatives with respect to the proper time. The four-velocity u is taken as a matrix column with components $u^\mu(\tau)$, while u^T corresponds to a matrix row with components $u_\mu(\tau)$. The ω is a 4×4 matrix with components $\omega^\mu{}_\nu = (e/m)F^\mu{}_\nu$, where $F^\mu{}_\nu$ is the usual electromagnetic field tensor. Finally $\omega_0 = 3m/2e^2$, which, for electrons, turns out to be $\omega_0 = 1.5958 \times 10^{23}\ \text{sec}^{-1}$.

As is well known,[2] Eq. (1) presents the problem of the existence of "run away" solutions which must be eliminated through the use of the asymptotic condition. Equation (1) plus the asymptotic condition is equivalent to the integro-differential equation[2]

$$\dot{u}(\tau) = \omega_0 \int_0^\infty d\tau'\, e^{-\omega_0 \tau'} \{ \omega u(\tau+\tau') + \omega_0^{-1}[\dot{u}^T(\tau+\tau')\,\dot{u}(\tau+\tau')] \times u(\tau+\tau')\} \ . \tag{2}$$

We must notice that the main contribution to the integral comes from $\tau' \leqslant \omega_0^{-1}$, which is an extraordinarily short time. Using a Taylor series expansion for $u(\tau+\tau')$ and $\dot{u}(\tau+\tau')$ and interchanging the orders of summation and integration, we obtain

$$\dot{u} = \sum_{n=0}^\infty \omega_0^{-n} u^{(n)} + \sum_{n,m,l=0}^\infty \frac{(n+m+l)!}{n!\,m!\,l!}\, \omega_0^{-(n+m+l+1)} \times [u^{(n+1)T} u^{(m+1)}]\, u^{(l)} \ . \tag{3}$$

From this equation we find that $\dot{u} = \omega u + O(\omega_0^{-1})$. Using this result and Eq. (3), an expression for $\dot{u}$ in terms of u up to terms of order ω_0^{-2} can be obtained and the procedure can be repeated. We find, for instance,

$$\begin{aligned}\dot{u} = \omega u &+ \omega_0^{-1}[\omega^2 u - (u^T \omega^2 u)\, u] \\ &+ \omega_0^{-2}[2\omega^3 u - 2(u^T\omega^2 u)\,\omega u] \\ &+ \omega_0^{-3}[5\omega^4 u - 5(u^T\omega^4 u)\,u + 6(u^T\omega^2 u)^2 u \\ &\quad - 6(u^T\omega^2 u)\,\omega^2 u] + O(\omega_0^{-4}) \ .\end{aligned} \tag{4}$$

From its derivation it is clear that this equation is valid for an arbitrary time-independent electromagnetic field. Furthermore we would like to point out that Eq. (4) can be derived more easily from (1) following a similar procedure.

Let us now consider the case of a constant magnetic field. We will choose the third axis along the direction of the field. In this case the matrix ω can be written as $\omega = \omega_H A$, where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

and $\omega_H = (e/m)H$. For electrons $\omega_H = 1.7588\times 10^7 H$ $\text{gauss}^{-1}\ \text{sec}^{-1}$. If $B = -A^2$ and we use the dimensionless proper time $x = \omega_H \tau$, then Eq. (4) can be written as

$$\frac{du}{dx} = [A - \alpha(S+B) - 2\alpha^2(1+S)A + \alpha^3(5+6S)(B+S) + \ldots]\, u \ , \tag{6}$$

where $S \equiv u_1^2(\tau) + u_2^2(\tau)$ and $\alpha = \omega_H\,\omega_0^{-1}$, which, for electrons, turns out to be $\alpha = 1.1021\times 10^{-16} H$. From Eq. (6) it follows that $u_3(x) = v_0 \cos\phi\, \gamma(x)$, where v_0 is the modulus of the initial velocity, and

ϕ is the angle between v_0 and the direction of the magnetic field. Then $S(x)=\epsilon^2\gamma^2(x)-1$, where $\epsilon^2=1-v_0^2\cos^2\phi$.

If we introduce the dimensionless laboratory time $z=\omega_H t$ we obtain the equations

$$\frac{d\gamma}{dz}=-\alpha(\epsilon^2\gamma^2-1)+\alpha^3(6\epsilon^4\gamma^4-7\epsilon^2\gamma^2+1)+\cdots,$$
$$\frac{d\tilde{v}}{dz}=\gamma^{-1}\{[-\alpha+\alpha^3(6\epsilon^2\gamma^2-1)+\cdots] + i\sigma_2[1-2\epsilon^2\gamma^2\alpha^2+\cdots]\}\,\tilde{v}(z)\,, \tag{7}$$

where $\tilde{v}(z)$ is a matrix column with components v_1 and v_2, and σ_2 is the usual Pauli matrix. Once the first equation (7) is solved, the velocity is given by

$$v_1(z)=v_0\sin\phi\exp[-h(z)]\cos g(z),$$
$$v_2(z)=-v_0\sin\phi\exp[-h(z)]\sin g(z)\,, \tag{8}$$

where

$$h(z)=\alpha I_{-1}-\alpha^3(6\epsilon^2 I_{+1}-I_{-1})+\cdots,$$
$$g(z)=I_{-1}-2\alpha^2\epsilon^2 I_{+1}+\cdots\,, \tag{9}$$
$$I_n=\int_0^z dz'\,\gamma^n(z')\,.$$

We would like to point out that in all these expansions the relevant parameter is, for large γ, $(\alpha\epsilon\gamma)^2$ and the series expansions will be useful when $(\alpha\epsilon\gamma)^2\ll 1$. For electrons this implies that $H\gamma\ll 10^{16}$ gauss. The Lorentz-Dirac equation is a classical equation which has no meaning when the quantum effects are important. In order for the quantum effects to be negligible two conditions must be satisfied[3]: The electron associated wavelength must be small compared with the other characteristic lengths of the problem, and the discrete nature of photon emission must be insignificant. The first condition implies that $\gamma(\gamma^2-1)\gg R_{cr}$, and the second implies that $R_{cr}\ll 1$, where $R_{cr}=\frac{3}{2}\gamma(e/m^2)H$. For electrons the second condition can be written as $H\gamma\ll 3\times 10^{13}$ gauss. Therefore when the classical equation is valid the expansion parameter is always very small ($\alpha\leqslant 10^{-3}$), and hence all higher-order terms in (7) are practically impossible to detect.

Since x is small we will solve the equation for $\gamma(z)$, keeping only first-order terms; then we obtain

$$\gamma(z)=\epsilon^{-1}\frac{(\epsilon\gamma_0+1)+(\epsilon\gamma_0-1)\exp(-2\epsilon\alpha z)}{(\epsilon\gamma_0+1)-(\epsilon\gamma_0-1)\exp(-2\epsilon\alpha z)}\,, \tag{10}$$

which is correct up to terms of α^3; (higher-order corrections can be calculated by Picard's Method). In this case $h(z)=\alpha g(z)=\alpha I_{-1}$, where now

$$I_{-1}=\epsilon z + \alpha^{-1}\ln\{(2\epsilon\gamma_0)^{-1}[(\epsilon\gamma_0+1)+(\epsilon\gamma_0-1)\exp(-2\epsilon\alpha z)]\}. \tag{11}$$

We would like to point out that the second-order terms in the Shen paper[1] are incorrect, but nevertheless this is not too important since in the present experimental situation their contribution is completely negligible.

[1] C. S. Shen, Phys. Rev. D 6, 2736 (1972). In this paper further references are given.

[2] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).

[3] T. Erber, Rev. Mod. Phys. 38, 626 (1966).