Comments and Addenda

The Comments and Addenda section is for short communications which are not of such urgency as to justify publication in **Physical Review Letters** *and are not appropriate for regular Articles. It includes only the following types of communications: (1) comments on papers previously published in* **The Physical Review** *or* **Physical Review Letters**; *(2) addenda to papers previously published in* **The Physical Review** *or* **Physical Review Letters**, *in which the additional information can be presented without the need for writing a complete article. Manuscripts intended for this section should be accompanied by a brief abstract for information-retrieval purposes. Accepted manuscripts will follow the same publication schedule as articles in this journal, and galleys will be sent to authors.*

Unitary multiperipheral model with isospin*

R. L. Sugar
Department of Physics, University of California, Santa Barbara, California 93106

(Received 30 November 1973)

A unitary, multiperipheral model is constructed in which isospin is taken into account. Unlike previous models of this type in which internal quantum numbers were neglected, it is possible to arrange both for saturation of forces between the multiperipheral chains and for a Pomeranchuk pole with intercept one.

Recently, a class of multiperipheral-like models was constructed for which the scattering operator satisfies full multiparticle unitarity in the direct channel.[1-3] The purpose of this note is to point out that the structure of these models can be altered significantly when the internal quantum numbers of the produced particles are taken into account.

The basic idea of these models is that the two incident high-energy particles interact by the exchange of multiperipheral-like chains along which secondary particles can be emitted or absorbed. Some typical chains are shown in Fig. 1. It is crucial for s-channel unitarity that one take into account the exchange of an arbitrary number of chains.[1-3] The inclusion of multichain effects constitutes the main difference between these models and the standard multiperipheral model. One of the most important of these multichain effects is that a secondary, which is produced on one chain, can either come off as a physical particle or be reabsorbed on a second chain, thereby giving rise to a force between the two chains.

Diagrams which contributed to the elastic-scattering amplitude are illustrated in Fig. 2. The exchange of a single chain gives rise to a pole in the angular momentum plane which will be referred to as the input pole. Two-chain exchange corresponds to the familiar ladder graphs whose leading l-plane singularity is also expected to be a pole. The diagrams which arise from the exchange of three or more chains can also have l-plane poles provided that the interchain forces are sufficiently strong. In previous work,[1-3] where the internal quantum numbers of the produced particles were ignored, one found that whenever the interchain forces were strong enough so that the pole of the ladder graphs was to the right of the input pole, the N-chain graphs had poles whose $t=0$ intercept grew at least as fast as N^2. In other words, the multichain forces did not saturate. The poles to the right of $l=1$ did not lead to a violation of the Froissart bound, since they were on an unphysical sheet of the l plane. Nevertheless, it was disconcerting that at least in the solvable models it was not possible to arrange for a pole on the physical sheet in the neighborhood of $l=1$. We shall now see that these difficulties can be overcome once isospin is taken into account.

Let us consider the scattering of two isospin-$\frac{1}{2}$ nucleons labeled a and b in Figs. 1 and 2. They interact by the exchange of a pair of exchange-degenerate isospin-1 Regge poles, the ρ-A_2, which are denoted by wavy lines. At each vertex along the Reggeon chain a pion (dashed line) can be emitted or absorbed.[4]

The unitary scattering operator can be written in the form

$$S=e^{iX}, \tag{1}$$

where the Hermitian operator X corresponds to the exchange of a single chain. X is a matrix in the isospin space of the nucleons and a functional of the creation and annihilation operators of the pions. It is convenient to write

$$X=\sum_{i,j=1}^{3}(\tau_a)_i\,\chi_{ij}(\tau_b)_j\equiv\vec{\tau}_a\cdot\chi\cdot\vec{\tau}_{bj}\tag{2}$$

and

$$\chi=\sum_{n=0}^{\infty}\chi_n\,.\tag{3}$$

Here n is the number of pions emitted or absorbed on the chain. Typical contributions to X are shown in Fig. 1. In order to illustrate the basic idea in the simplest possible setting, let us start by ignoring the transverse momenta. Working in the rest system of nucleon b and denoting the rapidity of nucleon a by Y, we have

$$(\chi_0)_{ij}=G_aG_be^{(\alpha-1)Y}\delta_{ij}\,,\tag{4}$$

$$(\chi_1)_{ij}=G_aG_be^{(\alpha-1)Y}\lambda\int_0^Y dt[\vec{t}\cdot\vec{a}(y)+\vec{t}^*\cdot\vec{a}^\dagger(y)]_{ij}\,,\tag{5}$$

$$\chi_n=G_aG_be^{(\alpha-1)Y}\lambda^n\int_0^Y dy_1\int_0^{y_1}dy_2\cdots\int_0^{y_{n-1}}dy_n[\vec{t}\cdot\vec{a}(y_1)+\vec{t}^*\cdot\vec{a}^\dagger(y_1)]\cdots[\vec{t}\cdot\vec{a}(y_n)+\vec{t}^*\cdot\vec{a}^\dagger(y_n)]\,.\tag{6}$$

α is the $t=0$ intercept of the Regge trajectory, which couples to nucleon a (b) with a factor of $G_a\vec{\tau}_a$ ($G_b\vec{\tau}_b$). λ is the effective pion coupling constant, and $a_i^\dagger(y)$ and $a_i(y)$ are the creation and annihilation operators for pions with rapidity y and isospin index i. They satisfy the commutation relations

$$[a_i(y),a_j^\dagger(y')]=\delta_{ij}\delta(y-y')\,.\tag{7}$$

The matrix $\vec{t}$ is the usual three-dimensional representation of the angular momentum operator $(t_k)_{ij}=-i\epsilon_{ijk}$. Notice that the form of the pion vertices given in Eqs. (5) and (6) is required in order that the operator X be Hermitian. The sum in Eq. (3) can be written formally as

$$\chi=G_aG_be^{(\alpha-1)Y}\mathcal{Y}\exp\left[\lambda\int_0^Y dy[\vec{t}\cdot\vec{a}(y)+\vec{t}^*\cdot\vec{a}^\dagger(y)]\right],\tag{8}$$

where the symbol $\mathcal{Y}$ stands for the y-ordered product. It is clear from Eqs. (2) and (8) that X is in fact Hermitian and therefore that S is unitary.

The elastic scattering amplitude T is obtained by taking the matrix element of S between states with no pions. Using units in which $m_a=m_b=1$

$$T=2ie^Y\langle 0|(1-S)|0\rangle\equiv\sum_{N=1}^{\infty}T_N\,.\tag{9}$$

The subscript N in the last term of Eq. (9) indicates the number of exchanged chains. One sees from Eqs. (2) and (6) that

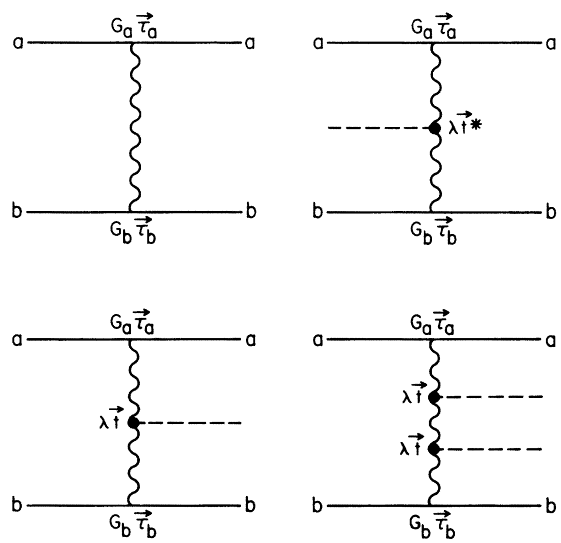

FIG. 1. Low-order contributions to the chain operator X.

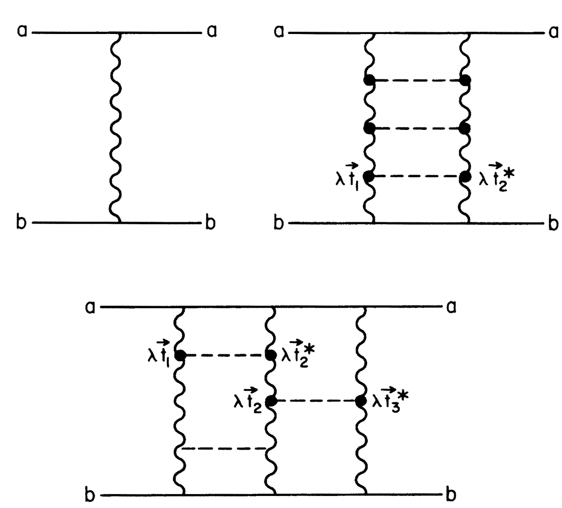

FIG. 2. Typical contributions to the elastic-scattering amplitude arising from one-, two-, and three-chain exchange.

$$T_1 = 2G_aG_b\vec{\tau}_a\cdot\vec{\tau}_b e^{\alpha Y} \tag{10}$$

$$T_2 = iG_a^2G_b^2e^{(2\alpha-1)Y}\sum_{i_1,i_2;j_1,j_2}(\tau_a)_{i_1}(\tau_a)_{i_2}\sum_{n=0}\frac{1}{n!}(\lambda^2 Y\vec{t}_1\cdot\vec{t}_2^*)^n{}_{i_1i_2;j_1j_2}(\tau_b)_{j_1}(\tau_b)_{j_2}\,. \tag{11}$$

Now $\vec{t}^* = -\vec{t}$ and the total isospin operator for the two exchanged Reggeons is $\vec{I} = \vec{t}_1 + \vec{t}_2$, so $\vec{t}_1\cdot\vec{t}_2^* = 2 - \frac{1}{2}\vec{I}^2$. As a result,

$$T_2 = iG_a^2G_b^2\vec{\tau}_a\vec{\tau}_a\cdot\exp\{Y[2\alpha - 1 + \lambda^2 - \tfrac{1}{2}\lambda^2\vec{I}^2]\}\cdot\vec{\tau}_b\vec{\tau}_b\,. \tag{12}$$

The only poles that couple to the external nucleons are those with isospin 0 and 1. They are located at

$$\alpha^{I=0}(2) = 1 + 2(\alpha - 1 + \lambda^2)\,,$$
$$\alpha^{I=1}(2) = 1 + 2(\alpha - 1 + \lambda^2) - \lambda^2\,. \tag{13}$$

The contribution arising from the exchange of N chains can be calculated in a similar fashion. There is a factor of $-\lambda^2\vec{t}_i\cdot\vec{t}_j$ for the emission of a pion from the jth chain and its reabsorption on the ith chain. The counting of the diagrams is the same as in nonrelativistic potential scattering, and because of the y ordering one obtains a factor of $Y^n/n!$ for diagrams involving the emission and reabsorption of n pions. As a result,

$$T_N \sim \exp\left\{Y\left[1 + N(\alpha - 1) - \tfrac{1}{2}\lambda^2\sum_{i\neq j}\vec{t}_i\cdot\vec{t}_j\right]\right\}$$
$$= \exp\{Y[1 + N(\alpha - 1 + \lambda^2) - \tfrac{1}{2}\lambda^2\vec{I}^2]\}\,. \tag{14}$$

In Eq. (14) use has been made of the fact that

$$\sum_{i\neq j}\vec{t}_i\cdot\vec{t}_j = \sum_{i,j}\vec{t}_i\cdot\vec{t}_j - \sum_i\vec{t}_i^{\,2}$$
$$= \vec{I}^2 - 2N\,, \tag{15}$$

with $\vec{t}_i^{\,2} = 2$ and $\vec{I} = \sum_i\vec{t}_i$. Thus, the N-chain exchange contains poles of isospin I located at

$$\alpha^I(N) = 1 + N(\alpha - 1 + \lambda^2) - \tfrac{1}{2}\lambda^2 I(I+1)\,. \tag{16}$$

Of course only the $I = 0, 1$ poles couple to the external nucleons. Equation (16) should be contrasted with the results of the solvable model of Ref. 1, which differed from the present model only in the fact that isospin was neglected. One then found that

$$\alpha(N) = 1 + N(\alpha - 1) + \tfrac{1}{2}\lambda^2 N(N-1)\,. \tag{17}$$

So, it is the inclusion of isospin which is responsible for the saturation of the interchain forces in the present model.[5]

Suppose we identify the $N=2$, $I=0$ pole as the Pomeron. Then setting $\alpha^{I=0}(2) = 1$ implies $\alpha - 1 + \lambda^2 = 0$. As a result,

$$\alpha^{I=0}(N) = 1\,, \quad N = 2, 3, \ldots$$
$$\alpha^{I=1}(N) = 1 - \lambda^2 = \alpha\,, \quad N = 1, 2, 3, \ldots. \tag{18}$$

Notice that $\alpha^{I=0}(2) = 1$ automatically means that the $I=1$ trajectory bootstraps itself.

In the present model all production arises from the exchange of $I=1$ objects. A pion couples to a set of N chains with a factor of $\lambda\sum_{i=1}^N\vec{t}_i = \lambda\vec{I}$. As a result, the emission of a pion cannot lead to a change in the total isospin of the exchanged object, nor can there be any emission from an $I=0$ object. One result of this fact is that there are no Pomeron-Pomeron forces.

The model just discussed can be generalized without altering its basic structure. First, transverse momenta and exact energy-momentum conservation can be included using the techniques developed in Ref. 1. Second, following the work of Chang and Segrè,[6] one can introduce along the incident lines not only the nucleons, but all nucleon resonances that can be excited from the incident particles by emission and absorption of the Reggeons. Let us denote the vertex of a Reggeon with isospin index i and transverse momentum $\vec{q}_\perp$ to two-nucleon states, n and n', by $M^i_{nn'}(\vec{q}_\perp)$. Then the superconvergence relations for Reggeon-nucleon scattering suggest that[5]

$$[M^i(\vec{q}_\perp), M^j(\vec{q}\,'_\perp)] = 0\,, \tag{19}$$

where $M^i(\vec{q}_\perp)$ stands for the matrix with elements $M^i_{nn'}(\vec{q}_\perp)$. The $M^i(\vec{q}_\perp)$ are to replace the $\vec{\tau}$ matrices in the construction of the chain operator X. Since they commute, one now has a true eikonal model with crossed as well as planar graphs.

If one makes the simplifying assumption that $M^i_{nn'}(q_\perp)$ can be written in the separable form $C^i_{nn'}f(q_\perp)$, then this model can also be solved in closed form. In this case the only allowed states for the exchanged chains are those which are totally symmetric under interchange of isospin indices. For the ladder graphs this means only $I=0$ exchange is allowed. These graphs can be summed by a multiperipheral-like integral equation, and the coupling constant can be chosen so that the leading trajectory, the Pomeron, has intercept one. Now consider three-chain exchange. If we first sum the interactions between a pair of chains, we again obtain the Pomeron, since any pair must be in an $I=0$ state. As explained above, with the present form of the coupling, the Pomeron cannot interact with the third chain, so the leading contribution to the elastic amplitude arising from three-chain exchange is the Mandelstam cut associated with a noninteracting Pomeron and a Reggeon. Similarly, the exchange of $2N$ chains will lead to the N-Pomeron cut and the exchange of $2N+1$ chains to the cut involving N Pomerons and one Reggeon.

Obviously, the models just considered are highly oversimplified; however, the main purpose of this note has merely been to exhibit a simple mechanism by which multichain forces can saturate. In more sophisticated models there must certainly be some interaction among Pomerons, although it may well be weak. Such interactions can be introduced into the present framework by allowing for the production along the chain of pairs of pions in an $I=0$ state,[2] and by allowing for the branching of the underlying chains.[7] It appears that neither of these mechanisms will destroy the saturation of the multichain forces.

I would like to thank J. C. Botke and D. J. Scalapino for helpful conversations.

---

*Work supported by the National Science Foundation.
[1]S. Auerbach, R. Aviv, R. Blankenbecler, and R. Sugar, Phys. Rev. Lett. 29, 522 (1972); Phys. Rev. D 6, 2216 (1972). The second of these papers will be referred to as I.
[2]S. Auerbach, Phys. Rev. D 8, 532 (1973).
[3]R. Sugar, Phys. Rev. D 8, 1134 (1973).
[4]Multiperipheral models with this isospin structure have been studied by L. Caneschi and A. Schwimmer, Phys. Rev. D 3, 1588 (1971), and by D. K. Campbell and S. J. Chang, Phys. Rev. D 8, 2996 (1973).
[5]However, a model without isospin in which the interchain forces saturate has been constructed by J. C. Botke, D. J. Scalapino, and R. L. Sugar, Phys. Rev. D 9, 813 (1974).
[6]L. N. Chang and G. Segrè, Phys. Rev. D 6, 2231 (1972).
[7]R. Blankenbecler, J. Fulco, and R. Sugar, Phys. Rev. D 9, 736 (1974).

PHYSICAL REVIEW D VOLUME 9, NUMBER 8 15 APRIL 1974

Spectral forms for three-point functions*

Julian Schwinger
University of California, Los Angeles, California 90024
(Received 31 December 1973)

A triple-spectral form is exhibited in a very simple situation.

It is characteristic of source theory that spectral forms for multifield couplings are inferred through physical rather than mathematical considerations by following the space-time propagation of various excitations. For tri-field couplings, in particular, where three momentum invariants are involved (say, p_α^2, p_β^2, p_γ^2), this has led to the development and application of spectral forms[1,2] in two of these variables, supplementing the more usual single-spectral forms (dispersion relations). As an example of their utility, we mention that the familiar anomalous-threshold situation encountered in the electromagnetic form factor of the deuteron is derivable from a double-spectral form in which only normal thresholds occur.[3] The possibility of representing any tri-field coupling, in three equivalent ways, by means of spectral forms that admit arbitrary values of two of the scalar momentum variables, while the third one can range arbitrarily only toward spacelike values, naturally suggests the existence of a still more fundamental representation of this multispectral type, in which all three variables appear on the same footing. The immediate obstacle to a direct attack on this problem is the absence of causal arrangements involving three independent timelike excitations (from which a triple-spectral form would be inferred). Accordingly, one is forced to fall back on more mathematical procedures.

The quite limited purpose of this note is to exhibit a triple-spectral form in a very special situation. It refers to the simplest nonlocal coupling of three scalar fields, as mediated by the exchange of spinless particles of *zero* mass. The essential structure of any of the three double-spectral forms[4] is illustrated by ($-i\epsilon$ is under-