Bootstrap equations with restricted SU(3) symmetry and the Cabibbo angle

J. A. Campbell*
CERN, *Geneva, Switzerland*

H. S. Green
Mathematical Physics Department, University of Adelaide, Adelaide, South Australia 5001, Australia

R. B. Leipnik
Michelson Laboratory, China Lake, California 93555
(Received 16 July 1973)

The bootstrap equations $v_i = cd_{ijk}(v_j v_k + a_j a_k) + V_i$, $a_i = \frac{1}{2}ed_{ijk}(a_j v_k + v_j a_k) + A_i$ for vector (v) and axial-vector (a) octet matrix elements, where V and A are driving terms, are studied. It is concluded that the equations in their most general form always have solutions with at least two free parameters, so that they may not be used to make unique physical predictions. The need for completeness of sums over repeated indices is questioned. As an example of what happens when the condition of completeness is relaxed, the equations exclusive of terms which represent weak neutral currents, and with no driving terms, are solved to give a Cabibbo angle which is 0.28 without adjustable parameters.

I. INTRODUCTION

The idea[1] that spontaneous breaking of SU(3) symmetry may be governed by a nonlinear equation in the relevant matrix elements has led to investigations of equations of a common over-all form by several authors.[2-6] This form may be summarized as

$$v_i = cd_{ijk}(v_j v_k + a_j a_k) + V_i\,, \tag{1}$$

$$a_i = \tfrac{1}{2}e\,d_{ijk}(a_j v_k + v_j a_k) + A_i\,, \tag{2}$$

where v and a stand for vector and axial-vector octet matrix elements, V and A are possible driving terms (nonzero if the symmetry breaking is supposed to be induced rather than spontaneous), and c and e are constants. The previous studies have covered the general behavior of (1) for only the vector octet,[2-4] with $a_j a_k$ absent, the general behavior of both equations without driving terms,[6] a particular model of the full set,[2] and a particular choice of driving term in (1), again with $a_j a_k$ absent.[5] Interest in this problem has fallen off lately following the establishment of two conclusions. Firstly, general solutions have the habit of depending on at least two free parameters as well as c or e, so that unique predictions for physical quantities such as the Cabibbo angle are not available. Secondly, with well-chosen driving terms in the equation for a single octet, it is possible to find estimates of a Cabibbo angle which are fairly close[3,5] to the experimental value.

In Sec. II, we fill a small gap by examining briefly the properties of the full set of Eqs. (1) and (2), and observe that once again the presence of at least two free parameters besides c or e means that no unique predictions are allowed to arise without extra assumptions.

While the fact that some driving terms generate good estimates of the Cabibbo angle is agreeable, it does not throw much light on the origin of the equations themselves. The extra physical assumptions which lead to uniqueness of solutions are all assumptions about V and A, and not about the remaining parts of the equations. Thus the question of how basic is the SU(3) content of (1) and (2) is postponed. In particular, there is no way to ask whether SU(3) is a foundation (axiom) on which the dynamics rest, or whether it is a happy accident of the underlying dynamics that pieces of what is derived from foundations not expressed in this group-theoretic language nevertheless exhibit some fossilized fragments which have SU(3) symmetry. If the first view is correct, then the repeated indices in (1) and (2) imply summation over all members of an octet, and the need to include all members is as pressing as, for example, the need to sum over all four space-time indices whenever one meets a repeated subscript in the special theory of relativity. Previous work on (1) and (2) has taken this view for granted. However, if the second view is nearer to the truth, then the "accidental" or secondary constraints stated by such equations may not apply to all members of an octet. In other words, there is a chance that invariance under SU(3) transformations here does not have the same protected status as invariance of form under Lorentz transformations for equations in the special theory of relativity. In Sec. III we investigate the situation in which the effects in (1) and (2) are spontaneous (i.e., no driving terms), and in which the contributions that refer to weak neutral currents are absent. This latter assumption, which excludes the indices 6 and 7, is reasonable in the sense that effects of these currents on their neighbors in the octet are generally regarded to be small. We are pleased to find a solution where the Cabibbo angle is approximately 0.28 compared to the experimental value of about 0.25.[6] We present this result, together with the assumption of noncompleteness of (1) and (2) which produces it, as a stimulus to further discussion.

II. PROPERTIES OF THE GENERAL EQUATIONS

Because the complete set of Eqs. (1) and (2) has not previously been examined, we study a model solution here far enough to show that it also suffers from the same type of nonuniqueness as the solutions obtained by Cronström and Noga[6] in the absence of driving terms.

If a_i and v_i are numbers, either one of the two bilinear terms in (2) may be discarded if the multiplier is changed from $\frac{1}{2}e$ to e. The actual form of (2) allows for the possibility that a_i and v_i may be noncommuting objects. A treatment of a solution for that case is reducible to the treatment of the commuting case, so that we consider an example of the more general problem here.

Firstly, only one of c or e need appear in the model equations, because we can redefine v_i and a_i as the old variables dilated by the reciprocal of the other constant. V_i and A_i can simultaneously be dilated in the same way. If we choose to make c disappear, then (1) and (2) in a tensor form [to emphasize the distinction between $a_j v_k$ and $v_j a_k$ in (2)] are

$$\begin{aligned} v^i{}_j &= v^i{}_k v^k{}_j + a^i{}_k a^k{}_j + V^i{}_j\,, \\ a^i{}_j &= \kappa(a^i{}_k v^k{}_j + v^i{}_k a^k{}_j) + A^i{}_j\,, \end{aligned} \tag{3}$$

where $v^i{}_j$ and $a^i{}_j$ are traceless tensors, and $\kappa = e/2c$. In this new form (3), an absence of driving terms means that $V^i{}_j$ and $A^i{}_j$ reduce to multiples of the unit tensor $\delta^i{}_j$. As matrix equations, we get

$$\begin{aligned} (v+a)^2 &= (v+a/\kappa) - (V+A/\kappa)\,, \\ (v-a)^2 &= (v-a/\kappa) - (V-A/\kappa) \end{aligned} \tag{4}$$

from (3). The solution of (4) follows (in the case of three-dimensional results, for the sake of illustration) from the exhaustive description of the combined V and A terms as

$$\begin{aligned} V+A/\kappa &= \alpha 1 + \beta z + \gamma z^2\,, \\ V-A/\kappa &= \alpha' 1 + \beta' z' + \gamma' z'^2\,, \end{aligned}$$

where α, β, γ, α', β', and γ' are numerical constants, and z and z' are matrices satisfying

$$z^3 = z\,, \quad z'^3 = z'\,. \tag{5}$$

Thus z^2 and z'^2 are idempotents, and z and z' are square roots of idempotents which are then used to construct $V+A/\kappa$ and $V-A/\kappa$. In the present case it is evident that solutions of (4) can be obtained in the form

$$\begin{aligned} v+a &= \phi 1 + \chi z + \psi z^2\,, \\ v-a &= \phi' 1 + \chi' z' + \psi' z'^2\,, \end{aligned} \tag{6}$$

where the six new quantities on the right-hand sides in (6) are constants.

For comparison with the results of Cronström and Noga,[6] we first consider an absence of driving terms ($\beta = \gamma = \beta' = \gamma' = 0$), and set $\kappa = 1$. When $\kappa = 1$, the equations for $v+a$ and $v-a$ are independent, and there is no reason why z and z' should be related in any particular way. The constants ϕ, χ, and ψ are given by

$$\begin{aligned} \phi &= \tfrac{1}{2} \pm (\tfrac{1}{4} - \alpha)^{1/2}\,, \\ \psi &= \begin{cases} 1 - 2\phi, & \chi = 0 \\ \tfrac{1}{2} - \phi, & \chi \neq 0\,, \end{cases} \\ \chi &= \pm(\psi - \psi^2 - 2\phi\psi)^{1/2}\,. \end{aligned} \tag{7}$$

A similar but independent solution exists in ϕ', ψ', χ', and α'. The three equations in (7) allow ϕ, ψ, and χ to be expressed only in terms of the free parameter α. The question of how many extra free parameters are necessary for the specification of the solutions in (6) is answered by examination of the solutions of (5). Here the answers are of three types, according to rank. The clearest statement of parameter dependence is found in the skew solutions

$$z = \begin{pmatrix} 0 & -i\lambda_1 & i\lambda_2 \\ i\lambda_1 & 0 & -i\lambda_3 \\ -i\lambda_2 & i\lambda_3 & 0 \end{pmatrix}$$

of rank 2, where $\lambda_3{}^2 = 1 - \lambda_1{}^2 - \lambda_2{}^2$. Rank-1 solutions, which are dyads $|\underline{x}\rangle\langle\underline{y}|$ where $\langle \underline{x} | \underline{y} \rangle = \pm 1$, and rank-3 solutions, which are adequately covered by Wedderburn[7] and other authors of standard texts on matrices, have the same free-parameter content for our purposes. Hence we are left with three free parameters $\alpha, \lambda_1, \lambda_2$, which are analogs of the three free parameters which Cronström and Noga[6] have found in their general scalar results.

When $\kappa \neq 1$, and there are still no driving terms, it is a direct consequence of (4) that v and a must commute, and z' can then be identified with z.

Our statements about free parameters above are unchanged, but now the actual solution of the equations is formally the same as if a driving term is present. We find the solution from the equations

$$\begin{aligned}
&\phi^2=\tfrac{1}{2}(\phi+\phi')+\tfrac{1}{2}(\phi-\phi')/\kappa+f\,,\\
&\phi'^2=\tfrac{1}{2}(\phi+\phi')-\tfrac{1}{2}(\phi-\phi')/\kappa+f'\,,\\
&2(\phi+\psi)\chi=\tfrac{1}{2}(\chi+\chi')+\tfrac{1}{2}(\chi-\chi')/\kappa+g\,,\\
&2(\phi'+\psi')\chi'=\tfrac{1}{2}(\chi+\chi')-\tfrac{1}{2}(\chi-\chi')/\kappa+g'\,,\\
&\chi^2+\psi^2+2\phi\psi=\tfrac{1}{2}(\psi+\psi')+\tfrac{1}{2}(\psi-\psi')/\kappa+h\,,\\
&\chi'^2+\psi'^2+2\phi'\psi'=\tfrac{1}{2}(\psi+\psi')-\tfrac{1}{2}(\psi-\psi')/\kappa+h'\,,
\end{aligned}\tag{8}$$

where

$$f=\tfrac{1}{2}(\alpha+\alpha')+\tfrac{1}{2}(\alpha-\alpha')/\kappa\,,$$
$$f'=\tfrac{1}{2}(\alpha+\alpha')-\tfrac{1}{2}(\alpha-\alpha')/\kappa\,,$$

and (g,g') and (h,h') are similarly related to (β,β') and (γ,γ'). The analogs of (7), obtained from (8) by elimination, are now quartics, from which numerical results may be computed. For example, the equation for ϕ is

$$[\phi^2-\tfrac{1}{2}(1+\kappa)\phi-f][\phi^2-\tfrac{1}{2}(1+\kappa)\phi-f-\tfrac{1}{4}(1-\kappa^2)]-\tfrac{1}{8}(1-\kappa)^3\phi-f'=0\,,\tag{9}$$

and an equation derived from (9) by the interchange of f and f' holds for ϕ'. The effort required for solution is at this stage rather great, but it suffices for our discussion to note that the free-parameter content of the general case is increased over that of the simplest case considered above by the presence of the constants β, β', γ, and γ'. Of course, a particular choice of driving term may fix these parameters along with α and α', but then there is still the degree of arbitrariness specified by the two quantities λ_1 [here identical to the λ_1' present in the solution of the second equation of (5)] and λ_2 (identical to λ_2').

Our conclusion is that the arbitrariness of the solutions of the model equations (1) and (2) remains even in the most general case, so that the prospects of finding unique predictions of physical quantities like the Cabibbo angle by this path of increasing generality are dim. Therefore, in the next section, we turn to a decrease of generality, by eliminating as much as is practicable of (1) and (2).

III. THE RESTRICTED EQUATIONS AND THEIR SOLUTION

Given the implications of earlier work that there are undetermined free parameters in solutions of (1) and (2), where $V_i=A_i=0$,[6] while a Cabibbo angle may be determined uniquely by suitable specific choices of V_i or A_i,[3,5] any analysis of the equations which excludes driving terms and still arrives at results not containing free parameters demands a little more respect than an approach which uses nonzero values of these terms. Additionally, analysis with $V_i=A_i=0$ more genuinely reproduces a "spontaneous" or "bootstrap" state of affairs, and concentrates attention on the physical consequences of the nonlinearity and the presence of the symmetric SU(3) structure constants d_{ijk}. We believe that the second is the stronger justification for excluding the driving terms from our analysis, but also that the first should not be overlooked.

If we return to our argument in Sec. I and wish to examine the behavior of (1) and (2) with some of the matrix elements removed, which of them should be omitted? The choice is not random, because of the physical associations of most of the members of an octet. Following Cronström and Noga,[6] we first write combinations of the matrix elements whose SU(3)-transformation properties are well known because of analogies with the transformations of particle states:

$$\begin{aligned}
&\pi^+=a_1+ia_2\,,\quad \pi^0=a_3\,,\\
&K^+=a_4+ia_5\,,\quad K^0=a_6+ia_7\,,\\
&\eta^0=a_8\,,
\end{aligned}\tag{10}$$

and remark that the corresponding combinations with v_i may be put in terms of π_V^+, π_V^0, K_V^+, etc. In this notation, if we assume that v_i and a_i are just numbers, the axial-vector Cabibbo angle is given by

$$\tan^2\theta_A=\frac{K^+K^-}{\pi^+\pi^-}=\frac{a_4{}^2+a_5{}^2}{a_1{}^2+a_2{}^2}\,,\tag{11}$$

and the corresponding vector angle by

$$\tan^2\theta_V=\frac{v_4{}^2+v_5{}^2}{v_1{}^2+v_2{}^2}\,.\tag{12}$$

Thus we need indices $i=1,2,4,5$ even in a restricted set if we are to calculate the Cabibbo angles without free parameters determined by constraints external to (1) and (2). Further, we have the familiar charge-hypercharge external relation

$$v_3=\sqrt{3}\,v_8\,,\tag{13}$$

which should hold exactly in the absence of weak interactions, and approximately in general.[6] Thus, unless (13) is to be entirely detached from influence on the quantities in a restricted version of (1), either one or both of $i=3$ and $i=8$ must be taken into account. We choose to include both indices, retaining (13) as a possible device to distinguish solutions of (1) and (2) later. If only $i=3$ or $i=8$ is included, the resulting equations give zero or contradictory answers.

The indices $i=6$ and $i=7$, which refer to weak neutral currents, remain unconsidered. Cronström and Noga[6] use these currents merely in the condition that

$$|K^0|^2 \ll |\pi_V^+|^2, \quad |K_V^0|^2 \ll |\pi_V^+|^2. \tag{14}$$

K^0 and K_V^0 are outside the primary scope of the Cabibbo theory in any case, and if we take the view that (1) and (2) are secondary manifestations of some underlying dynamics of all those matrix elements which are governed by the restricted bootstrap equations, the scale of the weak neutral currents may be fixed by a completely different dynamical process. It is encouraging then for proponents of (14) that K^0 and K_V^0 can only be built from the conventional weak-interaction Lagrangian by means of terms in G^2 or higher orders of the coupling constant. We choose to exclude $i=6$ and $i=7$ from consideration.

The restricted set of bootstrap equations from (1) and (2) is then

$$v_i = c d_{ijk}(v_j v_k + a_j a_k) \tag{15}$$

and

$$a_i = e d_{ijk} a_j v_k , \tag{16}$$

with $i, j, k \in (1,2,3,4,5,8)$. The two bilinear terms in (2) are combined into one in (16) if v_i and a_i are numbers.

Cronström and Noga[6] select only CP-conserving possibilities by requiring that $K^+ = K_V^+$, i.e., that $a_4 + i a_5 = v_4 + i v_5$, and then find that there are no physical solutions for the unrestricted equations unless $e = 2c$. In (15) and (16) there is the possibility of introducing parity-violating effects, which are of at least second order in $(e-2c)$ for small values of that difference, and which may be made as small as desired by comparison with parity-conserving effects by appropriate choices of e/c, but this leads to a dependence of results on free parameters, which we are anxious to avoid. Therefore we examine what happens when we impose CP conservation and put $e=2c$.

When written out in full, (15) and (16) contain a number of simple correspondences. In particular, $v_4 = a_4$ and $v_5 = a_5$ satisfy the equations for $i=4$ and $i=5$. Next, we may set $v = v_1 = v_2$ and $a = a_1 = a_2$, which reduces the equations for $i=1$ and $i=2$ to

$$v = \frac{2c}{\sqrt{3}}(v v_8 + a a_8), \quad a = \frac{2c}{\sqrt{3}}(v a_8 + a v_8). \tag{17}$$

This simplification also gives us

$$\tan^2\theta_V = v_4^2/v_1^2, \qquad \tan^2\theta_A = a_4^2/a_1^2 = v_4^2/a_1^2 \tag{18}$$

from (11) and (12).

From (17), it follows that

$$v_1 = a_1 a_8 \Big/ \left(\frac{\sqrt{3}}{2c} - v_8\right), \tag{19}$$

and that

$$a_8^2 = \left(\frac{\sqrt{3}}{2c} - v_8\right)^2. \tag{20}$$

The equations (19) and (20) lead to two possibilities: either

$$a_8 = \frac{\sqrt{3}}{2c} - v_8, \quad v_1 = a_1 \tag{21}$$

or

$$a_8 = v_8 - \frac{\sqrt{3}}{2c}, \quad v_1 = -a_1. \tag{22}$$

The case (21) produces trivial results, but (22) is more interesting. Its first by-product is that (18) is a relation for a single Cabibbo angle θ. Secondly, on substitution into the equation for v_8 in (15) and (16), it causes a cancellation of all factors with the subscript 8, leaving the equation

$$\frac{\sqrt{3}}{4c} - \frac{4c}{\sqrt{3}} v_1^2 = -\frac{\sqrt{3}}{2c} + \frac{4c}{\sqrt{3}} v_1^2,$$

or

$$v_1^2 = \frac{9}{32c^2}. \tag{23}$$

Next, we observe that the two $i=3$ equations in (15) and (16) are consistent with $v_3 = a_3$. Hence we are left with three unknowns v_3, v_4, and v_8, and three equations (for v_3, for v_4, and the equation originally written for a_8). The system is thus simplified to

$$\begin{aligned} v_3 &= \frac{2c}{\sqrt{3}} v_3 v_8 + c v_4^2, \\ \frac{1}{4c} &= v_3 - \frac{1}{\sqrt{3}} v_4, \\ v_3^2 - v_4^2 - v_8^2 &+ \frac{3}{16c^2} = 0. \end{aligned} \tag{24}$$

At present, to obtain the Cabibbo angle, we are interested mainly in v_4, although we may later wish to recover v_3 and v_8 to check Eq. (13). The elimination of v_3 and v_8 from (24) produces the single quartic equation

$$35 v_4^4 - \frac{16\sqrt{3}}{c} v_4^3 - \frac{9}{c^2} v_4^2 + \frac{31\sqrt{3}}{8c^3} v_4 + \frac{9}{8c^4} = 0. \tag{25}$$

Regarded as an equation for cv_4, (25) has two real roots at -0.154 and -0.432. From (24) it may be deduced that the ratio of the sizes of v_3

and v_8 is not exactly $\sqrt{3}$ for either root. In fact it is greater than two for the first root, but more than an order of magnitude smaller (about $\sqrt{3}/20$) for the second root. If the complex roots 0.689 $\pm 0.095i$ are admitted, then their performance as judged by (13) is even worse. Thus (13) favors the first solution. From (18) and (23) we then find that

$$\theta = \tan^{-1}\left[\frac{32}{9}(-0.154)^2\right]^{1/2} \approx 0.28\,, \tag{26}$$

the dependence on c having canceled. This value of θ is to be compared with the experimental value of about 0.25,[6] and the theoretical values of 0.26 (actually 15°) given by Solomon and Ne'eman,[3] and 0.14 to 0.22 given by Tanaka and Tarjanne.[5]

IV. DISCUSSION

By supposing that the bootstrap equations (1) and (2) without driving terms do not hold for all matrix elements in an octet, but exclude control over the matrix elements referring to weak neutral currents, we obtain a Cabibbo angle which is rather close to the accepted experimental value. The result does not depend on free parameters. The choice of the angle is unique if we use (13) as a means of expressing an external preference about the goodness of a solution to the restricted bootstrap equations. Otherwise [e.g., if we are discussing just the axial-vector angle θ_A, for which there is no constraint analogous to (13)] we find two possible Cabibbo angles, only one of which is near the experimental value. It therefore seems evident that some extra physical demand [like (13), but not necessarily this condition] outside the range of the bootstrap equations must come into play to establish uniqueness.

It may be argued that the truncation of (1) and (2) to (15) and (16) is equivalent to a choice of driving term which breaks the symmetry of the full equations and gives the definite value (26) of the Cabibbo angle. Starting from (1) and (2), however, we have been unable to find any such driving term.

The special interest of the result (26) lies in the implications of what we have assumed at the beginning. That is to say, we suppose that (1) and (2) are not equations invariant under all SU(3) transformations, i.e., that they are not basic to a dynamical theory. Instead, since we have excluded from consideration the terms involving weak neutral currents, we are effectively saying that those terms are governed by different dynamical considerations. The rest of the set of eight vector or axial-vector matrix elements may be presumed to have some basic dynamical properties which generate as a higher-level consequence the traces of SU(3) symmetry which are seen in (15) and (16).

Finally, we do not present the result of this artificial model calculation from Sec. III as firm evidence for the physical view advanced in the previous paragraph. Rather, we would like to suggest that Sec. III is evidence that such a view deserves some consideration side by side with the better-known alternatives.

---

*On leave from King's College, Cambridge, England. Senior Research Fellow in the King's College Research Centre.

[1] R. E. Cutkosky and P. Tarjanne, Phys. Rev. 132, 1354 (1963).

[2] A. Pais, in *Elementary Particle Physics*, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 215.

[3] A. I. Solomon and Y. Ne'eman, Nuovo Cimento 60A, 108 (1969).

[4] J. A. McClelland, K. Tanaka, and P. Tarjanne, Phys. Rev. 184, 1677 (1969).

[5] K. Tanaka and P. Tarjanne, Phys. Rev. Lett. 23, 1137 (1969).

[6] C. Cronström and M. Noga, Nucl. Phys. B17, 477 (1970).

[7] J. H. M. Wedderburn, *Lectures on Matrices* (American Mathematical Society, New York, 1934).