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Mathematical structure of the Bethe-Salpeter equation for massless exchange reinvestigated

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Solutions of the Bethe-Salpeter (BS) equation for two scalar particles of arbitrary masses interacting through an exchange of a massless scalar particle are reinvestigated by means of a new method, applicable to the problem irrespective of the metric (Euclidean or pseudo-Euclidean) of the underlying four-momentum space. This enables one to perform a step-by-step comparison of the solving procedure as applied on one hand to the "original" equation (with underlying pseudo-Euclidean metric), and to the corresponding Wick equation (obtained from the former by formally performing the Wick rotation without prior proof of its actual validity) on the other hand. At a certain point (compare the results of Secs. IV and V, respectively) the kernels of the appropriate transformed integral equations corresponding to the two cases become manifestly analytically different. This finding seems not only to render the Wick rotation—*a posteriori*—invalid, but also to preclude one—in the realistic (i.e., "original") case—from obtaining the well-known Wick-Cutkosky solutions (reproduced fully in the case of the Wick equation). Although the "original" version of the BS equation is thus found too difficult to solve exactly (due to the presence of an additional parameter in the kernel), the method developed leads in a most natural way to an exactly soluble model of the BS equation obtained by retaining the pseudo-Euclidean metric but replacing the Feynman propagator D_F for the exchange particle by the "relativistic Coulomb" propagator $\bar{D}$ (half the difference between "advanced" and "retarded" propagators). This model exhibits a marked correspondence—in its nonrelativistic limit—with the Schrödinger solution of the Coulomb problem. It should finally be noted that our method avoids any series expansions of the results whatsoever (partial-wave expansion included) which would otherwise tend to obscure clear-cut "analytic" conclusions by posing convergence problems, and thus aims always at obtaining a closed-form expression for the total (off mass shell) scattering amplitude.

I. INTRODUCTION

The basic motivation for the present investigation derives from an attempt to obtain a closed-form expression for the total scattering amplitude in the scattering region for the Bethe-Salpeter (BS) equation first investigated by Wick[1] and Cutkosky[2] more than 20 years ago. The problem is far from trivial. In spite of the fact that a nonrelativistic counterpart of such a closed-form expression is well known[3] and the fact that the Wick-Cutkosky solutions were also well known at that time, at least two early attempts in this direction—notably those by Nishijima[4] and Okubo and Feldman[5]—fell very short of the desired aim. The last three authors were able only to show that the problem of finding such a total scattering amplitude reduces to solving a rather complicated boundary-value problem in two variables, based on a *nonseparable* partial-differential equation[6] of the second order and for a rather complicated integral transform (more or less in the manner of Wick and Cutkosky) of this amplitude, at that. Nonetheless a feeling remains that even in the absence of an ansatz for the general solution a much simpler answer to the problem could still be obtained "by brute force" (i.e., synthetically) by somehow performing a closed-form summation [involving first of all (but perhaps not only) a partial-wave summation] on the already known Wick-Cutkosky solutions—or rather on the solutions of the inhomogeneous version of the Wick-Cutkosky equations (for noneigenvalue energies) with inhomogenieties equal to the appropriate projections of the so-called Breit term. That at least so defined a "program" can be carried out successfully was shown by Tang,[7] whose result represents an improvement on the Nishijima-Okubo-Feldman (NOF) result in that the problem of finding the total scattering amplitude now appears to reduce to a boundary-value problem in two variables, but one based this time on a *separable* partial differential equation obeyed (up to a known function as a factor) by the scattering amplitude itself.[8]

For what follows it is now of importance to notice that the apparent success of Tang's summation is in no small measure related to the use of an alternative and simplified method of deriving the Wick-Cutkosky equations discovered by Biswas and Green[9] which from the onset consistently avoids complicated integral transforms by introducing rather cleverly devised "bifocal coordinates" related directly to the four-momentum components (these bifocal coordinates seem not to be widely known in the literature). On the other hand, the Green-Biswas method relies much more heavily on the prior validity of the Wick rotation than does the Nishijima-Okubo-Feldman method. More specifically, while the bifocal coordinates can only be used in a meaningful way as integration variables in the *already Wick-rotated* version of the BS equation, the equalities (2.10) of Ref. 4 and (17) of Ref. 5 do not depend on the validity of the Wick rotation. Moreover, it is also extremely hard to visualize what sort of analytic continuation the Wick rotation actually implies in terms of those bifocal coordinates *in general*, i.e., quite apart from whether or not one proposes to use them as integration variables (compare discussions at the beginnings of Secs. IV and V).

The last remark has as its consequence that even if we simply *assumed* the validity of the Wick rotation we would still not be able to profit much from Tang's results, since we would not be able to meaningfully "translate" the content of the boundary-value problem of Sec. IV (where Tang's results are rederived; compare footnote 8) back into the realistic "pseudo-Euclidean world" (i.e., reverse the Wick rotation).[10]

Faced with this deadlock, but encouraged by the relative simplicity of Tang's result, we base present method essentially on the Green-Biswas type of coordinates and endeavor to solve both the original and the Wick-rotated BS equation concurrently but separately. The new method avoids entirely the partial-wave expansions, which feature it therefore shares with the above-mentioned methods of NOF (but not with the Tang method).

The order of the presentation and the main results are as follow: In Sec. II the necessarily rather extensively varied notation used throughout the paper is defined and summarized. This is followed in Sec. III by the presentation of the rudiments of the new method of approach insofar as it can be applied without variation to both the original and the Wick-rotated BS equation. Section IV is devoted to the Wick-rotated BS equation, where, as already mentioned, the Wick-Cutkosky-Green-Biswas results as well as the Tang results are reobtained, but the results of Sec. V seem unfortunately to *prove conclusively that the solution of the original BS equation must be analytically different from that of the Wick equation,* to the extent that the former (i.e., the solution of the original equation) must even exhibit a dependence on an additional parameter (ψ or ω) totally absent in the latter.[11] The conclusions of Sec. V seem in fact to be tantamount to a doubly negative result that (1) the Wick rotation is not valid, and that (2) although the closed-form solution of the original BS equation is not obtainable by our method (and therefore not even attempted), the Wick-Cutkosky solutions are definitely *not* the solutions of this equation. However, the considerations of Sec. V lead in the most natural way to the construction of an exactly soluble model of the BS equation of much greater analytical simplicity than even the Wick-Cutkosky-type solution (of the Wick equation). This model is obtained by *retaining the pseudo-Euclidean type of metric* but essentially[12] only by replacing the Feynman propagator for the exchange particle ("photon") by the "relativistic Coulomb" (half of the difference between the "retarded" and "advanced") propagator.[13]

Importantly, but at this point perhaps not too surprisingly, the solution of the model bears a striking resemblance to the solution of the nonrelativistic Schrödinger two-body problem and in fact becomes identical with it in the limit of nonrelativistic energies and small coupling constant, in further contrast with the Wick-Cutkosky eigenvalue problem, which seems to bear little or no resemblance[14] to the Schrödinger solution in this limit.

Returning briefly once more to the main issue of the validity of the Wick rotation, the mathematical findings of Sec. V can perhaps be best summarized by stating that in the Wick-rotated version of Eq. (47) the application of the "four-dimensional Laplacian" to $(p-p')^{-2}$ produces a Dirac δ function, which in the method devised by Biswas and Green[15] reduces (47) at once to a differential equation. However, upon closer examination, the Fourier transform $[(p-p')^2+i\epsilon]^{-1}$ of the Feynman propagator in the non-Euclidean *momentum* space is found *not* to exhibit such a simple property. This is of course connected with the existence of the momentum-space null cone instead of a point at $(p-p')^2=0$, so that the considerations of Sec. V are tantamount to a careful examination of what the term $+i\epsilon$ actually implies on such a null cone.[16] In this context, the exactly soluble model presented in Sec. VI is characterized by

$$\Box_p (p-p')^{-2}=0,$$

which however does not preclude the fact that the "relativistic Coulomb" propagator *does obey* the *inhomogeneous* Klein-Gordon equation in *position* space (i.e., with a Dirac δ function on the right-hand side).

II. NOTATIONS AND GENERAL DEFINITIONS OF VARIABLES USED

As mentioned in the Introduction, we shall preface the presentation of the new solving procedure of the BS equation by carefully defining the variables used throughout this paper. First, the notation will have to occasionally depart from that most commonly used in the literature of the BS equation. Second, because we will have to apply our method to the Wick equation and to the original BS equation *separately*, repetition will be avoided as well as uniformity preserved in spite of the widely different integration domains involved in these two cases. For these reasons also a number of new conventions must be made well in advance of the point, further in the text, where their usefulness will first become fully apparent.

A. The Lorentz frame used

Practically all derivations will be made in the barycentric system in which the over-all energy-momentum four-vector will have the components

$$r \equiv (\vec{0}, E), \tag{1}$$

where E denotes the total energy.

B. Labeling of states

All quantities pertaining to the initial, final, and intermediate (or "integrated-over") states will consistently be labeled by "no prime," "prime," and "double prime," respectively. Similarly all quantities definable only in terms of a pair of states (e.g., the scattering angle) will be labeled by "no prime," "prime," and "double prime," according to whether they refer to intermediate-final, final-initial, or initial-intermediate pairs of states, respectively.

C. Variables pertaining to a single state

Variables defined here will carry no prime, but, as explained above, all definitions will automatically be understood to be valid for primed and double-primed quantities.

Given r, another four-vector (related to the momentum transfer) is needed to uniquely determine the four-momenta p_A and p_B of the scattered particles A and B. As such we shall choose the four-vector p, defined by

$$p = -\beta_+ r + p_A = +\beta_- r - p_B, \tag{2}$$

where

$$\beta_+ + \beta_- = 1. \tag{3}$$

However, contrary to the convention adopted by most authors, we shall (except in the case of equal masses) not simply put $\beta_\pm = \frac{1}{2}$, but assign them the following values:

$$\beta_\pm = \frac{E^2 \pm m_A{}^2 \mp m_B{}^2}{2E^2}, \tag{4}$$

where m_A and m_B are the masses of the particles A and B. This is because we want to use, as presently explained, the so-called bifocal coordinates, first introduced and extensively used in connection with the Bethe-Salpeter equation by Green and Biswas.[9] Using these coordinates—in conjunction with new parameters introduced here and in Secs. II D and II E—will constitute a very important part of our new approach (see Introduction). However, because the usefulness of the Green-Biswas parameters as *integration variables* derives primarily from their application to the Wick case, we will define them in two steps, first introducing the parameters $v_\pm$, more useful as integration variables in the case of the original BS equation (i.e., with underlying pseudo-Euclidean metric). Those will be defined as

$$v_\pm = \frac{p_0 \pm |\vec{p}| - ic}{p_0 \pm |\vec{p}| + ic}, \tag{5}$$

where p_0 and $|\vec{p}|$ are respectively the timelike component and the magnitude of the three-momentum of p [as defined by (2)] in the frame of reference (1), and where c is given by

$$c = \frac{1}{2E}[-(E - m_A - m_B)(E - m_A + m_B) \times (E + m_A - m_B)(E + m_A + m_B)]^{1/2}. \tag{6}$$

The bifocal coordinates φ and ψ can then be defined by

$$v_\pm = e^{-i\varphi \mp i\psi}. \tag{7}$$

Solving (5) and (7) for p_0 and $|\vec{p}|$, we obtain the original definitions of these parameters by Green and Biswas:

$$|\vec{p}| = \frac{c \sin\psi}{\cos\varphi - \cos\psi}, \tag{8}$$

$$p_0 = \frac{-c \cos\varphi}{\cos\varphi - \cos\psi}. \tag{9}$$

Now, the particular values (4) assigned to $\beta_\pm$, as well as that of the constant c given by (6), are chosen so that

$$\frac{dp_0 d|\vec{p}|}{(p_A{}^2 - m_A{}^2)(p_B{}^2 - m_B{}^2)} = \frac{dp_0 d|\vec{p}|}{[(\beta_+ E + p_0)^2 - \vec{p}^2 - m_A{}^2][(\beta_- E - p_0)^2 - \vec{p}^2 - m_B{}^2]} = d\varphi\, d\psi \times (\text{function of } \varphi \text{ only}), \tag{10}$$

or equivalently, in terms of $v_\pm$,

$$\frac{dp_0 d|\vec{p}|}{(p_A{}^2 - m_A{}^2)(p_B{}^2 - m_B{}^2)}$$

$$= dv_+ \, dv_- \times (\text{function of the product } v_+ v_- \text{ only}),$$

the requirement (10) bringing about a great simplification of the problem. More precisely, because the Jacobian

$$\frac{\partial(p_0, |\vec{p}|)}{\partial(v_+, v_-)} = \frac{-2c^2}{(v_+ - 1)^2 (v_- - 1)^2} \qquad (11)$$

the condition (10) is equivalent to the following two conditions:

$$(\beta_\pm E \pm p_0)^2 - \vec{p}^{\,2} - m_\pm{}^2 = \frac{1}{(v_+ - 1)(v_- - 1)} \times f_\pm(v_+ v_-), \qquad (12)$$

where $f_\pm(v_+ v_-)$ have to be functions of the product $v_+ v_-$ only and where we have also used the abbreviations

$$m_A = m_+, \quad m_B = m_-. \qquad (13)$$

The conditions (12) lead to

$$m_\pm{}^2 = \beta_\pm E^2 + c^2. \qquad (14)$$

Solving the system of equations (3) and (14) for $\beta_\pm$ and c, we obtain (4) and (6). With the values of $\beta_\pm$ and c so determined the right-hand side of (10) becomes

$$-\frac{1}{2} \frac{dv_+ \, dv_-}{(E_+ - ic)(E_- + ic)(v_+ v_- - u_+{}^2)(v_+ v_- - u_-{}^2)}$$

$$= \frac{d\varphi \, d\psi}{4 m_A m_B \sin(\varphi + \alpha_+) \sin(\varphi - \alpha_-)}, \qquad (15)$$

where the quantities $u_+{}^2, u_-{}^2$ and α_+, α_- are defined by

$$u_\pm{}^2 = \frac{E_\pm \pm ic}{E_\pm \mp ic} = e^{\pm 2i\alpha_\pm} \qquad (16)$$

and where we have also introduced the abbreviation

$$E_\pm = \beta_\pm E. \qquad (17)$$

It should be noted that c is imaginary in the scattering region (in the direct channel for $E > m_A + m_B$ as well as in one of the crossed channels for $0 < E < |m_A - m_B|$), in which case we shall also use the notation

$$d = ic, \qquad (18)$$

with d assumed positive. Physically d is the absolute value of the three-momentum of each particle in the barycentric system (pertaining to the on-the-mass-shell situation, however, and therefore to be distinguished from the variable $|\vec{p}|$). In the bound-state region c is real and allows the "geometrical" interpretation as the height of the triangle depicted in Fig. 1, whereas $\alpha_\pm$ acquire the meaning of the angles at the base E of this triangle. The remaining notations pertaining to a single state are the following:

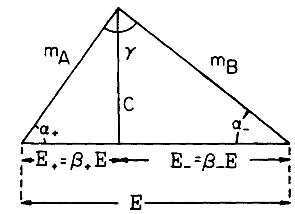

FIG. 1. Geometrical interpretation of the relationships between the total energy E, the masses m_A and m_B of the scattered particles, the angles $\alpha_\pm$ defined by (16), the parameters c and $\beta_\pm$ defined by (6) and (4), respectively, and finally the angle γ entering Eqs. (198) and (199). The situation depicted corresponds to E in the bound-state region.

$$w = \cos\psi = \frac{1}{2} \frac{v_+ + v_-}{(v_+ v_-)^{1/2}}, \qquad (19)$$

$$\omega = v_+ + v_-, \qquad (20)$$

$$u = e^{-i\varphi}, \qquad (21)$$

$$\rho = e^{-i\psi}. \qquad (22)$$

w and ω are used mainly as convenient notations in the otherwise too lengthy formulas of Sec. V, while the convenience of the notation (21) and (22) derives mainly from considerations pertaining to the Wick equation (Sec. IV).

D. Variables definable in terms of a pair of states

Variables defined here will again carry no primes and therefore will pertain to the intermediate-final pair of states (the latter denoted by "double prime" and "prime," respectively), but the definitions will be understood to be valid for the remaining two pairs of states by appropriate cyclic permutations of primes (see Ref. 17, however). Denoting by θ the scattering angle [i.e. the angle between the three-vectors $\vec{p}''$ and $\vec{p}'$ in the barycentric system (1)], we first introduce the notation

$$z = \cos\theta. \qquad (23)$$

Having thus defined what we shall henceforth refer to as the z variables or z's, we next introduce the y variables as exemplified by

$$y = \frac{|\vec{p}''|^2 + |\vec{p}'|^2 - (p_0'' - p_0')^2 - i\epsilon}{2|\vec{p}''|\,|\vec{p}'|}. \qquad (24)$$

The usefulness of this notation stems from the fact that the propagator for the exchange particle can then be written as

$$\frac{1}{(p''-p')^2}=\frac{1}{2|\vec{p}''||\vec{p}'|}\,\frac{1}{z-y}\,. \qquad (25)$$

With values of p_0'', $|\vec{p}''|$ and p_0', $|\vec{p}'|$ considered as given, y can therefore also be defined as the cosine of such a scattering angle as would correspond to the situation where the exchange four-momentum is a null vector (or lies on the mass shell of the exchange particle, since the mass of the latter is assumed to be $=0$).

Using the notations introduced in Sec. II C, we can either use the Green-Biswas parameters and write

$$y=\frac{-\cos\psi''\cos\psi'+\cos(\varphi'-\varphi'')}{\sin\psi'\sin\psi''} \qquad (26)$$

or use the parameters $v_\pm$ defined by (5). In the latter case the most noteworthy formula is perhaps not so much the expression for y itself as the relation

$$\frac{y-1}{y+1}=\frac{(v_+''-v_+')(v_-''-v_-')}{(v_+''-v_-')(v_-''-v_+')}\,. \qquad (27)$$

E. Special variables introduced to later replace the variables z and z'' (Ref. 18)

For the purpose of avoiding partial-wave expansions entirely it will be most convenient, as seen later in the text [compare the derivation of Eqs. (69) and (70) in Sec. III], to eliminate z and z'' in terms of one of the parameters τ, σ, t, s and $\tau'', \sigma'', t'', s''$, respectively, which we are now going to define. The defining relation for τ is modeled on (26) and reads

$$z=\frac{-\cos\psi''\cos\psi'+\cos(\varphi''-\varphi'+\tau)}{\sin\psi'\sin\psi''}\,, \qquad (28)$$

so that for $\tau=0$ z becomes equal to y. Likewise the parameter τ'' is defined by[18]

$$z''=\frac{-\cos\psi\cos\psi'+\cos(\varphi-\varphi'+\tau'')}{\sin\psi'\sin\psi}\,. \qquad (29)$$

Since Eqs. (28) and (29) do not define the quantities τ and τ'' uniquely, however, they shall at the same time be considered as defining relations for the quantities σ and σ'', the latter defined as the only other (modulo 2π) pair of solutions of (28) and (29), respectively. More precisely, starting with a particular pair of solutions τ and τ'' of (28) and (29), another pair of solutions σ and σ'' can be constructed by setting

$$\sigma=2(\varphi'-\varphi'')-\tau \qquad (30)$$

and

$$\sigma''=2(\varphi'-\varphi)-\tau''. \qquad (31)$$

With the choice as to which of the two basic solutions of, e.g., Eq. (28) to call τ and which σ remaining to a large extent optional, the τ and σ parameters are bound to play a highly symmetric role in our further considerations. For that reason, it will be ultimately most convenient to use τ's and σ's—or rather the presently defined t and s parameters—as *independent variables*, eliminating not only z and z'' but φ and φ'' through (30) and (31) as well. The t and s parameters are defined by

$$t=e^{i\tau},\quad s=e^{i\sigma}, \qquad (32)$$

and

$$t''=e^{i\tau''},\quad s''=e^{i\sigma''}, \qquad (33)$$

and the convenience deriving from the option to use them as independent variables will become especially apparent in the case of the quasi-Euclidean metric of the underlying four-momentum space, so that, beginning with Eq. (116) of Sec. V, all formulas are indeed written almost exclusively[19] in terms of these parameters. At this point it is perhaps also worth mentioning that, as a general rule, the exponentials, such as (32) and (33), and the $v_\pm$ parameters defined by (5), are better suited to the considerations pertaining to the case of the quasi-Euclidean metric, while the "angles" τ, σ, φ, and ψ are better suited to the case of the Wick-rotated BS equation.

For these reasons it is worth noting for the purpose of future reference that, because of (7), the relations (30) and (31) are equivalent to

$$v_+'v_-'\,st=v_+''v_-'' \qquad (34)$$

and

$$v_+'v_-'\,s''t''=v_+v_-. \qquad (35)$$

There is also an alternative option of defining the t and s parameters in terms of z's and v's *directly*, as pairs of solutions of the quadratic equations

$$\frac{z-1}{z+1}=\frac{(v_+''-tv_+')(v_-''-tv_-')}{(v_+''-tv_-')(v_-''-tv_+')}, \qquad (36)$$

$$\frac{z''-1}{z''+1}=\frac{(v_+-t''v_+')(v_--t''v_-')}{(v_+-t''v_-')(v_--t''v_+')}\,, \qquad (37)$$

with Eqs. (34) and (35) precisely the relations between the appropriate roots, and regarding (32) and (33) as defining relations for τ's and σ's. Moreover, Eqs. (36) and (37) can be considered modeled on (27) in precisely the same sense as (28) and (29) were modeled on (26).

It should finally be noted that there will be no need to introduce any auxiliary parameters—analogous to τ's and σ's defined above—to represent z'. z' will be completely eliminated at a comparatively early stage of the solving procedure of the BS equation by explicitly performing a Cauchy integration [compare Eqs. (54) and (55)] and will thus be replaced by y' everywhere in the appropriate integrand. In this connection the need will arise to further transform the expression

$$\Delta(z,y',z'')=1-z^2-y'^2-z''^2+2zy'z'', \qquad (38)$$

originally symmetric in the z's. For reasons of greater clarity, it will then be sometimes advantageous to formally restore the full symmetry with respect to permutations of primes by introducing the notations

$$x=\cos(\varphi''-\varphi'+\tau)=\frac{1}{2(st)^{1/2}}\,(t+s), \qquad (39)$$

$$x'=\cos(\varphi-\varphi'')=\frac{1}{2(sts''t'')^{1/2}}\,(st+s''t''), \qquad (40)$$

$$x''=\cos(\varphi-\varphi'+\tau'')=\frac{1}{2(s''t'')^{1/2}}\,(t''+s''), \qquad (41)$$

so that

$$z=\frac{-\cos\psi''\cos\psi'+x}{\sin\psi'\sin\psi''}, \qquad (42)$$

$$y'=\frac{-\cos\psi\cos\psi''+x'}{\sin\psi''\sin\psi}, \qquad (43)$$

$$z''=\frac{-\cos\psi'\cos\psi+x''}{\sin\psi\sin\psi'}. \qquad (44)$$

In conclusion it should be remarked that x's will appear most naturally in conjunction with w's defined by (19) in the otherwise too lengthy formulas of Sec. V.

F. Notations adopted to represent the scattering amplitude

The (total, off shell) scattering amplitude T belongs—from the point of view of the conventions so far adopted—to the same category of quantities as the s and t parameters in that it is defined for the initial-final and intermediate-final pairs of states only (the only meaningful and similar quantity pertaining to the remaining pair of states is the kernel of the BS equation). Therefore, the notations T'' and T suffice to identify the scattering amplitude as pertaining to these two situations, respectively. However, occasionally it will be desirable to indicate clearly the variables on which the scattering amplitude does depend. In such cases we will again depart from the notation most commonly used, e.g.,

$$T''=T(r;p,p'),\ \text{etc.}$$

(with the four-vectors r,p,p',p'' as defined in Sec. II C), and introduce a Dirac-type notation:

$$T''=(p|T(r)|p'),\ \text{etc.} \qquad (45)$$

It will have the advantage that the p's as arguments of bras and kets can be easily replaced by Green-Biswas parameters or v's defined by (5) without losing sight of the general structure of the transition amplitude. This would be more cumbersome to accomplish if the more conventional notation were used and the particular six independent parameters, momentarily most useful, were just listed inside a common bracket without clear distinction as to in which category (that of Sec. II C, II D, or II E) they belong. In the barycentric system (1) we will write, e.g.,

$$T''=(v_+,v_-|T(E,z'')|v'_+,v'_-), \qquad (46)$$

the general idea being that the variables of the category defined in Sec. II C should be used to designate the end states, while E (sometimes omitted) and one variable of the category defined in Sec. II D or II E should be written inside the brackets as if they were arguments of an operator T.[20] The same type of notations will also be used for the "auxiliary" amplitudes Λ and Φ defined later in the text [compare (49), (50) and (67), (68)].

III. THE SOLVING PROCEDURE

Using the notations just described, the Bethe-Salpeter equation

$$\frac{1}{(p-p')^2+i\epsilon}=\frac{(2\pi)^4 i}{4g^2}\,(p|T(r)|p')+\int\frac{d^4p''(p''|T(r)|p')}{(p_A{}^2-m_A{}^2+i\epsilon)(p_B{}^2-m_B{}^2+i\epsilon)[(p''-p')^2+i\epsilon]}, \qquad (47)$$

if written in the barycentric system (1), becomes

$$\frac{1}{z''-y''}=\frac{(2\pi)^4 i}{2g^2}\Lambda'' + \int dp''_0\int_0^\infty d|\vec{p}|''\int d^2\Omega''\,\frac{1}{z'-y'}\,\frac{\Lambda}{[(\beta_+E+p''_0)^2-|\vec{p}''|^2-m_A{}^2+i\epsilon][(\beta_-E-p''_0)^2-|\vec{p}''|^2-m_B{}^2+i\epsilon]}, \qquad (48)$$

where z's and y's are defined by (23) and (24), the four-vectors p and $\beta_\pm$ by (2) and (4), respectively, and the auxiliary amplitude Λ by

$$\Lambda = |\vec{p}''|\,|\vec{p}'|(v_+'', v_-''|T(E,2)|v_+', v_-'), \quad (49)$$

$$\Lambda'' = |\vec{p}|\,|\vec{p}'|(v_+, v_-|T(E,z'')|v_+', v_-'). \quad (50)$$

The integration limits in $\int dp_0''(\cdots)$ are suppressed on purpose to indicate that (48) comprises both the original BS equation (integration from $-\infty$ to $+\infty$ along the real axis) and the Wick equation (integration from $-i\infty$ to $+i\infty$ along the imaginary axis, together with assigning imaginary values to the remaining p_0 and p_0').[21] $\int d^2\Omega''(\cdots)$ denotes the usual angular integration in the barycentric system (1). To perform it, however, we prefer to use the two cosines of the scattering angles z and z' as independent variables rather than the usual polar angles χ'' (or χ') and θ (see Fig. 2).[22] As easily seen, we have

$$\int d^2\Omega''(\cdots) = \int_{-1}^{+1} dz \oint dz' \frac{(\cdots)}{(1-z^2-z'^2-z''^2+2zz'z'')^{1/2}}\,, \quad (51)$$

where the contour integration in z' denotes the integration around the cut of

$$(1-z^2-z'^2-z''^2+2zz'z'')^{1/2}$$

extending from

$$z_-' = zz'' - |(1-z^2)^{1/2}|\,|(1-z''^2)^{1/2}| \quad (52)$$

to

$$z_+' = zz'' + |(1-z^2)^{1/2}|\,|(1-z''^2)^{1/2}|\,, \quad (53)$$

and where the proper branch of the square root is understood as that which remains positive on the lower lip of the cut. Now, since according to (49) the "unknown" function Λ does not depend on z', the z' integration can be performed explicitly using the Cauchy theorem; thus

$$\oint \frac{1}{z'-y'}(1-z^2-z'^2-z''^2+2zz'z'')^{-1/2} = -2\pi i(1-z^2-y'^2-z''^2+2zy'z'')^{-1/2}, \quad (54)$$

whereby Eq. (48) becomes

$$\frac{1}{z''-y''} = \frac{(2\pi)^4 i}{2g^2}\Lambda'' - 2\pi i \int dp_0'' \int_0^\infty d|\vec{p}''| \int_{-1}^{+1} dz \frac{(1-z^2-y'^2-z''^2+2zy'z'')^{-1/2}\Lambda}{[(\beta_+E+p_0'')^2-|\vec{p}''|^2-m_A{}^2+i\epsilon][(\beta_-E-p_0'')^2-|\vec{p}''|^2-m_B{}^2+i\epsilon]}\,, \quad (55)$$

so that the number of integrations involved is reduced from the original four to three. It is also evident that by using the Green-Biswas parameters or v's defined by (5) a substantial simplification of the kernel can be achieved because of (10) and (15). The chief difficulty, however, is how to deal with the residual dependence on the angular variable z without resorting to partial-wave analysis. In order to resolve this difficulty we proceed as follows.

We begin by transforming the expression (38). Using x's defined by Eqs. (39)–(41) and w's defined by (19), we have

$$1-z^2-y'^2-z''^2+2zy'z'' = \frac{Q}{\sin^2\psi\,\sin^2\psi'\,\sin^2\psi''}\,, \quad (56)$$

where

$$\begin{aligned} Q &= (1-x^2-x'^2-x''^2+2xx'x'') + (x^2-1)w^2 \\ &\quad + (x'^2-1)w'^2 + (x''^2-1)w''^2 + 2(x-x'x'')w'w'' \\ &\quad + 2(x'-x''x)w''w + 2(x''-xx')ww' \\ &\equiv Aw''^2 + Bw'' + C, \end{aligned} \quad (57)$$

i.e., with Q a *quadratic* form in w'' for $x = \text{const}$. Consequently

$$\frac{1}{\sqrt{Q}} = \frac{1}{\sqrt{A}}\frac{\partial \ln\xi}{\partial w''}\,, \quad (58)$$

where

$$\xi = \frac{2}{[B^2-4AC]^{1/2}}\left[\sqrt{A}\left(w''+\frac{B}{2A}\right)+\sqrt{Q}\right] \quad (59)$$

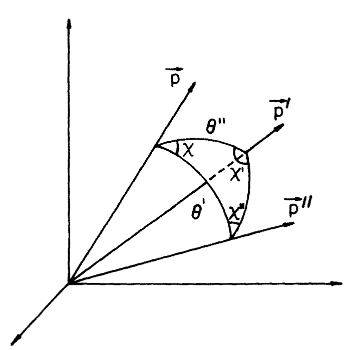

FIG. 2. Angular variables used to perform the integration indicated by $\int d^2\Omega''$ in Eqs. (48) and (51).

and where

$$A=(x''^2-1), \tag{60}$$

$$B=2(x-x'x'')w'+2(x'-x''x)w, \tag{61}$$

$$C=B^2-4AC$$
$$=4(1-w^2-w'^2-x''^2+2ww'x'')$$
$$\times(1-x^2-x'^2-x''^2+2xx'x''). \tag{62}$$

Defining the proper branch of $\sqrt{Q}$ in terms of that of $(1-z^2-y'^2-z''^2+2zy'z'')^{1/2}$ by

$$\sqrt{Q}=\sin\psi\sin\psi'\sin\psi''(1-z^2-y'^2-z''^2+2zy'z'')^{1/2} \tag{63}$$

and the proper branch of $\sqrt{A}$ by

$$\sqrt{A}=i\sin(\varphi-\varphi'+\tau''), \tag{64}$$

[compare (60) and (41)] and noting that for φ'', ψ'' = const

$$dz=-\frac{\sin(\varphi''-\varphi'+\tau)}{\sin\psi''\sin\psi'}d\tau \tag{65}$$

[compare (28)], we finally have because of (58) and (19) (see Ref. 23)

$$\frac{dz}{(1-z^2-y'^2-z''^2+2zy'z'')^{1/2}}$$
$$=-i\frac{\sin\psi\sin(\varphi''-\varphi'+\tau)}{\sin\psi''\sin(\varphi-\varphi'+\tau'')}\frac{\partial\ln\xi}{\partial\psi''}d\tau. \tag{66}$$

If we now introduce another auxiliary scattering amplitude

$$\Phi=\frac{\sin(\varphi''-\varphi'+\tau)}{\sin\psi''\sin\psi'}\Lambda, \tag{67}$$

and, consistently with it,

$$\Phi''=\frac{\sin(\varphi-\varphi'+\tau'')}{\sin\psi'\sin\psi}\Lambda'', \tag{68}$$

the integrand in (55) becomes

$$\frac{\sin\psi\sin\psi'}{\sin(\varphi-\varphi'+\tau'')}\times(\text{function of }\varphi''\text{ only})$$
$$\times\Phi d\varphi''\,d\psi''\,d\tau\,\frac{\partial\ln\xi}{\partial\psi''}$$

(or a corresponding expression if $v''_\pm$ and t were used), so that further simplification can be made by dividing (55) throughout by

$$\frac{\sin\psi\sin\psi'}{\sin(\varphi-\varphi'+\tau'')}.$$

Because of widely different integration domains involved, it is best, from this point on, to write the correspondingly transformed Eq. (55) separately for the Wick case and for the original BS equation, using the parameters best suited to the particular situation as integration variables. They are

$$\frac{\sin(\varphi-\varphi'+\tau'')}{\cos(\varphi-\varphi'+\tau'')-\cos(\varphi-\varphi')}=\frac{(2\pi)^4 i}{2g^2}\Phi''-\frac{\pi}{2}\frac{1}{m_A m_B}\int\int_{D_1}\int d\varphi''\,d\psi''\,d\tau\,\frac{\Phi(\partial/\partial\psi'')\ln\xi}{\sin(\varphi''+\alpha_+)\sin(\varphi''-\alpha_-)} \tag{69}$$

for the Wick case and

$$-\frac{1}{t''-1}+\frac{1}{s''-1}$$
$$=\frac{(2\pi)^4}{2g^2}\Phi''+\frac{\pi i}{(\beta_+E-ic)(\beta_-E+ic)}\int\int_{D_2}\int dv''_+\,dv''_-\,dt\,\frac{1}{t}\,\frac{\Phi}{(v''_+v''_--u_+^2)(v''_+v''_--u_-^2)}\left(v''_+\frac{\partial}{\partial v''_+}-v''_-\frac{\partial}{\partial v''_-}\right)\ln\xi \tag{70}$$

for the case of the original BS equation and where the relation

$$\frac{1}{z''-y''}\,\frac{\sin(\varphi-\varphi'+\tau'')}{\sin\psi\sin\psi}$$
$$=\frac{\sin(\varphi-\varphi'+\tau'')}{\cos(\varphi-\varphi'+\tau'')-\cos(\varphi-\varphi')}$$
$$=-i\left(\frac{1}{t''-1}-\frac{1}{s''-1}\right) \tag{71}$$

was also used to represent the inhomogeneity (Born term). $\alpha_\pm$ and $u_\pm$ are defined by (16), $\beta_\pm$ by (4), and c by (6). The integration domains D_1 and D_2 are explained in detail in Secs. IV and V, where parallel attempts are made to solve the Wick equation and the original BS equation, respectively.

To conclude the preliminary remarks of the present section, it should finally be pointed out that the analytic properties of the quantity ξ defined by (59) are obviously going to play an essential role in all further considerations. These analytic properties are derived in detail in Appendix A.

IV. SOLUTION OF THE WICK EQUATION

The continuation of the solving procedure from the Sec. III must now—out of necessity—differ when applied to (69) as compared to (70). We propose therefore to solve (69) as a simpler case first and then, in Sec. V, to try to follow as closely as possible all the same essential steps in order to solve (70). We shall begin by determining the integration domain D_1. First of all notice that, because c defined by (6) is real in the bound-state region and imaginary for the scattering region, the nature of the integration domain involved (i.e., D_1 as well as D_2 of Sec. V) will additionally depend on whether $m_A+m_B \geq E \geq |m_A-m_B|$ or $E > m_A+m_B$ ($|m_A-m_B| \geq E \geq 0$). From now on we shall therefore work exclusively in the scattering region; thus

$$E \geq m_A+m_B \text{ or } |m_A-m_B| \geq E \geq 0;$$

and we shall only occasionally (compare the exactly soluble model of Sec. VI) analytically continue the results into the bound-state region, e.g., in order to determine the positions of the bound-state poles, etc. This makes c imaginary and d, [defined by (18)] real and—according to the convention already made—positive. Since in the Wick-rotated situation

$$p_0 = ip_4, \quad p_4 \text{ real} \tag{72}$$

we see that according to (8) and (9) φ must be real and ψ imaginary. To determine the integration limits in φ and ψ it now suffices to realize that with p_4 real Eqs. (8) and (9) represent the conformal mapping

$$\zeta_1 = i \ln \frac{\zeta_2+d}{\zeta_2-d}, \tag{73}$$

also depicted in Fig. 3, where

$$\zeta_1 = \varphi + i\chi, \tag{74}$$

$$\zeta_2 = |\vec{p}| + ip_4, \tag{75}$$

with χ defined by

$$\psi = i\chi \tag{76}$$

and therefore real. Since the analytic right half plane in ζ_2 represents the integration domain in $|\vec{p}''|$ and p_4'' (we shall now again start using double primes to indicate the actual integration variables), the integration limits in φ'' and ψ'' are from 0 to 2π and from 0 to $+i\infty$, respectively (see Fig. 3). In order to make the left-hand side of (28) real τ must now be in general complex. Its real part remains constant and equal to $\varphi' - \varphi''$ for φ'' and χ'' constant and its imaginary part varies between

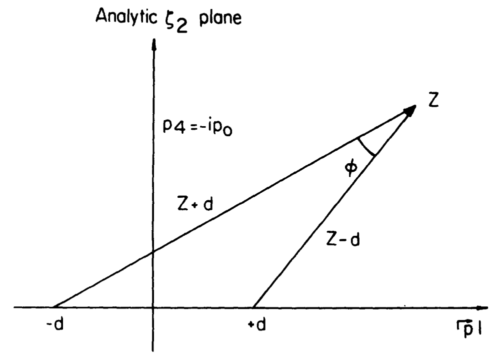

FIG. 3. Graphical representation of the conformal mapping (73), obtaining in the case of the Wick-rotated BS equation, supplying in part the motivation for introducing the bifocal coordinates φ and ψ of Green and Biswas.

the limits $-i(\psi' \pm \psi'')$ as z varies between ∓ 1, respectively. We thus have

$$\iiint_{D_1} d\varphi''\, d\psi''\, d\tau(\cdots) \equiv \int_0^{2\pi} d\varphi'' \int_0^{i\infty} d\psi'' \int_{\varphi'+\psi'-\varphi''-\psi''}^{\varphi'+\psi'-\varphi''+\psi''} d\tau(\cdots). \tag{77}$$

For what follows it is of importance to also show at this point that in the presently considered Wick case (and contrary to the situation encountered in Sec. V) the v parameters defined by (5) would be completely useless as integration variables. This is easily seen from (7) in conjunction with the fact that φ's are real and ψ's imaginary; thus

$$v_+'' = \frac{1}{(v_-'')^*} \tag{78}$$

so that the point representing v_+'' in its analytic plane becomes completely determined by that of v_-'' and to speak of an integration path in v_+'' "while v_-'' remains constant" becomes meaningless.

To solve (69) we now proceed as follows: The form of this equation suggests that if Φ were not to depend on ψ'', but only on the remaining variables φ'' and τ, the integrand could be rewritten as

$$\frac{\partial}{\partial\psi''}\left(\frac{\Phi \ln\xi}{\sin(\varphi''+\alpha_+)\sin(\varphi''-\alpha_-)}\right), \tag{79}$$

so that the "volume" integral over D_1 could be reduced to a "surface" integral over the "boundary" of this domain by a "Green theorem." However, for reasons of internal consistency, Φ'' would then have to be independent of ψ, and since the inhomogeneity also does not contain this variable the outcome of the above surface integration would likewise have to be ψ-independent. In other words, the possibility of a further reduction of the number of integrations involved from three to two depends on the compatibility of (69) with the solving ansatz

$$\Phi = \Phi(\varphi'', \tau), \tag{80}$$

$$\Phi'' = \Phi(\varphi, \tau''), \tag{81}$$

emphasizing that the auxiliary scattering amplitude Φ defined by (67) and (68) is independent of the ψ parameter.[24]

We shall now proceed to show that such compatibility indeed obtains and leads to a solution of (69). However, a slight change of the integration variables is first indicated (i.e., going slightly beyond the general "register" of parameters of Sec. II and definable only in the Wick-rotated case). Mainly because of a rather complicated way the integration with respect to τ has to be performed [compare (77)], the boundaries of the domain D_1 are much more conveniently established in terms of the parameters φ'', ρ'', and α—the latter two are defined by (22) and by

$$\alpha = |t| \tag{82}$$

[with t defined by (32)], respectively—than in terms of φ'', ψ'', and τ. Namely, since the real part of τ is equal to $\varphi' - \varphi''$ we can write

$$t = |t| e^{i(\varphi' - \varphi'')} = \alpha e^{i(\varphi' - \varphi'')}, \tag{83}$$

while, because of (7), (21), and (22), Eq. (36) becomes

$$\frac{z-1}{z+1} = \frac{(\alpha - \rho''/\rho')(\alpha - \rho'/\rho'')}{(\alpha - \rho'\rho'')(\alpha - 1/\rho'\rho'')}, \tag{84}$$

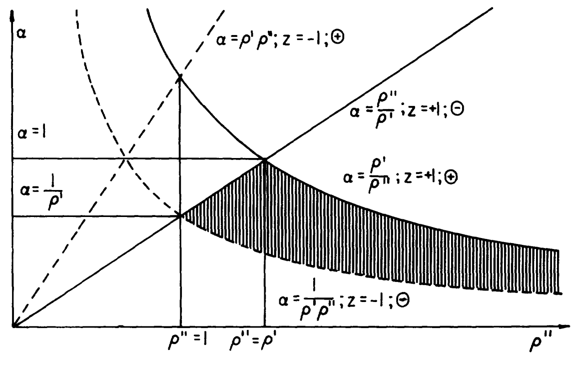

FIG. 4. Integration domain in α and ρ'' (shaded area) in (85) and (88). ⊕ and ⊖ denote the signatures of the boundary lines in the sense defined in Appendix A and correspond to $-\epsilon_1\epsilon_2$ set equal to +1 and −1, respectively, where ϵ_1 and ϵ_2 are defined by (A9) and (A16).

representing a relation between *all real numbers* in the Wick-rotated case. Consequently, to fully cover the domain D_1, ρ'' must vary from 1 to ∞ (corresponding to the variation of ψ'' from 0 to $i\infty$), and φ'', as before, must vary from 0 to 2π, while α must vary from $1/\rho'\rho''$ to ρ''/ρ' for $\rho'' < \rho'$ and from $1/\rho'\rho''$ to ρ'/ρ'' for $\rho'' > \rho'$ as z varies from −1 to +1. The integration region in α and ρ'' is represented graphically by the shaded area in Fig. 4, where the solid lines $\alpha = \rho''/\rho'$ and $\alpha = \rho'/\rho''$ correspond to $z = +1$ and the dashed lines $\alpha = \rho'\rho''$ and $\alpha = 1/\rho'\rho''$ correspond to $z = -1$ and represent the boundaries of this domain. Equation (69) written in terms of φ'', ρ'', and α instead of φ'', ψ'', and τ now becomes

$$\frac{1}{s''-1} - \frac{1}{t''-1} = \frac{(2\pi)^4}{2g^2}\,\phi'' + \frac{\pi}{2m_A m_B}\int_0^{2\pi} d\varphi'' \left\{ \int_1^{\rho'} d\rho'' \int_{1/\rho'\rho''}^{\rho''/\rho'} \frac{d\alpha}{\alpha} + \int_{\rho'}^{\infty} d\rho'' \int_{1/\rho'\rho''}^{\rho'/\rho''} \frac{d\alpha}{\alpha} \right\} \times \frac{\Phi}{\sin(\varphi'' + \alpha_+)\sin(\varphi'' - \alpha_-)} \frac{\partial}{\partial \rho''} \ln\xi, \tag{85}$$

where t'' and s'' expressed in terms of α's and φ's are

$$t'' = \alpha'' e^{i(\varphi' - \varphi)}, \quad \alpha'' = |t''| \tag{86}$$

[compare (83)] and, because of (31) and (33),

$$s'' = \frac{1}{\alpha''} e^{i(\varphi' - \varphi)}. \tag{87}$$

Thus, reversing the order of integrations with respect to α and ρ'',

$$\left\{ \int_1^{\rho'} d\rho'' \int_{1/\rho'\rho''}^{\rho''/\rho'} d\alpha + \int_{\rho'}^{\infty} d\rho'' \int_{1/\rho'\rho''}^{\rho'/\rho''} d\alpha \right\} (\cdots) = \left\{ \int_0^{1/\rho'} d\alpha \int_{1/\alpha\rho'}^{\rho'/\alpha} d\rho'' + \int_{1/\rho'}^{1} d\alpha \int_{\alpha\rho'}^{\rho'/\alpha} d\rho'' \right\} (\cdots) \tag{88}$$

(see Fig. 4), making the ansatz (80), (81), and finally performing the integration with respect to ρ'' explicitly, Eq. (85) becomes

$$\frac{1}{s''-1} - \frac{1}{t''-1} = \frac{(2\pi)^4}{2g^2}\,\Phi(\varphi, \alpha'') + \frac{\pi}{2m_A m_B}\int_0^{2\pi} d\varphi'' \int_0^1 \frac{d\alpha}{\alpha} \frac{\Phi(\varphi'', \alpha)\ln(\xi_+/\xi_-)}{\sin(\varphi'' + \alpha_+)\sin(\varphi'' - \alpha_-)}, \tag{89}$$

where we have now written $\Phi(\varphi, \alpha'')$ instead of $\Phi(\varphi, \tau'')$, and $\Phi(\varphi'', \alpha)$ instead of $\Phi(\varphi'', \tau)$, and where $\xi_\pm$ denote the values of ξ corresponding to the same values of φ'', α, and ρ, but taken at $z=\pm 1$, or, more precisely, where the subscript $\pm$ of ξ is identical with the signature $-\epsilon_1\epsilon_2$ in the usage of Appendix A, with ϵ_1 and ϵ_2 defined by Eqs. (A9) and (A16), respectively. The internal consistency of (87) now obtains because, as proved in Appendix A [Eqs. (A29) and (A30)],

$$\frac{\xi_+}{\xi_-}=\frac{(t-t'')(s-s'')}{(s-t'')(t-s'')}, \tag{90}$$

which if expressed in terms of α's and φ's reads

$$\frac{\xi_+}{\xi_-}=\frac{(1/\alpha)|\alpha'' e^{i\varphi}-\alpha e^{i\varphi''}|^2}{\alpha|\alpha'' e^{i\varphi}-(1/\alpha)e^{i\varphi''}|^2} \tag{91}$$

and is thus indeed ρ- (or ψ-) independent. Equation (89) represents by far the most important and useful analytic property of ξ. It should be borne in mind, however, that the validity of the above result depends to no lesser extent on the fact that ξ remains single-valued throughout the domain D_1, which in turn stems mainly from the positive-definiteness of Q, defined by (57), in the Wick-rotated case. The latter is no longer true in the situation under discussion in Sec. V (see Appendix A for details). To further transform (89), we first notice that because of (86), (87), and (91) both expressions

$$\frac{1}{s''-1}-\frac{1}{t''-1}$$

and

$$\ln\frac{\xi_+}{\xi_-}$$

change sign under the substitution $\alpha''\to 1/\alpha''$, so that any solution of (89) must exhibit the additional symmetry

$$\Phi(\varphi'', \alpha)=-\Phi(\varphi'', 1/\alpha). \tag{92}$$

Consequently, (89) becomes

$$\frac{1}{s''-1}-\frac{1}{t''-1}=\frac{(2\pi)^4}{2g^2}\Phi(\varphi,\alpha'')+\frac{\pi}{m_A m_B}\int_0^{2\pi}d\varphi''\int_0^{\infty}\frac{d\alpha}{\alpha}\,\frac{\ln|\alpha e^{i\varphi''}-\alpha'' e^{i\varphi}|-\frac{1}{2}\ln\alpha}{\sin(\varphi''+\alpha_+)\sin(\varphi''-\alpha_-)}\Phi(\varphi'',\alpha), \tag{93}$$

where the integration in α has been formally extended to infinity by the substitution $\alpha\to 1/\alpha$ in the expression pertaining to ξ_-, since the denominator of (91) becomes equal to the numerator and vice versa under this substitution. Equation (93) is now equivalent to a two-dimensional boundary-value problem. To see this it suffices to introduce the rectangular coordinates

$$\alpha_1=\alpha\cos\varphi'',\quad \alpha_2=\alpha\sin\varphi'', \tag{94}$$

$$\alpha_1''=\alpha''\cos\varphi,\quad \alpha_2''=\alpha''\sin\varphi, \tag{95}$$

whereby

$$\int_0^{2\pi}d\varphi''\int_0^{\infty}\frac{d\alpha}{\alpha}(\cdots)=\int_{-\infty}^{+\infty}d\alpha_1\int_{-\infty}^{+\infty}d\alpha_2\frac{(\cdots)}{\alpha_1^{\,2}+\alpha_2^{\,2}}, \tag{96}$$

and also to realize that

$$\ln|\alpha^{i\varphi''}-\alpha'' e^{i\varphi}|=\ln[(\alpha_1''-\alpha_1)^2+(\alpha_2''-\alpha_2)^2]^{1/2} \tag{97}$$

and

$$\frac{1}{s''-1}-\frac{1}{t''-1}=\alpha''\frac{\partial}{\partial\alpha''}\ln\left[\frac{(\alpha_1''-\cos\varphi')^2+(\alpha_2''-\sin\varphi')^2}{\alpha''}\right]=\frac{1-\alpha''^2}{(\alpha_1''-\cos\varphi')^2+(\alpha_2''-\sin\varphi')^2}. \tag{98}$$

Applying the Laplacian

$$\frac{\partial^2}{\partial\alpha_1''^2}+\frac{\partial^2}{\partial\alpha_2''^2}=\frac{1}{\alpha''^2}\left[\left(\alpha''\frac{\partial}{\partial\alpha''}\right)^2+\frac{\partial^2}{\partial\varphi^2}\right] \tag{99}$$

to both sides of (93) and using the well-known two-dimensional "potential theory" formula[25]

$$\left(\frac{\partial^2}{\partial\alpha_1''^2}+\frac{\partial^2}{\partial\alpha_2''^2}\right)\int_{-\infty}^{+\infty}d\alpha_1\int_{-\infty}^{+\infty}d\alpha_2\ln[(\alpha_1-\alpha_1'')^2+(\alpha_2-\alpha_2'')^2]^{1/2}f(\alpha_1,\alpha_2)=2\pi f(\alpha_1'',\alpha_2''), \tag{100}$$

we get

$$\left[\left(\alpha\frac{\partial}{\partial\alpha}\right)^2+\frac{\partial}{\partial\varphi''^2}+\frac{g^2}{(2\pi)^2}\frac{1}{m_A m_B}\frac{1}{\sin(\varphi''+\alpha_+)\sin(\varphi''-\alpha_-)}\right]\Phi=0, \tag{101}$$

which is a *homogeneous* equation satisfied everywhere except at the point

$$\alpha_1'' = \cos\varphi', \quad \alpha_2'' = \sin\varphi', \tag{102}$$

notwithstanding the presence of the Born term in (93) since, according to (98),

$$\left(\frac{\partial^2}{\partial \alpha_1''^2} + \frac{\partial^2}{\partial \alpha_2''^2}\right)\left(\frac{1}{s''-1} - \frac{1}{t''-1}\right) = 0, \tag{103}$$

satisfied likewise everywhere except at the point (102). The proper boundary conditions to be imposed on Φ satisfying (101) in order that it should satisfy (93) are therefore that

(i) ϕ should develop a singularity of the type (99) at the point (102), or more precisely that the expression

$$\phi(\varphi, \alpha'') - \frac{4g^2}{(2\pi)^4}\left(\frac{1}{s''-1} - \frac{1}{t''-1}\right)$$

should remain regular at that point,[26]

(ii) ϕ/α should remain regular at $\alpha = 0$ [because of the factor $1/\alpha$ in the integrand of (93)], and

(iii) the solution should exhibit the over-all symmetry (92).

Equation (101) is separable in α and φ'', so that the solution of our boundary-value problem may be sought in the form of a sum of products

$$\phi_n = (\alpha^n - \alpha^{-n})\, Y_n(\varphi''), \tag{104}$$

where the functions Y_n satisfy

$$\left[\frac{d^2}{d\varphi''^2} + n^2 + \frac{g^2}{(2\pi)^2}\,\frac{1}{m_A m_B}\,\frac{1}{\sin(\varphi'' + \alpha_+)\sin(\varphi'' - \alpha_-)}\right] Y_n = 0. \tag{105}$$

Equations (105) are identical with the "radial" equations of Green and Biswas[9] and are therefore also equivalent—as shown by these authors—to the well-known Wick-Cutkosky results. Having thus established an essential equivalence of our results concerning the Wick equation with those known in the literature, we shall now attempt to solve the "original" BS equation (70).

V. THE "ORIGINAL" BETHE-SALPETER EQUATION

A. Integration domain

As in the previous section, we shall first determine the integration domain D_2. To begin with, since p_0'' is now real and $ic = d$ is also real (i.e., for E in the scattering region), the parameters $v_\pm''$ defined by (5) are real. Moreover, they can be considered independent variables since $p_0'' \pm |\vec{p}''|$ are also real and independent—a situation very different from that of the previous Sec. IV, where this was not true because of (78). Consequently, in the case of the original BS equation, *$v_\pm''$ represent a good choice of integration variables.* Inversely, the Green-Biswas parameters are practically useless for the following reasons:

(i) Since for real p_0'' the conformal mapping (73) becomes meaningless, the integration domain in φ'' and ψ'' becomes very involved even if the quantities E, m_A, m_B which enter the definition (6) of c are all "strictly real."

(ii) If small imaginary parts are assigned to m_A and m_B, as entailed by the use of Feynman propagators, the above difficulty is compounded enormously—to such an extent, in fact, that the "integration domain" can now only be understood as a certain two-dimensional "surface" embedded in the four-dimensional space [Re(φ''), Im(φ''), Re(ψ''), Im(ψ'')] and, in a fashion similar to what was true for v_+'' and v_-'' in Sec. IV, even the concept of, e.g., "an integration contour in φ'' while ψ'' is kept constant" loses its meaning.[27]

The parameter t defined in Sec. II E represents a third "natural" choice of an integration variable because it is real and—most importantly—because now Eq. (36) becomes a relation between all real numbers. To actually determine the integration domain D_2 in the underlying three-dimensional

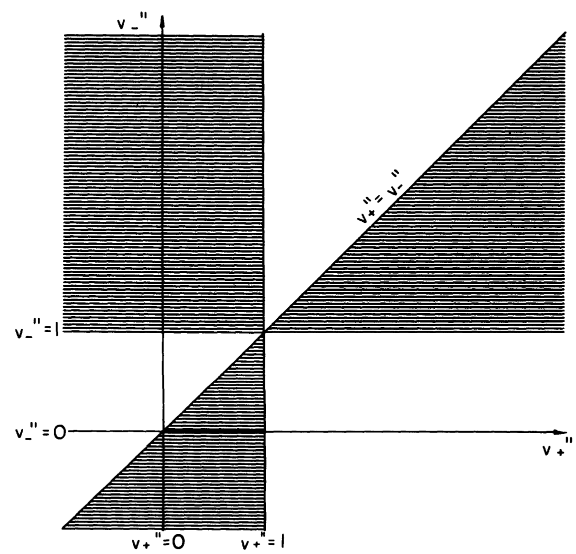

FIG. 5. Allowed values of $v_\pm''$ (shaded area) by virtue of the fact that the integration domains in p_0'' and $|\vec{p}''|$ extend from $-\infty$ to $+\infty$ and from 0 to $+\infty$, respectively. This means that $p_0'' + |\vec{p}''| \geq p_0'' - |\vec{p}''|$ and consequently [compare (5)] that $1/(v_+'' - 1) \leq 1/(v_-'' - 1)$.

(v''_+, v''_-, t) space, it is important to realize from the onset that one should expect it to be significantly more involved than was the case with the domain D_1 of Sec. IV, in its underlying $(\varphi'', \rho'', \alpha)$ space. The boundaries of D_2 consist not only of the planes

$$v''_+ - tv'_+ = 0 \text{ and } v''_- - tv'_- = 0, \tag{106}$$

corresponding to $z = +1$, and

$$v''_+ - tv'_- = 0 \text{ and } v''_- - tv'_+ = 0, \tag{107}$$

corresponding to $z = -1$, but also of the planes

$$v''_+ = 1, \tag{108}$$

corresponding to $p''_0 + |\vec{p}''| = \infty$, with $p''_0 - |\vec{p}''|$ finite and

$$v''_- = 1 \tag{109}$$

corresponding to $p''_0 - |\vec{p}''| = \infty$ with $p''_0 + |\vec{p}''|$ finite and finally of the plane

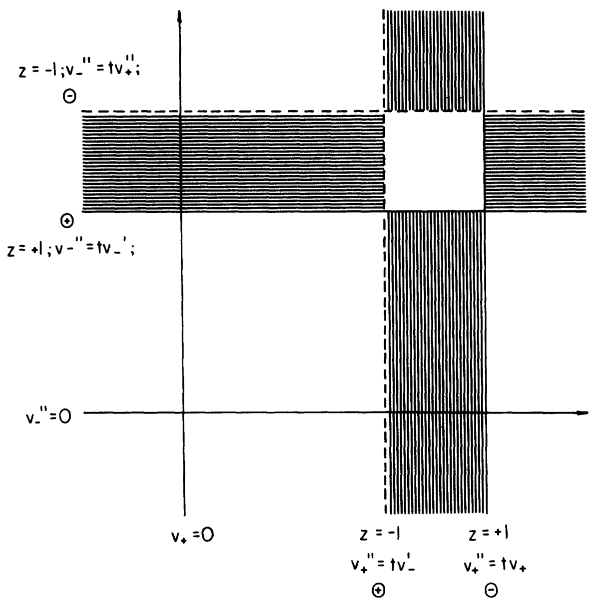

FIG. 6. Allowed values of $v''_\pm$ (shaded areas) by virtue of the fact that the integration domain in the cosine of the scattering angle z extends from -1 to $+1$. They derive from the condition that (for a given t) the right-hand side of (36) must always be negative. The distinction between areas shaded in horizontal solid lines and those shaded in vertical solid lines is that they must be counted positively and negatively, respectively. This depends on whether—proceeding in the direction of increasing t [i.e., in the *three*-dimensional (v''_+, v''_-, t) space] and keeping $v''_\pm$ constant—the surface corresponding to $z = -1$ is encountered first and that corresponding to $z = +1$ second, or vice versa. By combining the information contained in Figs. 5 and 6 (the latter redrawn for different values of t) we obtain Figs. 7–13. The meaning of signatures ⊕ and ⊖ is the same as in Fig. 4.

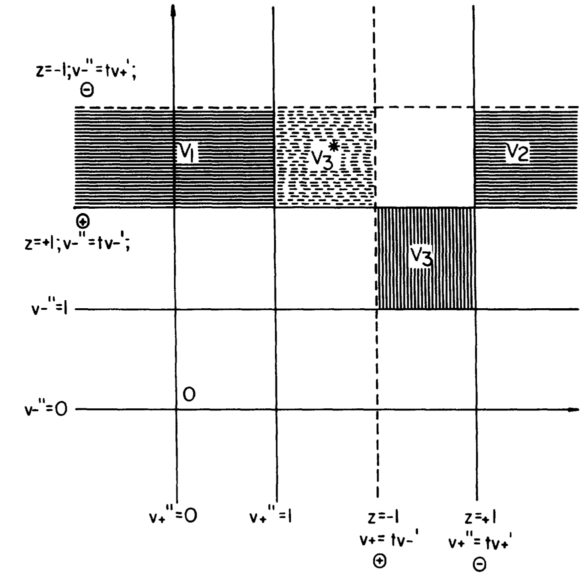

FIG. 7. Cross sections of the "rearranged" (shadings in the horizontal solid and horizontal dashed lines) and "original" (shadings in the vertical solid and vertical dashed lines) integration domains D'_2 and D_2 by a t $=$const plane for $t > 1/v'_-$.

$$v''_+ = v''_-, \tag{110}$$

corresponding to $|\vec{p}''| = 0$ [compare (5) and the integration limits in (48)]. While the boundaries (106) and (107) clearly have their analogs in D_1 (compare, e.g., the solid and dashed lines in Fig. 4), *the boundaries of the type (108), (109), and (110) are simply absent in the Wick-rotated case*. It is now a matter of a little painstaking but perfectly straightforward geometry (consisting of determining the overlap regions of the shaded areas depicted in Figs. 5 and 6 and representing the allowed regions of integration by virtue of the domains of variation in p''_0, $|\vec{p}''|$, and z, respectively) to realize that the domain D_2 can be represented symbolically by

$$D_2 = V_1 + V_2 - V_3 - V_4 - V_5 + V_6, \tag{111}$$

where, to avoid lengthy verbal descriptions, the "subdomains" or "volumes" $V_1, \ldots, V_6$ are represented graphically in Figs. 7–13, and where the minus signs in front of V_3, V_4, and V_5 denote that the corresponding integrals over these volumes should be counted negatively. Figures 7–13 represent cross sections of the integration domains by the t = const planes for t in the intervals $t > 1/v'_- > 1/v'_+$, $t = 1/v'_-$, $1/v'_- > t > 1/v'_+$, $t = 1/v'_+$, $1/v'_- > 1/v'_+ > t > 0$, $t = 0$, and $t < 0$, respectively, and are all drawn for a particular situation where $v'_+ > v'_- > 0$.[28]

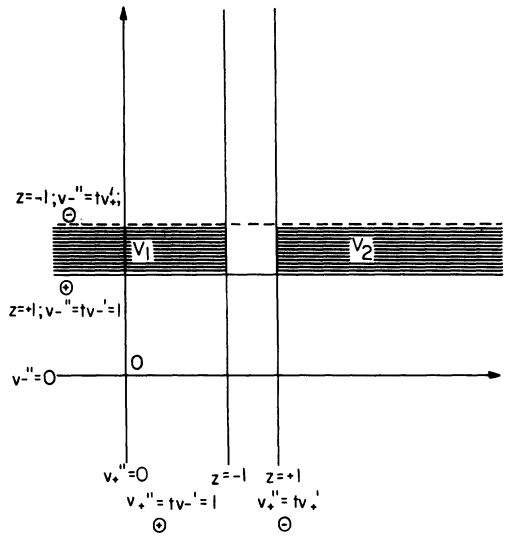

FIG. 8. Cross sections of the "rearranged" and "original" integration domains D_2' and D_2 by the $t=1/v'_-$ plane.

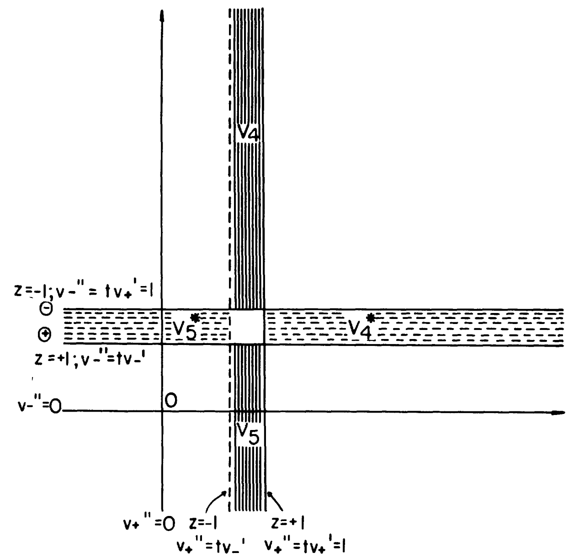

FIG. 10. Cross sections of the "rearranged" and "original" integration domains D_2' and D_2 by the $t=1/v'_+$ plane.

Thus, as anticipated, the detailed structure of D_2 is indeed rather involved. Before going into details of a more analytic nature, we shall therefore first try to improve the geometry of this over-all integration domain, i.e., try to replace D_2 with an equivalent domain of a simpler shape. The possibility of doing so stems from the fact that the integration over a particular V_i can always be replaced by that over a corresponding "mirror reflection" V_i^* in the $v_+'' = v_-''$ plane by performing the formal substitution $v_+'' \rightleftarrows v_-''$ in the integrand. A glance at Figs. 7–13, where the positions of V_3^*, V_4^*, and V_5^* are also indicated, shows that a particularly simple choice of a over-all domain equivalent in that sense to D_2 is

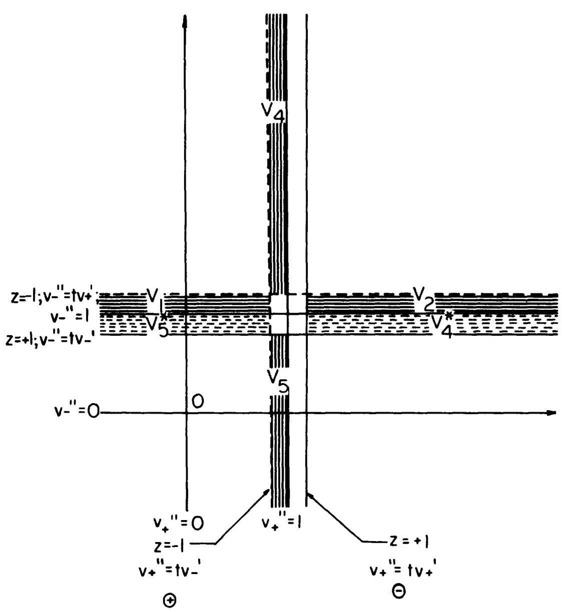

FIG. 9. Cross sections of the "rearranged" and "original" integration domains D_2' and D_2 by a $t=\text{const}$ plane $1/v'_- > t > 1/v'_+$.

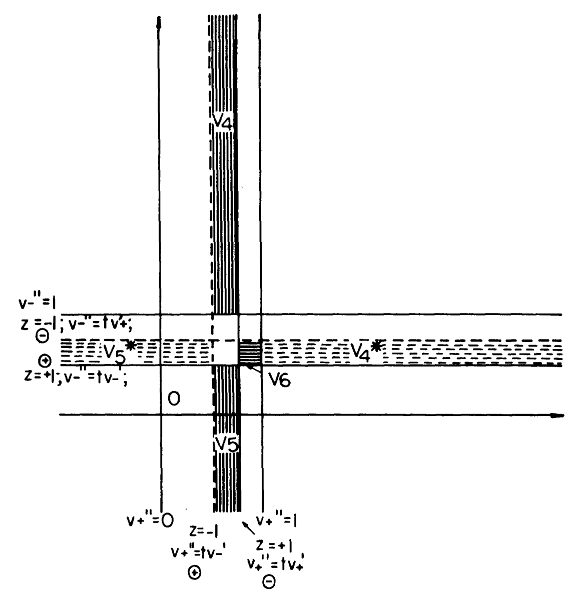

FIG. 11. Cross sections of the "rearranged" and "original" integration domains D_2' and D_2 by a $t=\text{const}$ plane for $1/v'_+ > t > 0$.

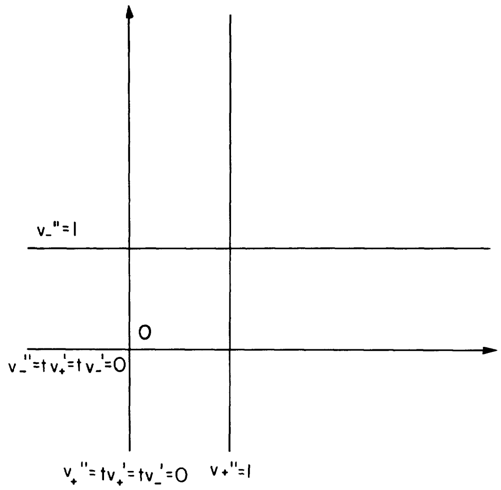

FIG. 12. Cross sections of the "rearranged" and "original" integration domains D_2' and D_2 by the $t=0$ plane.

$$D_2' = V_1 + V_2 + V_3^* + V_4^* + V_5^* + V_6, \tag{112}$$

in which therefore we shall actually perform all the necessary integrations. It should be noticed that, for reasons to become clear later,[29] the definition of D_2' also includes the change of sign in front of the V^*'s which means that the sign of the integrand should also be changed in addition to the substitution $v_+'' \rightleftarrows v_-''$. As again seen from Figs. 7–13, where the domain D_2' is indicated by the shaded areas, D_2' is not only more compact than D_2, but—most importantly—its *external* boundaries (unlike those of D_2) *consist now solely of the planes (106) and (107)*, while those of the type (108), (109), and (110) are relegated to the roles of *interfaces* between the constituent volumes (to emphasize this fact D_2' is additionally represented by a three-dimensional picture, Fig. 14).

More "analytic" aspects of the replacement of D_2 by D_2' in conjunction with the special form of the integrand in (70) entail now the following: First of all, the substitution $v_+'' \rightleftarrows v_-''$ poses no problem, even as far as the unknown Φ is concerned, since on general grounds [and entirely independently on whether or not the ansatz (114), (115) is made—see later] it has to be a symmetric function of v_+'' and v_-''. This follows from the fact that the substitution $v_+'' \rightleftarrows v_-''$ implies $|\vec{p}''| \to -|\vec{p}''|$ and $z \to -z$ [compare (5) and (36), respectively], which in turn is compatible with Eq. (48) provided that $\Lambda \to -\Lambda$ under this substitution.[30] That Φ is a symmetrical function in the variables v_+'' and v_-'' is then a consequence of the definition (67), (68).

As far as ξ is concerned, the substitution $v_+'' \rightleftarrows v_-''$ can of course be performed explicitly since ξ is known in a closed form (59); in fact, the sym-

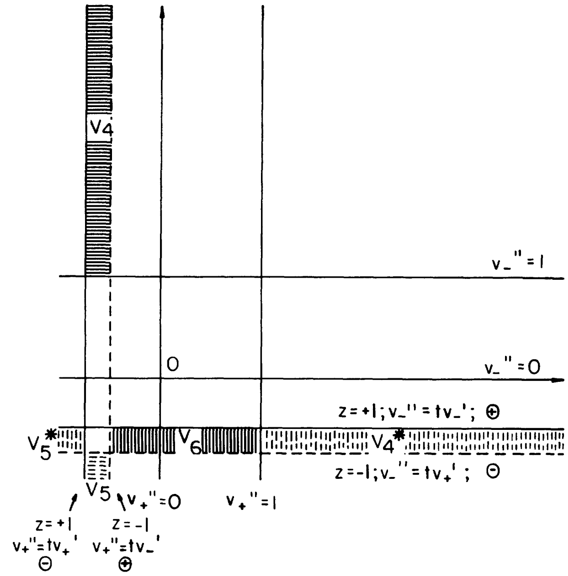

FIG. 13. Cross sections of the "rearranged" and "original" integration domains D_2' and D_2 by a $t=$const plane for negative t. Note the "reversal of sign of the subdomains" V_i as compared, with, e.g., Fig. 11. However, this is "compensated for" when the integration is finally performed in the t, s, and ω'' variables by the appearance of the factor $|t|$ in the Jacobian $\partial(t,s,\omega'')/\partial(v_+'', v_-'', t)$, so that this integration continues always smoothly into the range of negative t (or s).

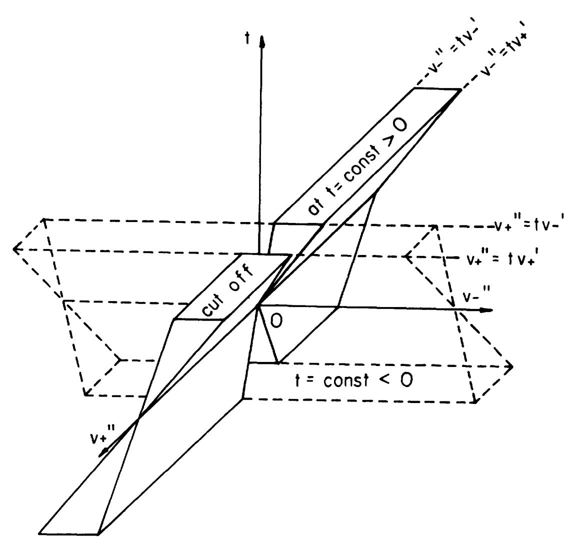

FIG. 14. Over-all three-dimensional picture of the ("rearranged") integration domain D_2' of (70) (outlined in solid lines). For greater clarity, the position of $(D_2')^*$ (obtained from D_2' by reflection in the $v_+''=v_-''$ plane) is also indicated (dashed lines). Notice the gap in D_2' in the region where the two would otherwise (i.e., if this gap were absent) intersect. To preserve maximal readability, however, the positions of individual V_i's and those of the otherwise very important $v_+''=1$ and $v_-''=1$ planes (interfaces) are no longer included.

metry properties of its logarithm ($\ln\xi$) play a predominant role in our future considerations, as we shall presently see. In this connection it is of particular interest to notice that if $\ln\xi$ were a *symmetric* function of v_+'' and v_-'', the integrand of (70) would be antisymmetric in these variables, so that—taking into account the additional change of sign of this integrand implied in the definition (112)[30]—*it would be continued smoothly through the interfaces (108), (109), and (110).* In practical terms this means that the detailed behavior of the integrand in the vicinity of the surfaces (108), (109), and (110), *i.e., precisely those that have no counterparts in the Wick-rotated case,* will be apt to influence our results *only* if $\ln\xi$ has a nonvanishing *antisymmetric* part at these surfaces, in which case the corresponding part of the integrand would have to change its sign discontinuously while crossing them.

B. Analytic difference between Eqs. (69) and (70)

In an actual attempt to solve (70) we shall now try, as emphasized before, to follow as closely as possible the same methods which in Sec. IV successfully led to the solution of (69), since, even in spite of the fact that we were forced to use different integration variables, the situation which confronts us here still seems to bear much resemblance to that of Sec. IV. What we have primarily in mind is the fact that if the ansatz (80), (81), which we now prefer to write as

$$\Phi = \Phi(v_+''v_-'', t)\,, \tag{113}$$

$$\Phi'' = \Phi(v_+v_-, t'') \tag{114}$$

to emphasize the dependence of Φ on the v parameters only through their product, could be made, then in complete analogy to (79) the integrand in (70) could be written as

$$\left(v_+''\frac{\partial}{\partial v_+''} - v_-''\frac{\partial}{\partial v_-''}\right)\left[\frac{\Phi \ln\xi}{(v_+''v_-'' - u_+{}^2)(v_+''v_-'' - u_-{}^2)}\right]. \tag{115}$$

Consequently, the volume integral of a divergence term (115) could again be converted into a sum of surface integrals over the boundaries of D_2' by a Green's theorem. Unfortunately, as we shall presently see, some of these surface integrals *will* now in general be ψ-dependent (or in terms of the v parameters will depend not only on the product but also on the ratio of these parameters), so that the compatibility of (113), (114) with (70) will *not* obtain (at least without changes made as to the types of the propagators used—see later). From the preceding geometrical considerations it is of course immediately clear that the only boundaries that *could* contribute to such ψ dependence (a term that we shall continue to use for the sake of brevity in spite of the present emphasis on the use of the v parameters) are the interfaces (108), (109), and (110) since the contributions from the external boundaries (106) and (107) corresponding to $z=\pm1$, respectively, remain [i.e., *contingent on* the ansatz (113), (114)] ψ-independent because of (89). Anticipating this, in a more detailed analysis which now follows we shall use the ansatz (113), (114) to transform only the right-hand side of (70), but shall not equate it to the Breit term $1/(s''-1) - 1/(t''-1)$ lest this should lead to inconsistencies. At this point it is convenient to finally go over to t, s, and ω'' [defined by (36), (34), and (20), respectively] as integration variables in preference to v_+'', v_-'', and t, keeping in mind, however, that the integration domain should correspond to D_2' defined by (112). The ansatz (113), (114) will therefore now mean that

$$\Phi = \Phi(t, s)\,, \tag{116}$$

$$\Phi'' = \Phi(t'', s'')\,, \tag{117}$$

emphasizing the (desired) lack of dependence on the ψ parameters.

The right-hand side of (70) becomes

$$\frac{(2\pi)''}{2g^2}\Phi(t'', s'') + \frac{\pi i v_+'v_-'}{(E_+ - ic)(E_- + ic)}\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} ds \frac{\Phi(t, s)}{(tsv_+'v_-' - u_+{}^2)(tsv_+'v_-' - u_-{}^2)}\int_{\omega_-''}^{\omega_+''} d\omega''\, \frac{\partial \ln\xi}{\partial\omega''}\,,$$

where the integration limits $\omega_\pm$ correspond to $z=\pm1$, respectively, and are

$$\omega_+'' = tv_-' + sv_+' \text{ or } \omega_+'' = tv_+' + sv_-'\,, \tag{119}$$

$$\omega_-'' = tv_+' + sv_-' \text{ or } \omega_-'' = tv_-' + sv_+' \tag{120}$$

[see also Eqs. (A10)–(A15) of Appendix A] according to whether the particular end point corresponds to a point in one of the $V_{1,2,6}$'s or one of the $V_{3,4,5}$'s constituting the original D_2, so that in D_2' only the *first* alternative applies. To evaluate the last integral in (118) we now proceed as follows: First of all we prefer to write it as

$$\int_{\omega_-''}^{\omega_+''} d\omega''\,\frac{\partial \ln\xi^{(1)}}{\partial\omega''}\,, \tag{121}$$

replacing ξ defined by (59) by a new function $\xi^{(1)}$ differing from ξ by a factor independent of ω'' and so chosen that

$$\xi_+^{(1)} \equiv \xi^{(1)}(\omega_+'') = (t - t'')(s - s'') , \tag{122}$$

$$\xi_-^{(1)} \equiv \xi^{(1)}(\omega_-'') = (t - s'')(s - t'') \tag{123}$$

[compare (90)]. For arbitrary ω'' and also using the t and s parameters (see Appendix A for details), the so-normalized function $\xi^{(1)}$ is then given by

$$\xi^{(1)}(\omega'') = \frac{-(t'' - s'')^2 t'' v_+'}{2\delta_+} \{\omega'' - \tfrac{1}{2}(\tilde{\omega}_+'' + \tilde{\omega}_-'') + [(\omega'' - \tilde{\omega}_+'')(\omega'' - \tilde{\omega}_-'')]^{1/2}\} , \tag{124}$$

where $\tilde{\omega}_\pm''$ [not to be confused with $\omega_\pm''$ of (119)] are given by

$$\tfrac{1}{2}(\tilde{\omega}_+'' + \tilde{\omega}_-'') = \frac{1}{(t'' - s'')^2} \{[(ts + t''s'')(t'' + s'') - 2t''s''(t + s)]\omega' + [(t + s)(t'' + s'') - 2(ts - t''s'')]\omega\} \tag{125}$$

and

$$\tfrac{1}{2}(\tilde{\omega}_+'' - \tilde{\omega}_-'') = \frac{2}{(t'' - s'')^2 t''} \left(\frac{\delta_+\delta_-}{v_+' v_-'}\right)^{1/2} (\xi_+^{(1)} \xi_-^{(1)})^{1/2} , \tag{126}$$

where the expressions $\delta_\pm$ are given by

$$\delta_\pm = (v_+ - t'' v_\pm')(v_- - t'' v_\pm') . \tag{127}$$

Finally, the proper branch of

$$[(\omega'' - \tilde{\omega}_+'')(\omega'' - \tilde{\omega}_-'')]^{1/2} ,$$

which is of course nothing else but $\sqrt{Q}$ defined by (63) multiplied by a factor independent of ω'', is defined in terms of that of

$$(1 - z^2 - y'^2 - z''^2 + 2zy'z'')^{1/2}$$

by

$$[(\omega'' - \tilde{\omega}_+'')(\omega'' - \tilde{\omega}_-'')]^{1/2} = -i \frac{(v_- + v_+)(v_-' - v_+')(v_-'' - v_+'')}{2v_+ v_-(s'' - t'')} \times (1 - z^2 - y'^2 - z''^2 + 2zy'z'')^{1/2} . \tag{128}$$

In evaluating (120) special attention must be given to the following two questions: Does the integration with respect to ω'' between ω_-'' and ω_+'' entail crossing of an interface? If such crossing does occur, what is the behavior of $\ln\xi^{(1)}$ at this interface?

The first problem is a purely geometrical one and is again best discussed with help of the (v_+'', v_-'') diagrams of the type in Figs. 7–13. The integration path in the (v_+'', v_-'') plane corresponding to the ω'' integration from ω_-'' to ω_+'' with s and t kept constant is represented by an arc of the hyperbola $v_+''v_-'' = tsv_+'v_-'$ [compare (34)] contained within D_2'. A typical situation is shown in Fig. 15, which is actually the same as Fig. 9, but where hyperbolas corresponding to $s > 1/v_-'$, $s = 1/v_-'$, $1/v_-' > s > 1/v_+'$, $s = 1/v_+'$, and $s > 1/v_+'$ are also drawn. We can see that in this particular situation interface crossing occurs only for $s > 1/v_-'$ (namely, proceeding in the direction of the increasing v_-'' from V_4^* into V_2) and for $s < 1/v_+'$ (from V_5^* into V_1), while for $1/v_-' > s > 1/v_+'$ the whole integration path lies entirely within either V_4^* or V_1. Repeating the same for all the other situations depicted in Figs. 7–13, the full answer to the question concerning the interface crossing is contained in Table I. The meaning of the symbols used in Table I is as follows: A simple + or – denotes that no interface crossing occurs. The + sign also indicates that the integration path lies entirely inside a single V_i (V_i^*) if this region lies above (below) the diagonal $v_+'' = v_-''$. The – sign indicates just the converse. Situations where interface crossing does occur are marked + → – or – → +,

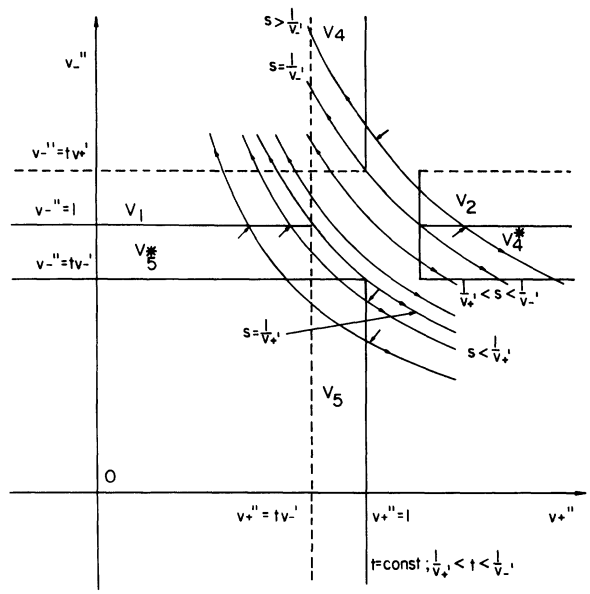

FIG. 15. Graphical illustration of the conditions under which interface crossings occur (indicated by ↗ arrows; the arrows ↙ indicate the corresponding points in the "original"—or "unreflected"—V_i's). The situation depicted corresponds to the case represented in Fig. 9 ($1/v_-' > t > 1/v_+'$) only. Table I is verified by repeating the construction for the remainder of Figs. 7–13. Arrows along the hyperbolas $v_+''v_-'' = tsv_+'v_-' =$const indicate the direction of increasing ω''.

TABLE I. Areas in the (t,s) plane where interface crossings occur (marked by arrows).

$s \backslash t$	$t>\frac{1}{v'_-}$	$\frac{1}{v'_-}>t>\frac{1}{v'_+}$	$t<\frac{1}{v'_+}$
$s>\frac{1}{v'_-}$	−	+ → −	+
$\frac{1}{v'_-}>s>\frac{1}{v'_+}$	− → +	+	+ → −
$s<\frac{1}{v'_+}$	+	− → +	−

respectively, according to whether, proceeding in the direction of increasing v''_-, the integration path crosses from a + area into a − area or inversely from a − area into a + area.

The importance of the above distinction between + areas and − areas is connected with the behavior of $\ln\xi^{(1)}$ and derives from the fact that if *Feynman propagators* are used for the exchange particle,[31] then, in situations where

$$z'_+>y'>z'_- \tag{129}$$

($\bar\omega''_+$ and $\bar\omega''_-$ both real and $\bar\omega''_+\geq\omega''\geq\omega''_-$), with $z'_\pm$ defined by (52), (53), we have

$$\operatorname{sgn}\{i[(\omega''-\bar\omega''_+)(\omega''-\bar\omega''_-)]^{1/2}\}=+\operatorname{sgn}\left[\frac{(v_--v_+)(v'_--v'_+)}{v_+v_-(s''-t'')}\right] \tag{130}$$

for the + areas and

$$\operatorname{sgn}\{i[(\omega''-\bar\omega''_+)(\omega''-\bar\omega''_-)]^{1/2}\}=-\operatorname{sgn}\left[\frac{(v_--v_+)(v'_--v'_+)}{v_+v_-(s''-t'')}\right] \tag{131}$$

for the − areas. Therefore discontinuities of $\ln\xi^{(1)}$ *can* and, as we shall see, *will* occur if an interface crossing takes place with the situation (129) prevailing. This and the equally important fact that an interface crossing proceeds smoothly (i.e., without a discontinuity in $\ln\xi^{(1)}$) if rather than (129) either

$$y'<z'_- \tag{132}$$

or

$$y'>z'_+ \tag{133}$$

obtains, follow from an at first paradoxical-looking theorem to the effect that the proper branch of $[(\omega''-\bar\omega''_+)(\omega''-\bar\omega''_-)]^{1/2}$ behaves like an *antisymmetric* function in v''_+ and v''_- for situation (129), but becomes a *symmetric* function in these variables for situations (132) or (133).

To prove the first part of the theorem notice that the situation (129) remains unchanged under the substitution $v''_+\rightleftarrows v''_-$, since the latter implies $y'\to -y'$ *and* $z\to -z$, so that $1-z^2-y'^2-z''^2+2zy'z''$ remains unchanged. Now, for a *Feynman* propagator, y' defined by (24) is real except for a small and *always negative* imaginary part, which for situations (129) makes $(1-z^2-y'^2-z''^2+2zy'z'')^{1/2}$ *always positive,* since the proper branch of this square root as a function of y' was defined [compare (51), (52), and (53)] as that which is positive at the lower lip of the cut extending from z'_- to z'_+. Consequently, not only the expression $1-z^2-y'^2-z''^2+2zy'z''$ itself but also the sign of its square root must remain unchanged by the substitution $v''_+\rightleftarrows v''_-$, which proves, according to (128), that for a situation (129) $[(\omega''-\bar\omega''_+)(\omega''-\bar\omega''_-)]^{1/2}$ is indeed an antisymmetric function in v''_+ and v''_-.

To prove the second part of our theorem it now suffices to observe that if a situation (132) or (133) obtains, the expression $1-z^2-y'^2-z''^2+2zy'z''$ again remains unchanged under the substitutions $v''_+\rightleftarrows v''_-$, but its square root *does* change sign since under this substitution the situation (132) becomes that of (133) and vice versa and the signs of the proper branch of $i[1-z^2-y'^2-z''^2+2zy'z'']^{1/2}$ to the left and to the right of the above cut are opposite. Q. E. D.

In order to establish in detail what trajectory will be described by $\xi^{(1)}$ in its analytic plane as ω'' varies between ω''_- and ω''_+ for a given pair of values of t and s, or in other words what will be the *shape* of this trajectory as a function of t and s, we must now supplement the information contained in Table I with that stemming from (125) and (126), regarding the properties of $\bar\omega''_+$ and $\bar\omega''_-$ as functions of t and s. Noting that according to (124) and (37)

$$\delta_+\delta_-=\frac{z''+1}{z''-1}(v_+-t''v'_+)^2(v_--t''v'_-)^2\leq 0 \tag{134}$$

and also assuming, consistently with our previous geometrical constructions, that $v'_+v'_->0$,[32] we first of all see that if $\xi^{(1)}_+$ and $\xi^{(1)}_-$ are either both positive or both negative then $\bar\omega''_+$ and $\bar\omega''_-$ are complex, the expression $(\omega''-\bar\omega''_+)(\omega''-\bar\omega''_-)$ is positive-definite, and, consequently, the situation (129) can never occur. This means that, regardless of whether or not the interface crossing occurs, $\xi^{(1)}$ continues smoothly from $\xi^{(1)}_-$ to $\xi^{(1)}_+$, remains real, and also retains the same sign, since for finite ω'', $\xi^{(1)}$ must [compare (124)] remain finite and $\neq 0$. Consequently, whenever

$$\xi^{(1)}_+\xi^{(1)}_-=(t-t'')(s-s'')(t-s'')(s-t'')>0\ , \tag{135}$$

we have simply

$$\int_{\omega''_-}^{\omega''_+} d\omega'' \frac{\partial \ln\xi^{(1)}}{\partial\omega''} = \ln\left|\frac{(t-t'')(s-s'')}{(t-s'')(s-t'')}\right| \quad . \qquad (136)$$

The situation is much more complicated if $\xi_+^{(1)}$ and $\xi_-^{(1)}$ have opposite signs, i.e., if

$$\xi_+^{(1)}\xi_-^{(1)} = (t-t'')(s-s'')(t-s'')(s-t'') < 0 \ . \qquad (137)$$

First of all, the situation (129) can and in fact must then prevail part of the way between ω''_- and ω''_+, which means that the interval $(\tilde{\omega}''_-, \tilde{\omega}''_+)$ must be contained within the interval (ω''_-, ω''_+). This is because as long as (129) prevails, the point representing $\xi^{(1)}$ in its analytic plane must remain on a circle around the origin of radius

$$\left|\frac{t''-s''}{2(s''t'')^{1/2}}\left(\frac{v'_+\delta_-}{v'_-\delta_+}\,\xi_+^{(1)}\xi_-^{(1)}\right)^{1/2}\right|$$

[compare (124)] and therefore either the lower or the upper part of this circle must be traversed in order to connect a positive (negative) $\xi_+^{(1)}$ with negative (positive) $\xi_-^{(1)}$ by a continuous "trajectory" consisting only of intervals of the real axis [$\xi^{(1)}(\omega'')$ remains real for situations (132) and (133)] and arcs of the above circle without passing either through zero or infinity [compare Fig. 16, representing such a trajectory for a situation where $s''>s>t''>1/v'_->1/v'_+>t>0$, $v_+>v_-$, and $\delta_+<0$]. However, whether $\xi^{(1)}$ will follow a continuous trajectory of this type or will jump discontinuously from the upper (lower) to the lower (upper) semicircle upon reaching an interface in a manner indicated in Fig. 17 [drawn for a situation where $s''>s>t''>1/v'_->t>1/v'_+>0$, $v_+>v_-$, and $\delta_+<0$] will depend on the following further circumstances: First of all a continuous trajectory will always obtain if the values of s and t in addition to satisfying (137) also correspond to regions marked + or − in Table I. In this case we will have instead of (135)

$$\int_{\omega''_-}^{\omega''_+} d\omega'' \frac{\partial \ln\xi^{(1)}}{\partial\omega''} = \ln\left|\frac{(t-t'')(s-s'')}{(t-s'')(s-t'')}\right| \pm i\pi, \qquad (138)$$

where the + sign applies to a situation where $\xi_+^{(1)}>0$ ($\xi_+^{(1)}<0$) and the lower (upper) semicircle was traversed, and the − sign to $\xi_+^{(1)}>0$ ($\xi_+^{(1)}<0$), $\xi^{(1)}$ describing the upper (lower) semicircle. Pro-

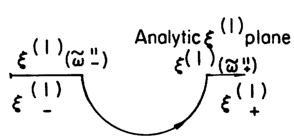

FIG. 16. Trajectory described by the point representing $\xi^{(1)}$ in its analytic plane, as ω'' varies between ω''_- and ω''_+ in the case of $s''>s>t''>1/v'_->1/v'_+>t>0$, $v_+>v_-$, and $\delta_+<0$. There is no interface crossing in this instance.

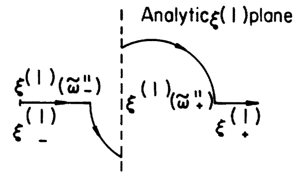

FIG. 17. Trajectory described by the point representing $\xi^{(1)}$ in its analytic plane, as ω'' varies between ω''_- and ω''_+ in the case of $s''>s>t''>1/v'_->t>1/v'_+>0$, $v_+>v_-$, and $\delta_+<0$. Notice the discontinuous jump from the lower to the upper semicircle upon reaching the interface at $\omega''=\omega''_0$, where ω''_0 is given by (141).

vided that (137) is satisfied, continuous trajectories and therefore also the result (138) will still obtain in *parts* of the regions marked +→ − and −→ + in Table I as long as either

$$\tilde{\omega}''_- > \omega''_0 \qquad (139)$$

or

$$\tilde{\omega}''_+ < \omega''_0 \ . \qquad (140)$$

Here

$$\omega''_0 = 1 + v'_+ v'_- st \ , \qquad (141)$$

and denotes the value of ω'' for which an interface crossing occurs [i.e. for which *either* $v''_+=1$ *or* $v''_-=1$; compare (20) and (34)]. Finally, a discontinuous trajectory of the type depicted in Fig. 17 will always obtain if (137) is satisfied and in addition s and t correspond to subregions of the regions marked +→ − or −→ + in Table I for which

$$\tilde{\omega}''_+ > \omega''_0 > \tilde{\omega}''_- \ . \qquad (142)$$

If the latter is the case, we will have

$$\int_{\omega''_-}^{\omega''_+} d\omega'' \frac{\partial \ln\xi^{(1)}}{\partial\omega''} = \ln\left|\frac{(t-t'')(s-s'')}{(t-s'')(s-t'')}\right| \pm [2\ln\xi(\omega''_0) - i\pi] \ , \qquad (143)$$

where $\xi(\omega''_0)$ denotes the "old" function ξ defined by (59) (see Ref. 33) at $\omega''=\omega''_0$. The over-all results of the above analysis are best represented graphically in Fig. 18, drawn for a particular situation where $s''>t''>1/v'_->1/v'_+>0$, $v_+>v_-$, and $\delta_+<0$, which makes (130) positive. The important "phase term" $P(t,s;t'',s'')$ is defined by

$$\int_{\omega''_-}^{\omega''_+} d\omega'' \frac{\partial \ln\xi^{(1)}}{\partial\omega''} = \ln\left|\frac{(t-t'')(s-s'')}{(t-s'')(s-t'')}\right| + P(t,s;t'',s'') \ . \qquad (144)$$

The boundaries of the doubly shaded regions inside which (143) applies are of course given by

$$(\omega''_0 - \tilde{\omega}''_+)(\omega''_0 - \tilde{\omega}''_-) \equiv [\omega''_0 - \tfrac{1}{2}(\tilde{\omega}''_+ + \tilde{\omega}''_-)]^2 - \tfrac{1}{2}(\tilde{\omega}''_+ - \tilde{\omega}''_-)^2 = 0 \ , \qquad (145)$$

where $\bar{\omega}_{\pm}''$ and ω_0'' are given by (125), (126), and (141), respectively. If written in terms of the t and s parameters, (145) reads

$$\{(t''-s'')^2(1+stv_+'v_-')-[(ts+t''s'')(t''+s'')-2t''s''(t+s)](v_+'+v_-')-[(t+s)(t''+s'')-2(ts+t''s'')](v_++v_-)\}^2$$

$$-\frac{4}{t''^2v_+'v_-'}(v_+-v_+'t'')(v_--v_+'t'')(v_+-v_-'t'')(v_--v_-'t'')(t-t'')(t-s'')(s-t'')(s-s'')=0\ . \quad (146)$$

This is obviously symmetric in t and s, and for $s=\text{const}$ ($t=\text{const}$) represents a *quadratic* equation for t (s). Equation (146) shows also that the curve (or rather curves) it represents must always be tangential to the lines $s=t''$ and $s=s''$ ($t=t''$ and $t=s''$). These properties allow us to immediately draw phase diagrams pertaining to topologies different from that depicted in Fig. 18, i.e., to types of inequalities between the initial and final parameters different from those hitherto assumed (compare Figs. 19 and 20). Continuing with the situation depicted in Fig. 18, however, they imply in particular that the roots $t=l_1(s)$ and $t=l_2(s)$ [$s=l_1(t)$ and $s=l_2(t)$] of (146) must be real for $s''>s>t''$ ($s''>t>t''$), "merging" at the ends of this interval, where for future reference we shall also assume that $l_2 \geq l_1$.

From Fig. 18 it is now clear that by substituting

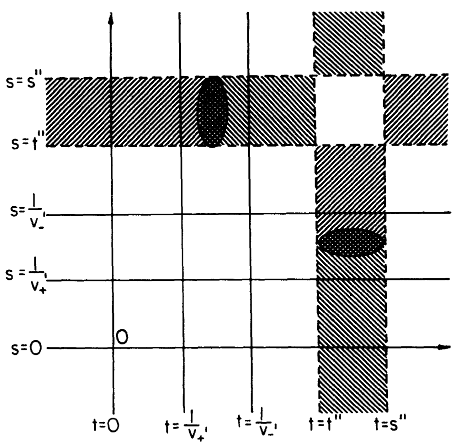

FIG. 18. A diagram depicting the main characteristics of the dependence of the phase P defined by (144) on the parameters t, s, and ω'', drawn for the particular case ($s>t''>1/v_-'>1/v_+'$, $v_+>v_-$, and $\delta_+<0$) worked out in detail in the main text. $P=0$ everywhere outside the shaded areas. The two different shadings, namely that in continuous lines inclined under 45° with respect to the horizontal direction and that in continuous lines inclined under 135° with respect to the horizontal direction, indicate regions where P is constant and equal to $+i\pi$ and $-i\pi$, respectively (or more generally, where P is equal to $-i\epsilon\pi$ and $+i\epsilon\pi$ and ϵ stands for $\text{sgn}[(s''-t'')(v_--v_+)(v_-'-v_+')/v_+'v_-'\delta_+]$). The doubly shaded areas represent "transition regions" where P becomes ω-dependent and is equal to F given by (153). The shape of the boundaries of the latter regions is also ω-dependent and is given by (146).

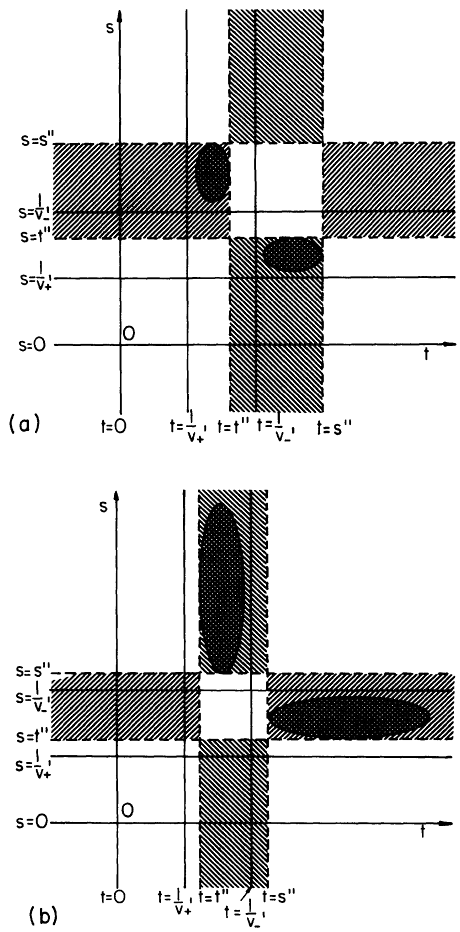

FIG. 19. Diagrams of the same type as Fig. 18, but drawn for a general situation where $s''>1/v_-'>t''>1/v_+'$, $v_+>v_-$, and $\delta_+<0$. Additional—though of no particular interest—inequalities between the parameters involved distinguish between the special cases (a) and (b). Somewhere between (a) and (b) a situation arises where the "transition regions" degenerate to two points at the intersections of the lines $t=t''$, $s=s''$ and $t=s''$, $s=t''$ and the general pattern becomes accidentally identical to that of Fig. 22.

(144) into (118), the latter will in general become ψ-dependent (ω-dependent) on account of the ψ-dependence of $\ln\xi(\omega_0'')$, so that the compatibility of (116), (117) with (70), needed not only as practical means of solving the latter but also *representing a necessary condition for the Wick-Cutkosky results to follow*, becomes indeed questionable. In order to analyze more closely this important point, it will now be advantageous to represent $P(t, s; t'', s'')$ defined by (144) in the form of either of the following two alternative "decompositions":

$$P = P_W + R_{\text{I}} \tag{147}$$

and

$$P = P_M + R_{\text{II}}. \tag{148}$$

P_W and P_M are defined by

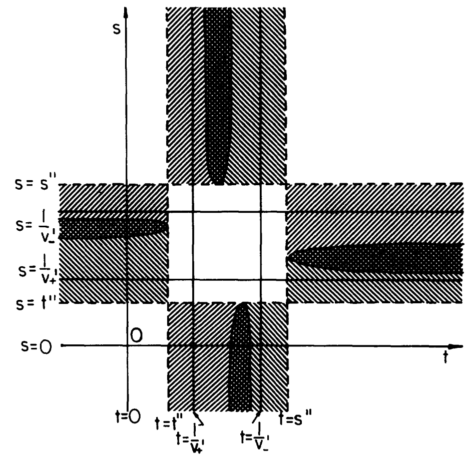

FIG. 20. Another phase diagram belonging to the same general category as Figs. 18 and 19, drawn for a situation where $s'' > 1/v_-' > 1/v_+' > t''$, $v_+ > v_-$, and $\delta_+ < 0$.

$$P_W(t, s; t'', s'') = -i\pi[\Theta(t''-t)\Theta(s''-s) - \Theta(s''-t)\Theta(t''-s) + \Theta(t-t'')\Theta(s-s'') - \Theta(t-s'')\Theta(s-t'')] \tag{149}$$

and

$$P_M(t, s; t''s'') = -i\pi[\Theta(s''-t) - \Theta(t''-t) - \Theta(s''-s)\Theta(t''-s)]. \tag{150}$$

They are represented graphically in Figs. 21 and 22, respectively, and are obviously ψ-independent. The "rest terms" R_{I} and R_{II} are given by

$$R_{\text{I}}(t, s; t'', s''; \omega) = -[\Theta(s''-s) - \Theta(t''-s)] \times\{[F(t, s; t'', s''; \omega) - 2\pi i]\Theta(l_1(s)-t) - F(t, s; t'', s''; \omega)\Theta(l_2(s)-t)\} - (s \rightleftarrows t) \tag{151}$$

and

$$R_{\text{II}}(t, s; t'', s''; \omega) = -[\Theta(s''-s) - \Theta(t''-s)] \times\{2\pi i[\Theta(s''-t) - \Theta(l_1(s)-t)] + F(t, s; t'', s''; \omega)[\Theta(l_1(s)-t) - \Theta(l_2(s)-t)]\} - (s \rightleftarrows t), \tag{152}$$

where $(s \rightleftarrows t)$ denotes terms obtained from the preceding by substituting s for t and vice versa. The term $2\ln\xi(\omega_0'')$ is now denoted by $F(t, s; t'', s'', \omega)$, which, to better emphasize its pertinent analytic properties as a function of t, can be written as

$$F(t, s; t'', s''; \omega) \equiv 2\ln\xi(\omega_0'') = 2\ln\{at + b + [(at+b)^2 - (t-t'')(t-s'')]^{1/2}\} - \ln[(t-t'')(t-s'')], \tag{153}$$

where

$$a = \frac{t''}{2[\delta_+\delta_-(s-t'')(s-s'')]^{1/2}}\{(t''-s'')^2 s v_+' v_-' - [(t''+s'')s - 2t''s''](v_+' + v_-') - [(t''+s'') - 2s](v_+ + v_-)\}, \tag{154}$$

$$b = \frac{t''}{2[\delta_+\delta_-(s-t'')(s-s'')]^{1/2}}\{(t''-s'')^2 - [t''s''(t''+s'') - 2t''s''s](v_+' + v_-') - [(t''-s'')s - 2t''s''](v_+ + v_-)\}. \tag{155}$$

The analytic structure of F as a function of t is represented graphically in Fig. 23, and the proper branch of F in the analytic cut t plane is defined as that which is $=0$ at $t = l_1$ and which exhibits a $-4\pi i$ (emphasis on the minus sign) discontinuity across the real axis for $\text{Re}(t) > l_2$ (i.e., the value of F at the *lower* lip minus that at the *upper* lip of the cut $= -4\pi i$).

The decomposition (147) is directly motivated by the problem of elucidating the analytic difference between (69) and (70), while that of (148) will to a certain extent serve a similar purpose in comparing the "realistic" equation (70) with the exactly soluble model constructed in Sec. VI.[34]

C. Failing of the Wick-Cutkosky solutions [study of the "decomposition" (147)]

Addressing ourselves to the first problem first, the most important property of P_W, which accord-

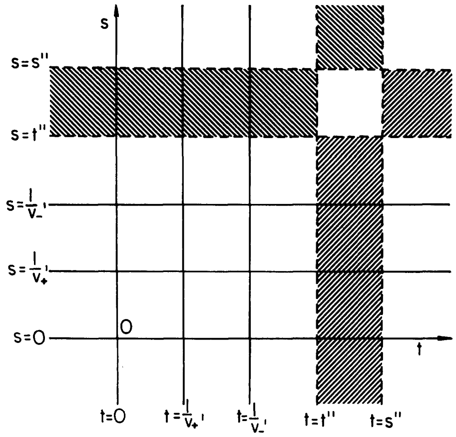

FIG. 21. A "simplified" diagram of Fig. 18 for which the Wick-Cutkosky type of solutions *would* obtain. The t and s dependence in this diagram is given by Eq. (149).

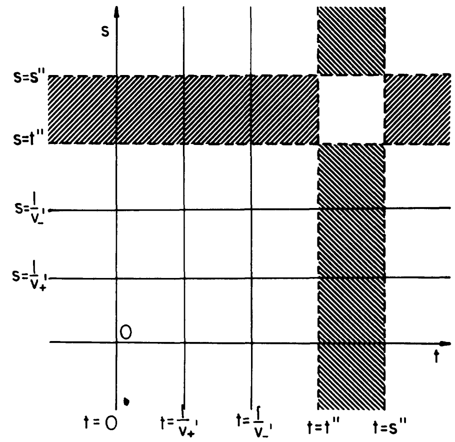

FIG. 22. A "simplified" diagram of Fig. 18 that would lead to (174) and is closer related to the exactly soluble model of the BS equation considered in Sec. VI. The t and s dependence in this diagram is given by Eq. (150).

ing to Fig. 21 represents—roughly speaking—an "extension to infinity" of a situation prevailing in Fig. 18 in the vicinity of the square where the lines $t=t''$, $t=s''$, $s=t''$, and $s=s''$ intersect, is

$$\frac{\partial^2}{\partial t''\partial s}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}ds\,P_W(t,s;\,t'',s'')f(t,s)=-2\pi i[f(t'',s'')-f(s'',t'')]\,. \quad (156)$$

The importance of (156) derives from the fact that if reasons could be found in support of the conjecture that the ω-dependent "rest term"

$$\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}ds\frac{\Phi(t,s)}{(tsv_+'v_-'-u_+^2)(tsv_+'v_-'-u_-^2)}\times R_1(t,s;\,t'',s'';\,\omega)=0\,, \quad (157)$$

or could otherwise be neglected [to which we shall refer in a sequel as the orthogonality conjecture (157)], then not only would the ansatz (116), (117) again become compatible with (70), but also the solution of the resulting integral equation would have to satisfy a second-order partial differential equation *identical with (101)*, thus leading once more to results essentially equivalent to those of Wick and Cutkosky. To see this, replace P by P_W in (144), substitute the result in (118), and equate the so-obtained expression to the Breit term $1/(s''-1)-1/(t''-1)$. The resulting integral equation is

$$\frac{1}{s''-1}-\frac{1}{t''-1}=\frac{(2\pi)^4}{2g^2}\Phi(t'',s'')+\frac{\pi i v_+'v_-'}{(E_+-ic)(E_-+ic)}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}ds\frac{\Phi(t,s)}{(tsv_+'v_-'-u_+^2)(tsv_+'v_-'-u_-^2)}\times\left[\ln\left|\frac{t-t''}{s-t''}\right|-\ln\left|\frac{t-s''}{s-s''}\right|+P_W(t,s;\,t'',s'')\right] \quad (158)$$

and has its closest analog in either (89) or (93) of Sec. IV. From (158) it now follows that

$$\Phi(t,s)=-\Phi(s,t)\,, \quad (159)$$

which is exactly the relation (92) written in terms of t and s. Applying the operator $\partial^2/\partial t''\partial s''$ to both sides of (158), we finally get because of (156) and (159)

$$\left[\frac{\partial^2}{\partial t''\partial s''}-\left(\frac{g}{2\pi}\right)^2\frac{v_+'v_-'}{(E_+-ic)(E_-+ic)}\frac{1}{(t''s''v_+'v_-'-u_+^2)(t''s''v_+'v_-'-u_-^2)}\right]\Phi=0\,, \quad (160)$$

where again the method parallels closely that of obtaining (101) from (93) with help of (100), since even the Laplacian (99) is, up to a numerical factor, identical with the operator $\partial^2/\partial t''\partial s''$. Reintroducing the α and φ variables through Eqs. (83), (34), (35), and (7) and also using (16) and (14) we can see in fact that (160) is identical with (101), so that (157) with Φ satisfying (160) becomes

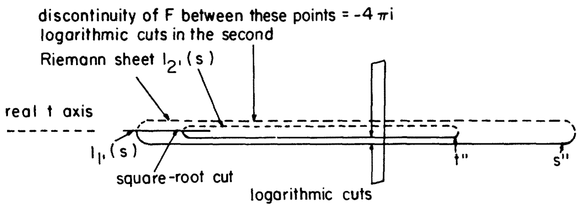

FIG. 23. Analytic structure of F given by (153).

indeed a necessary condition for the Wick-Cutkosky results to obtain and becomes, by the same token, also an *a posteriori necessary condition for the validity of the Wick rotation.*

Having thus succeeded in reducing the problem of analytic difference between (69) and (70) to that of satisfying the orthogonality condition (157), we will now finally proceed to show that it in fact *cannot be satisfied.* The possibility of such a clear-cut negative proof [as contrasted with the general feeling that it would be hard to satisfy (157) on account of its ψ dependence] stems mainly from a hitherto-unexplored property of the phase P defined by (144). According to (124), and (128) [compare also the detailed sign determination implied by (130)], and (131) the change of [35]

$$\operatorname{sgn}\left[\frac{(v_- - v_+)(v'_- - v'_+)}{-v_+ v_- \delta_+}\right] \tag{161}$$

brings about an over-all reversal of sign of this phase. To correctly assign the proper values of (161) to different parts of the physical domain of the space of initial states (v_+, v_-, t'') [or equivalently (t'', s'', ω) space], it is best to again subdivide it into volumes V_i in exactly the same way as was done with the intermediate-state space D_2 at the beginning of the present section (compare again Figs. 7–13 with $v_\pm$ instead of $v''_\pm$ and t'' instead of t). Having done so we immediately see (compare Table II) that (161) is positive in V_1, V_2, and V_6 and negative in V_3, V_4, and V_5.

Now, in order to prove that (157) cannot be satisfied at all points of the physical domain of the (v_+, v_-, t'') space, assume—to the contrary—that it is satisfied, e.g., at least in the region of V_2 immediately bordering on V_4^*, whose representative point A is depicted in Fig. 24 (which is a "replica" of Fig. 9 in the space of the initial states). Consider also a point B lying in V_4 such that the distance between A and B^*, the "mirror image" of B in the $v_+ = v_-$ plane (lying therefore in V_4^* across the interface $v_- = 1$ from A), can be chosen arbitrarily small. In the limit as both A and B^* approach the same point C of the interface $v_- = 1$, the phase structures—as depicted in Fig. 18—pertaining to A and B will become exactly the same, *except for the above-mentioned over-all*

TABLE II. Signs of expressions: $v_- - v_+$, δ_+, and (161), obtaining in different subdomains of the space of initial states, tabulated in order to demonstrate the existence of the discontinuous reversal of the sign of P at the interfaces.

sign of \ "volume"	V_1	V_2	V_3	V_4	V_5	V_6
$v_- - v_+$	+	−	−	+	−	−
δ_+	+	−	+	−	+	−
(161)	+	+	−	−	−	+

change of sign. This is because on one hand (146), (154), and (155) are symmetric in v_+ and v_-, but on the other hand the signs (161) in V_2 and V_4 are opposite. More precisely, *retaining* the same signs in the definitions (149) and (151) of P_W and R_I we will have on one hand

$$P_W(C) = P_W(C^*), \tag{162}$$

$$R_I(C) = R_I(C^*), \tag{163}$$

where C^* is the "mirror image" of C in the $v_+ = v_-$ plane, whereas on the other hand

$$P(A) \underset{A\to C}{\longrightarrow} P_W(C) + R_I(C) \tag{164}$$

and, most importantly,

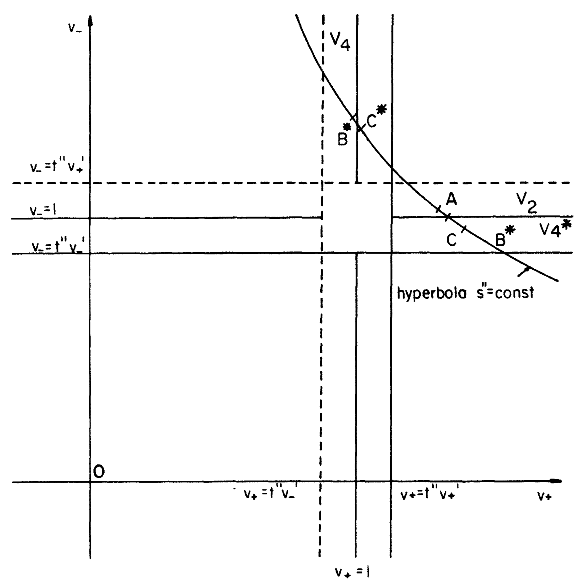

FIG. 24. A diagram corresponding to the same situation as Fig. 9, but drawn for the space of *initial* states ($v_\pm$ instead of $v''_\pm$ and t'' instead of t), referred to in proving the theorem that there is an over-all discontinuous change of the sign of the phase P at the interface crossing at the point C.

$$P(B) \underset{B \to C^*}{\longrightarrow} -P_W(C) - R_I(C)\,. \qquad (165)$$

Considering the contribution from $R_I(C)$ as $=0$, i.e., invoking (157), we can therefore only conclude that, upon the above-mentioned interface crossing from V_2 into V_4^*, the sign of the integral term in (158) should also suffer a discontinuous reversal and therefore should lead to, instead of (160),

$$\left[\frac{\partial^2}{\partial t''\partial s''} + \left(\frac{g}{2\pi}\right)^2 \frac{v'_+ v'_-}{(E_+ - ic)(E_- + ic)} \frac{1}{(t''s''v'_+v'_- - u_+{}^2)(t''s''v'_+v'_- - u_-{}^2)}\right]\Phi = 0\,. \qquad (166)$$

However, the assumed ω independence would then lead us to believe that in fact *both (166) and (160) should be satisfied simultaneously* since, as we saw before, the interface crossing can very well proceed along an arc of the hyperbola $v_+ v_- = v'_+ v'_- / s'' t'' = \text{const}$, so that, even at finite distances from each other, all the points A, B, B^*, C, and C^* could be assigned exactly the same values of t'' and s''.

Repeating the above argument for all the areas of the (t'', s'') space marked $+--$ or $--+$ in Table I, i.e., for all the areas where the interface crossing is possible, the simultaneous validity of (160) and (166) can of course only mean that $\Phi = 0$ in these areas. The assumed validity of (157) in the remaining areas of the (t'', s'') space must on the other hand at least imply the validity of *either (160) or (166)*, so that by the usual continuity arguments we are driven to an inevitable conclusion that $\Phi = 0$ should prevail over the whole (t'', s'') plane. Needless to say, such a "solution" is not only physically unacceptable, but is also simply incompatible with (70), whose inhomogeneity (Breit term) does not vanish identically. The final conclusion can therefore only be that (157) cannot be satisfied and therefore *the Wick rotation is not valid.* Q.E.D.

In addition to furnishing the proof that (69) and (70) are indeed analytically different and that a solution of the Wick-Cutkosky type cannot satisfy the latter, the above negative result is also practically tantamount to admitting our inability to solve (70) by the methods developed so far. Therefore, no further attempt will be made in this work at obtaining a closed-form solution of (70), which, if at all possible, must in all probability involve a ω dependence of Φ. Instead we shall devote the remainder of the present section to considerations leading to the construction of an exactly soluble model of the BS equation (see Sec. VI), since the currently used t and s parameterization seems to be particularly suited for this purpose. As we shall see later, the construction of this model will be tantamount to changing the type of propagators involved, in such a manner, however, that some basic physical features of the "exact" problem hopefully will be retained. In conjunction with the already proven lack of equivalence between (69) and (70), the solution of the model might therefore conceivably represent even a better approximation to the "true" scattering amplitude than that based on the Wick-Cutkosky results.

D. Study of other "decompositions"–in particular, (148)

A good starting point for the construction of such a model is provided by returning once more to the general problem of the compatibility of the solving ansatz (116), (117) with (70) *per se*, i.e., independently of whether or not the former leads to a solution of the Wick-Cutkosky type. Notwithstanding the fact that the latter possibility is now definitely excluded, it is still *a priori* possible (though admittedly not very likely, in view of our recent experience) that a "decomposition" of the "phase" P defined by (144) into a ω-independent "main part" and a ω-dependent "rest term"—*other than (147)*—could still be found in such a way, that an equation obtained by replacing P by this main part *would* admit of a solution orthogonal [in the sense similar to (157)] to the rest term (for arbitrary ω), so that it would also satisfy (70) itself. Unfortunately, it seems almost impossible to furnish even a conclusive *negative* proof pertaining to this more general problem of compatibility (i.e., to the effect that it is impossible to find such a decomposition), let alone—to the contrary—to actually find a decomposition which *does* satisfy the above conditions and thus be able to solve (70) after all. This can be partly ascribed to the apparent lack of a valid guiding principle as to the proper choice of trial decompositions, though each such trial would be apt to be very complicated itself, involving as its *preliminary* step a necessity to solve an integral equation of a type presumably varying widely from case to case.

However, if we were to be guided in this context by a principle of *analytic simplicity* alone, then the decomposition (148) would certainly seem to deserve our very special attention, so much so in fact that had we not known about the existence of the Wick-Cutkosky solutions we would probably be inclined to investigate it first, even in preference to that given by (147). The greater analytic simplicity of (148) as compared to (147) obtains, as it should, separately in both the "main parts," i.e., in P_M as compared with P_W, and the "rest terms," i.e., in R_{II} as compared with R_I. As seen from Fig. 22, P_M, given by (150), represents an extension of the pattern from Fig. 18 prevailing at the far-off tails of the latter. Its far greater simplicity in comparison to P_W given by (149) is almost self-evident in that, unlike the latter, it represents a sum of a function of t'' alone and a function of s'' alone, which property it therefore shares with the logarithmic term in (144). Consequently, in sharp contrast to (156), we have simply

$$\frac{\partial^2}{\partial t''\partial s''} P_M(t,s;\, t'',s'')=0\,. \qquad (167)$$

The greater analytic simplicity of R_{II} given by (152) as compared with R_I shows itself in the fact that

$$\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} ds\, R_{II}(t,s;\, t'',s'';\, \omega) f(t,s)$$
$$=\tfrac{1}{2}\int_{t''}^{s''} ds \oint dt\, F(t,s;\, t'',s'';\, \omega) f(t,s)$$
$$-\tfrac{1}{2}\int_{t''}^{s''} dt \oint ds\, F(t,s;\, t'',s'';\, \omega) f(t,s)\,, \qquad (168)$$

where F is given by (153), the emphasis being on the appearance of *contour integrals*, which seem to fit naturally with the system of analytic cuts of F depicted in Fig. 23. By contrast, an expression corresponding to the left-hand side of (168) with R_{II} replaced by R_I would involve only open path integrals in both variables t and s, i.e., integrals taken from some initial value (l_1, l_2, t'', or s'') to ∞, where ∞ represents a *regular* point of F. In more detail, the contour integral in t of the first term on the right-hand side of (168) is understood as that taken—in the usual anticlockwise direction—around the whole system of cuts depicted in Fig. 23, or equivalently around a cut extending from $l_1(s)$ to s'' and passing through $l_2(s)$ and t''. It should be noted that the subsequent integration with respect to s between the values t'' and s'' could also be converted into a contour integral (though of a different and slightly more complicated type and therefore avoided here), since these two points represent also (logarithmic) branch points of F (this time as a function of s). The integrations involved in the second term of the right-hand side of (168) are understood in precisely the same sense as above, only with the roles of t and s reversed (F is a symmetric function of t and s).

If we now wanted to undertake seriously the problem of solving (70) by the ansatz (116), (117) again, but decided to base our procedure on the decomposition (148) instead of (147), Eq. (158) would have to be replaced by

$$\frac{1}{s''-1}-\frac{1}{t''-1}=\frac{(2\pi)^4}{2g^2}\Phi(t'',s'')+\frac{\pi i v'_+ v'_-}{(E_+-ic)(E_-+ic)}\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} ds\, \frac{\Phi(t,s)}{[tsv'_+v'_- -u_+^2][tsv'_+v'_- -u_-^2]}$$
$$\times\{[\ln(t-t'')-\ln(s-t'')]-[\ln(t-s'')-\ln(s-s'')]\}\,, \qquad (169)$$

since, because of (150),

$$\ln\left|\frac{t-t''}{s-t''}\right| - \ln\left|\frac{t-s''}{s-s''}\right| + P_M(t,s;\, t'',s'') = [\ln(t-t'')-\ln(s-t'')]-[\ln(t-s'')-\ln(s-s'')]\,, \qquad (170)$$

where the proper branch of $\ln x$ for real x is understood as that for which

$$\ln x = \ln|x| + i\pi\theta(-x)\,. \qquad (171)$$

The integrations with respect to t and s in (169) can therefore also be understood as performed along a straight line parallel to the real axis, but lying slightly above this axis. Because of (168), the orthogonality condition (157) would have to be replaced by

$$\int_{t''}^{s''} ds \oint dt\, \frac{F(t,s;\, t'',s'';\, \omega)\Phi(t,s)}{[tsv'_+v'_- -u_+^2][tsv'_+v'_- -u_-^2]}=0\,, \qquad (172)$$

the appearance of only one (double) integral instead of two being caused by the fact that $\Phi(t,s)$ satisfying (169) would again [i.e., like that satisfying (158)] have to satisfy the antisymmetry relation (159).

The most important property of (169) deriving from (167) is now the fact that the application of the operator $\partial^2/\partial t''\partial s''$ gives

$$\frac{\partial^2\Phi(t'',s'')}{\partial t''\partial s''}=0\,, \qquad (173)$$

which, within the framework of our new approach, replaces (160). Equation (173), in conjunction with (159), leads immediately to the conclusion that

$$\Phi(t,s)=G(t)-G(s) \tag{174}$$

and brings about an enormous simplification of the problem of solving (169), which reduces, upon substituting (173) into (169) and differentiating with respect to t'' (mostly in order to convert the "unpleasant" logarithmic terms in the kernel into rational expressions), to that of solving a single integro-differential equation *in one variable*:

$$\frac{1}{(t''-1)^2}=\frac{(2\pi)^4}{2g^2}\frac{dG(t'')}{dt''} + \frac{\pi v'_+ v'_-}{Ec}\int_{-\infty}^{+\infty} dt K(t'',t)G(t), \tag{175}$$

with the kernel K given by[36]

$$K(t'',t)=\int_{-\infty}^{+\infty} ds\left(\frac{1}{tsv'_+v'_- -u_+{}^2}-\frac{1}{tsv'_+v'_- -u_-{}^2}\right) \times\left(\frac{1}{t''-t-i\epsilon}-\frac{1}{t''-s-i\epsilon}\right). \tag{176}$$

Of course, the orthogonality condition—or more properly conjecture—(172) also undergoes a further simplification [i.e., in addition to the already mentioned fact that, unlike (175), it can be expressed in terms of contour integrals] as a result of (174), in that one of the integrations in (172) can now be performed explicitly, before the exact form of $G(t)$ satisfying (175) is known.

Unfortunately, the above encouraging aspects of the approach based on the decomposition (148) notwithstanding, if *Feynman* propagators are used to represent the *scattered* particles A and B the exact form of $G(t)$ is not easy to obtain, since (175) is still very difficult to solve, and consequently no definite answer as to whether or not (172) can be satisfied can be given (the negative answer is of course still the most "probable") no matter how much more "plausible" a *positive* answer to that question might seem as compared to (157) on account of the appearance of *contour* integrals.

However, as we shall see in Sec. VI, it is not altogether too difficult to solve an equation of the general type (175) exactly, if the Feynman propagators representing the scattered particles are replaced by advanced (or retarded) propagators. Once the type of any propagator has been arbitrarily changed, on the other hand, we are already dealing with *models* of the BS equation rather than still trying to solve the original equation exactly, so that, in search for always valuable exact solubility, we might feel free—subject to certain limitations,[37] of course—to change the type of propagator representing the exchange particle as well. If this is done properly, however, then—as we shall presently see—*no orthogonality condition* of either type (157) or (172) is any longer required, since the solving ansatz (116), (117) becomes once more strictly valid.

VI. AN EXACTLY SOLUBLE MODEL OF THE BETHE-SALPETER EQUATION

Considering exact solubility as the most important guiding principle for the moment, we saw in Sec. V that our inability to solve (70) exactly stems mainly from the very complicated behavior of the phase P defined by (144). The most radical way to be free from this difficulty is therefore to simply put

$$P=0, \tag{177}$$

which is also tantamount to assuming the validity of (136) for all values of t and s. From the viewpoint of a more immediate physical interpretation, however, it comes at this point as a rather *pleasant* surprise that the validity of (177) is ensured by merely replacing the *Feynman* propagator for the exchange particle by its rather close cousin the so-called *relativistic Coulomb potential* (equal to half of the sum of the retarded and advanced propagators), so that we can still claim that *at least some essential physical features do remain included in the interaction.*[37] To see this it suffices to first realize that the above-defined Coulomb propagator (denoted sometimes by Δ_P) can also be represented as half the sum of Δ_{1R} and Δ_{1A}, the first denoting the Feynman propagator proper, involving the integration with respect to p_0 performed in the usual fashion, i.e., *below* the left singularity and *above* the right, and the second denoting, inversely, a propagator where the integration path in the complex p_0 plane bypasses the left singularity *above* and the right *below* the real axis.[38] Returning briefly to the pertinent part of the argument from Sec. V, we see that as long as (135) obtains, the outcome of the integration (136) remains of course the same whether or not the Feynman propagator is replaced by Δ_P, since it is in this case immaterial how the above singularities are to be avoided [situation (129) never obtains]. If on the other hand (137) holds, then, upon substituting Δ_P for Δ_{1R}, the integral (121) becomes equal to half the sum of its "old" value and the value corresponding to Δ_{1A} whose different integration path with respect to p_0 means that the point representing $\xi^{(1)}$ should describe *precisely the opposite semicircle* in its analytic plane. This happens because the signs of $(1-z^2-y'^2-z''^2+2zy'z'')^{1/2}$ corresponding to Δ_{1R} and Δ_{1A} are *always opposite* as long as the situation (129) prevails, with the result that the phase contributions stemming from these terms always

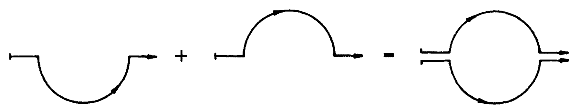

FIG. 25. Diagrammatic way of showing that using the "relativistic Coulomb" propagator for the exchange particle (right-hand side), considered as half of the sum of Feynman (first "term" on the left) and Δ_{1A} (second "term" on the left), has the effect of principal part integration in ω'' and thus leads to $P=0$. The drawings symbolize the trajectories described by the point representing $\xi^{(1)}$ in its analytic plane and correspond to the situation depicted in Fig. 16.

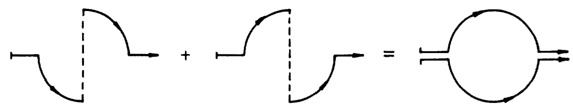

FIG. 26. A diagram showing that exactly the same is true as in the case depicted in Fig. 25, if there *is* an interface crossing. This diagram corresponds to the situation depicted in Fig. 17.

cancel, so that (177), or equivalently (136), indeed prevails throughout the whole (t,s) plane (detailed graphical illustration of this fact, pertaining separately to situations depicted in Figs. 16 and 17, is furnished by Figs. 25 and 26, respectively). The general validity of (136) means of course first of all the *disappearance of the* ω *(or* ψ*) dependence* from our equations, so that the ansatz (116), (117) is obviously valid and leads again to an equation of the general type (169), which is, however, now exact and need not be supplemented by any orthogonality condition. The only difference between the "old" and the "new" Eq. (169) [the latter referred to from now on as (169M), where M stands for "model"] is in the shape of the integration paths in t and s. Replacing the Feynman propagator by the Coulomb propagator for the exchange particle in (70) [the so-modified equation is referred to from now on as (70M)] means that the integrations with respect to both t and s in (169M) should be understood as *principal part integrations* (half sums of the corresponding integrals taken along two straight lines, one slightly below, the other slightly above the real axis) because only the absolute values of the expressions like $t-t''$, etc. enter Eq. (136). Another equally important consequence of the general validity of (136) is the fact that its right-hand side represents a sum of two functions, one of t'' alone and the other of s'' alone, so that the same is again true for the whole integral term in (169M) and, consequently, *(174) obtains*. This, in turn, by exactly the same manipulations as before, leads once more to the integro-differential equation of the type (175), only now its kernel is given by

$$K(t'',t)=\int_{-\infty}^{+\infty}\left(\frac{1}{tsv'_+v'_- -u_+^2}-\frac{1}{tsv'_+v'_- -u_-^2}\right)\times \mathrm{P}\left(\frac{1}{t''-t}-\frac{1}{t''-s}\right)ds\,, \tag{178}$$

where

$$\mathrm{P}\left(\frac{1}{t''-t}\right)=\frac{1}{2}\left(\frac{1}{t''-t+i\epsilon}+\frac{1}{t''-t-i\epsilon}\right), \quad \text{etc.} \tag{179}$$

instead of (176). To actually solve (175) with the kernel (178), the question concerning the type of propagators to be used to represent the *scattered* particles becomes once again of importance in connection with the valid "recipe" as to how to avoid the singularities at $s=u_+^2/tv'_+v'_-$ and $s=u_-^2/tv'_+v'_-$.[39] As explained in some detail at the end of Appendix B, this recipe becomes rather complicated in the case of the Feynman propagators, leading to the appearance of "step functions" (θ's) of rather complicated arguments if an attempt is made to perform the integration indicated in (178) explicitly, which in turn—as already mentioned—precludes obtaining a closed-form solution of (175) with such a kernel. We shall therefore include as part of the definition of our model that the retarded (or, equally well, the advanced) propagators shall be used to represent the scattered particles A and B instead of the Feynman propagators.[39] If this is done, and if we also assume as usual that $v'_+v'_->0$, then, if retarded (advanced) propagators are actually used, the singularity at $s=u_-^2/tv'_+v'_-$ should be bypassed always (i.e., independently of the value of the t parameter) *below* (above) and that at $s=u_+^2/tv'_+v'_-$ always *above* (below) the real axis (see Appendix B for detailed proof). Having from now on decided to consistently use only the retarded propagators to represent the scattered particles, we can therefore make the following identifications:

$$\frac{1}{tsv'_+v'_- -u_\pm^2}\equiv\frac{1}{tv'_+v'_-}\,\frac{1}{s-u_\pm^2/tv'_+v'_-\pm i\epsilon}\,. \tag{180}$$

Because of (180) the kernel K given by (178) now becomes

$$K(t'',t)=\frac{\pi i}{v'_+v'_-}\left[\frac{1}{t}\left(\frac{1}{t-t''-i\epsilon}+\frac{1}{t-t''+i\epsilon}\right)+\frac{1}{t}\left(\frac{1}{t''-u_+^2/tv'_+v'_-+i\epsilon}+\frac{1}{t''-u_-^2/tv'_+v'_- -i\epsilon}\right)\right], \tag{181}$$

so that instead of (175) [or rather what we now call (175M), where M stands for "model"] we can simply write

$$\frac{1}{(t''-1)^2}=\frac{(2\pi)^4}{2g^2}\frac{dG(t'')}{dt''}+\frac{\pi^2 i}{Ec}\int_{-\infty}^{+\infty}\frac{dt}{t}\left(\frac{1}{t-t''-i\epsilon}+\frac{1}{t-t''+i\epsilon}+\frac{1}{t''-u_+^2/tv_+'v_-'+i\epsilon}+\frac{1}{t''-u_-^2/tv_+'v_-'-i\epsilon}\right)G(t)\,. \tag{182}$$

Equation (182) can now be solved exactly, as follows: We start with the well-known integral representation for the inhomogeneous term of (182)

$$\frac{t''}{(t''-1)^2}=\frac{i}{2}\int_{-i\infty}^{+i\infty}\frac{l(-t'')^l}{\sin\pi l}\,dl\,, \tag{183}$$

where the integration can be carried out strictly along the imaginary l axis—i.e., without "detours"—since the point $l=0$ is regular. Correspondingly, the solution of (182) can be sought in the form

$$G(t)=\frac{i}{2}\int_{-i\infty}^{+i\infty}\frac{l}{\sin\pi l}\left[f(l)(-t)^l+h(l)\left(-\frac{1}{tv_+'v_-'}\right)^l\right]d_l\,. \tag{184}$$

This particular form of the integral representation is furthermore motivated by the fact that by the Cauchy theorem

$$\int_{-\infty}^{+\infty}\frac{1}{t}\left(\frac{1}{t-t''-i\epsilon}+\frac{1}{t-t''+i\epsilon}\right)(-t)^l\,dt=+\frac{2\pi i}{t''}(-t'')^l\,, \tag{185}$$

$$\int_{-\infty}^{+\infty}\frac{1}{t}\left(\frac{1}{t-t''-i\epsilon}+\frac{1}{t-t''+i\epsilon}\right)\left(-\frac{1}{t}\right)^l dt=-\frac{2\pi i}{t''}\left(\frac{-1}{t''}\right)^l\,, \tag{186}$$

$$\int_{-\infty}^{+\infty}\frac{dt}{t}\left(\frac{1}{t''-u_+^2/tv_+'v_-'+i\epsilon}+\frac{1}{t''-u_-^2/tv_+'v_-'-i\epsilon}\right)(-t)^l=-\frac{2\pi i}{t''}\left(-\frac{1}{t''v_+'v_-'}\right)^l\,, \tag{187}$$

$$\int_{-\infty}^{+\infty}\frac{dt}{t}\left(\frac{1}{t''-u_+^2/tv_+'v_-'+i\epsilon}+\frac{1}{t''-u_-^2/tv_+'v_-'-i\epsilon}\right)\left(-\frac{1}{t}\right)^l=+\frac{2\pi i}{t''}\left(-\frac{t''v_+'v_-'}{u_+^2}\right)^l\,, \tag{188}$$

where the proper branch of $(-x)^l$ is understood as that which for real x is given by

$$(-x)^l=\exp\{i[\ln|x|-i\pi\theta(+x)]\}\,. \tag{189}$$

The important point here is that (187) and (188) generate the $(-1/t)^l$ term from $(-t)^l$, and vice versa. This, combined with the fact that the operation $t''\,d/dt''$, like those of (185) and (186), simply reproduces the same type of term in both cases, leads to an easy identification of terms on both sides of the equation, when (184) is substituted into (182). Keeping in mind that for noninteger l the functions $(-1/t)^l$ and $(-t)^{-l}$ must be regarded as linearly independent, since

$$(-1/t)^l=(-t)^{-l}e^{2\pi i l\theta(t)} \tag{190}$$

leads to the following *two* relations:

$$1=\left[\frac{(2\pi)^4}{2g^2}l-\frac{2\pi^3}{Ec}\right]f(l)-\frac{2\pi^3}{Ec}(u_+)^{-2l}h(l) \tag{191}$$

and

$$0=\frac{2\pi^3}{Ec}(u_-)^{2l}f(l)-\left[\frac{(2\pi)^4}{2g^2}l-\frac{2\pi^3}{Ec}\right]h(l)\,. \tag{192}$$

Solving (191) and (192) for $f(l)$ and $h(l)$, substituting the result back into (184), and also using the relations (16), where the angles $\alpha_\pm$ and γ are once again those depicted graphically in Fig. 1, the exact solution of (182) finally becomes

$$G(t)=\frac{i}{2}\int_{-i\infty}^{+i\infty}\frac{l}{\sin\pi l}\frac{(Ml-N)(-t)^l+N(u_-)^{2l}(-1/tv_+'v_-')^l}{\{Ml-N[1+(u_-/u_+)^l]\}\{Ml-N[1-(u_-/u_+)^l]\}}\,dl\,, \tag{193}$$

where

$$M=\frac{(2\pi)^4}{2g^2}=\frac{8\pi^4}{g^2} \tag{194}$$

and

$$N=\frac{2\pi^3}{Ec}=\frac{2\pi^3}{m_Am_B\sin\gamma}\,. \tag{195}$$

Strictly speaking, it is at once evident that (193) cannot be the only solution of (182), since it is not symmetric in α_+ and α_- (or m_A and m_B), while the problem we undertook to solve obviously was. Another solution of (182) with the roles of α_+ and α_- exactly reversed can indeed be found by simply using another branch of $(-x)^l$ (x standing for t or $1/t$) in our ansatz (184), i.e., that which for real x is defined not by (189), but by

$$(-x)^l=\exp\{l[\ln|x|+i\pi\theta(x)]\}\,. \tag{196}$$

It appears therefore that the general solution of (182) is a linear combination of (193) and this other solution, whose coefficients C_1 and C_2 must satisfy the relation $C_1+C_2=1$. The circumstance that the solution of (182) is thus not uniquely determined (up to one arbitrary constant) need not concern us unduly, however, from either a strictly mathematical or physical point of view (i.e., quite apart from the fact that uniqueness is achieved by postulating the symmetry with respect to m_A and m_B in the solution, which does not seem too unreasonable *per se*). From the mathematical point of view, the above nonuniqueness has as its consequence the existence of a nontrivial solution of a homogeneous version of (182) for *all* values of the eigenvalue parameter (which is here the coupling constant g^2 rather than the total energy E). However, this is excused because the eigenvalue spectrum of an equation with the kernel (181) need not be discrete since the latter is most obviously a non-Fredholm kernel [in fact $|K^2|$ is proportional to the expression for the (divergent) self-energy of the composite particle of A and B]. The *physically* most pertinent properties of the solution, on the other hand, derive almost exclusively from the behavior of the bracketed expressions in the denominator of (193), which remain the same if the roles of the particles A and B are reversed. The energy eigenvalues of the problem correspond in fact to the zeros of these bracketed expressions for integer l, since, as already most strongly implied by the integral representations (183), (184), our method of solving (182) *is tantamount to performing the Regge analysis* of the scattering amplitude. This statement might seem a little surprising in view of the fact that we have many times emphasized the avoidance of a partial-wave decomposition. In order to show that it is indeed true, and, more importantly, to discuss more fully the method of obtaining the positions of bound states and/or resonances in our model, we shall therefore conclude this work by going into a little more detail on this slightly modified Regge analysis in t instead of in z. The crucial point in this connection is the relation between the cosine of the scattering angle z'' and t'', (36), which when solved for z'' reads

$$z''=1-\frac{2v_+v_-}{(v'_+-v'_-)\,(v_+-v_-)}\,t''-\frac{2v'_+v'_-}{(v'_+-v'_-)\,(v_+-v_-)}\,\frac{1}{t''},\tag{197}$$

where the coefficients multiplying t'' and $1/t''$ are constant for constant time and radial parameters p_0, $|\vec{p}|$, p_0', $|\vec{p}'|$ [compare (5); also recall the convention about primes, Sec. II B—III D]. As a consequence of (197) the Legendre (or for that matter any) polynomial $P_{l'}(z'')$, if expressed in terms of t'', will in general contain all positive as well as negative integer powers of t'' in the range $-l'$ to $+l'$ (though of course not inversely, i.e., a finite-power expansion in t'' is not always a polynomial in z''). From this it in turn follows that an energy resonance occurring in the lth term (i.e., for l equal to any positive *as well as negative* integer) of the power expansion of the scattering amplitude in t will necessitate an occurrence of a resonance in at least one partial-wave amplitude of order l' with $l'\geq|l|$. Equivalently, a resonance occurring in the lth (angular momentum) partial-wave amplitude must be caused by at least one "Regge pole in t" crossing the real l axis at a (positive or negative) integer value of l in the range $-l'$ to $+l'$. This proves our contention that our new "Regge analysis in t" is indeed essentially equivalent to the standard Regge analysis in the angular momentum, only that now both positive and negative integer values of l must be considered equally indicative of energy resonances. It is of importance also to notice another formal departure from a more orthodox Regge approach. This is the fact that the integral in (193) *as it stands*, i.e., obtained directly as a solution of (182), has to be taken along the imaginary l axis and is *not*—like the so-called background integral of the standard Regge analysis—obtained only secondarily by deforming the original integration path along the real positive l axis. Consequently a question arises as to which of the poles corresponding to zeros of the bracketed expressions in the denominator of (193) can in fact be regarded as "true" Regge poles, the latter defined as those capable of generating resonances when their trajectories cross the real l axis at integer (here positive *or* negative) values of l. To answer this question, it should of course first of all be borne in mind that our solution was obtained, as it actually should be, for the *scattering* region ($E\geq m_A+m_B$), so that the poles (in energy) of the scattering amplitude corresponding to bound state or resonances are expected to be found as a result of an *analytic continuation* of $G(t)$, given by (193), into the bound-state region ($m_A+m_B\geq E\geq|m_A-m_B|$). From this point of view it is then readily realized that the "true" Regge poles will be only those from among an *a priori* larger class of poles, which in the process of this analytic continuation will first migrate from the left half of the analytic l plane to the right (or vice versa) and *only then* cross the real l axis. If the latter crossing occurs at an integer value of l, a "true" Regge pole will thus pinch the integration path against a static pole stemming from $1/\sin\pi l$. Only by this process can a pole (in energy) of the scattering amplitude be generated. This pinching of course would not occur if the given pole were to proceed towards the real l axis "directly," e.g., its trajectory lying entirely in the same (left or right) analytic half plane of l [see Figs. 27(a) and 27(b)]. However, bearing in mind that there is more than one way in which an analytic continuation from the scattering into the bound-state region can be accomplished, we are at liberty to also adopt a slightly changed point of view, according to which *any* pole corresponding to a zero of the above-mentioned bracketed expressions can in fact be regarded as a "true" Regge pole with respect to at least one *suitably chosen* route of such an analytic continuation. This is certainly true, since we can obviously *prescribe* a trajectory of a given pole in the l plane—in particular to be that of the general type depicted in Fig. 27(a)—and *then* determine the corresponding trajectory in E. Given a sufficiently complicated relation between l and E, the latter might of course turn out to be quite involved, entailing, e.g., multiple encirclings of the thresholds at E equal to $-m_A-m_B$, $-m_A+m_B$, m_A-m_B, and m_A+m_B [compare the analytic structure of c given by Eq. (6)] and thus leading to resonances lying in much "higher" Riemann sheets [provided that such exist, i.e., that the analytic structure of $G(E,t)$ given by (193) is essentially more complicated than that of c given by Eq. (6)] of the scattering amplitude than the "physical" Riemann sheet[40] corresponding to the bound-state region. Although it might at the first sight seem that calling a trajectory a true Regge trajectory even though it might lead to resonances only in such higher Riemann sheets is tantamount to a *sui generis* sophistry, a little reflection suffices to realize that, given again a rather complicated relation between l and E, this approach is once again essentially not different from a more orthodox version of the Regge analysis where all trajectories are *a priori* "true" Regge trajectories (i.e., are not subject to the precondition that they first cross the imaginary axis in the l plane), and yet a situation may well arise where a given trajectory may not be able to cross the (positive) real l axis at all, unless we are prepared to venture into higher Riemann sheets in energy (i.e., encircle the corresponding thresholds more than once).

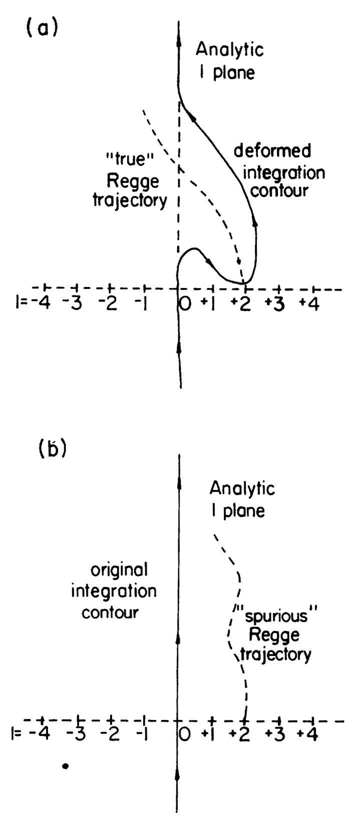

FIG. 27. An illustration of a slight difference between our Regge analysis and a more standard one: A Regge pole must first "migrate" across the imaginary l axis to produce a resonance.

Bearing the above in mind, we now see that according to (193)–(195) the Regge trajectories have, in our case, to obey the equations

$$l\sin\gamma-\left(\frac{g^2}{2\pi m_A m_B}\right)\tfrac{1}{2}\left(1+e^{il(\gamma-\pi)}\right)=0 \qquad (198)$$

or

$$l\sin\gamma-\left(\frac{g^2}{2\pi m_A m_B}\right)\tfrac{1}{2}\left(1-e^{il(\gamma-\pi)}\right)=0, \qquad (199)$$

where $\pi-\gamma\equiv\alpha_+ + \alpha_-$ and where $\alpha_\pm$ are related to the total energy E through (16) and (18) and thus indeed represent highly complicated relations between l and E. Determining a trajectory in E corresponding to even the simplest trajectory in l of the general type depicted in Fig. 27(a) in order to find out in which Riemann sheet a given resonance actually obtains seems therefore almost impossible without resorting to numerical computations, which we however forego since they are of necessity lengthy and generally out of tune with the predominantly analytic character of this work. In the hope that at least some of the resonances will nevertheless lie in the "physical" sheet, we shall therefore conclude this brief discussion of our exactly soluble model by at least trying to determine their *positions* for integer l as they follow from Eq. (198). Although even that seems to pose a difficult problem for finite $\pi-\gamma$, if we additionally assume

$$\frac{g^2}{2\pi m_A m_B}\ll 1 \qquad (200)$$

($g^2/2\pi m_A m_B$ is in fact the analog of the fine-structure constant α in electrodynamics, we can consistently keep l finite and yet assume $\pi - \gamma \approx \sin(\pi-\gamma) = \sin\gamma \ll 1$, replacing the exponential expression in (198) by 1. With help of Fig. 1 and Eq. (6) we then have

$$\{[E^2-(m_A-m_B)^2][(m_A+m_B)^2-E^2]\}^{1/2} = g^2/\pi l\,, \tag{201}$$

which finally reduces to the nonrelativistic Balmer formula

$$E - m_A - m_B = -\left(\frac{g^2}{2\pi m_A m_B}\right)^2 \frac{m_B}{2l^2} \tag{202}$$

if we additionally assume $m_A \gg m_B$ (smallness of the mass of the "electron" compared with that of the "proton"). Equation (202) furnishes an important link with the "nonrelativistic hydrogen atom," and in this connection it is of interest to again recall that many authors[14] found it rather difficult to establish a similar correspondence with the Wick-Cutkosky solutions, which in turn seems to bear out the already-made contention that our model may be closer to physical reality than these solutions, even as an approximation.

A little embarrassing may seem the fact that if we proceed to higher approximations where we can no longer put $e^{il(\gamma-\pi)} \approx 1$, the energy eigenvalues (even that of the ground state) become complex so that one may be inclined to think that the very appearance of $e^{il(\gamma-\pi)}$ is a rather artificial feature, brought about by the arbitrary changes of the types of propagators in our model, and may disappear if other changes are performed. It turns out, however, that it is not easy to do so.[41] On the other hand, the appearance of $e^{il(\gamma-\pi)}$ may not be all that unphysical on account of at least the following fact: One would expect that more precise energy-eigenvalue formulas than (201) or (202) would involve at least another quantum number in addition to l (i.e., something analogous to the Sommerfeld fine-structure formula in which our l will be cast more in the role of the principle quantum number, but where another quantum number j also appears), causing a degeneracy within a cluster of states. However, precisely that seems to be indicated by the "exact" equation (198), which for given l is an algebraic equation of the $(l+1)$th degree in the variable $e^{i\gamma}$ and therefore should in general exhibit as many solutions.

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APPENDIX A: PROPERTIES OF ξ AND $\xi^{(1)}$ DEFINED BY (59) AND (122),(123)

This appendix is devoted to a systematic presentation of the pertinent analytic properties of ξ defined by (59), as well as to the properties of a slightly differently normalized $\xi^{(1)}$, defined by (122) or (123), as functions of various sets of parameters introduced in Sec. II, with special emphasis on their dependence on the parameters w'' and ω'', defined by (19) and (20), respectively.

Equation (59) together with (57), (60), (61), and (62) defines ξ first in terms of the w and x variables, which, as mentioned earlier [Sec. II, Eqs. (38)–(44)], was done to preserve the invariance of (38) [and therefore also of (57)] with respect to permutations of primes and which constitutes probably the most economical (as far as the length of the mathematical expressions is concerned) way to define this very important quantity. The x parameters, however, are never used as integration variables. Also, the very important problem concerning the correct choice of proper analytic branches of ξ or $\xi^{(1)}$ [see especially the part of the main text which follows Eq. (121) of Sec. V] and therefore of the proper branches of $\sqrt{A}$, $\sqrt{Q}$, and $(B^2-4AC)^{1/2}$, is more conveniently discussed in terms of the ψ, φ, and τ parameters [due to the indeterminacy of signs of $(1-x^2)^{1/2}$, $(1-w^2)^{1/2}$, etc.], as best evidenced by the assignment (63) and (64), the latter constituting in fact an important part of the definition of ξ itself. Going over to ψ, φ, and τ and/or other parameters defined in Sec. II seems therefore mandatory. This must, however, proceed according to a well-defined plan, whereby maximum advantage is to be taken of a number of useful algebraic relations to be established, as one of the early experiences with ξ is that more or less haphazard substitutions can very well lead to mathematical expressions of considerable length, practically precluding the necessary interpretational clarity.

It appears that the best way to proceed is to start with the expression for B^2-4AC given by (62) and realize that both $1-x^2-x'^2-x''^2+2xx'x''$ and $1-w^2-w'^2-x''^2+2ww'x''$ are subject to further factorization, as follows [see (39)–(41)]:

$$\begin{aligned}1-x^2-x'^2-x''^2+2xx'x'' &= 1-\frac{1}{4st}(t+s)^2-\frac{1}{4sts''t''}(st+s''t'') \\ &\quad -\frac{1}{4s''t''}(t''+s'')+\frac{1}{4sts''t''}(t+s)(st+s''t'')(t''+s'') \\ &= -\frac{1}{4sts''t''}(t-t'')(s-s'')(t-s'')(s-t'')\end{aligned} \tag{A1}$$

and also

$$1-w^2-w'^2-x''^2+2ww'x''$$
$$=-[x''-ww'+(1-w^2)^{1/2}(1-w'^2)^{1/2}]\times[x''-ww'-(1-w^2)^{1/2}(1-w'^2)^{1/2}],$$

which because of (19) and (41) equals

$$-[\cos(\varphi-\varphi'+\tau'')-\cos(\psi+\psi')]\times[\cos(\varphi-\varphi'+\tau'')-\cos(\psi-\psi')],$$

and because of (32), (33), and (7) equals

$$-\frac{1}{4s''t''^3v_+'^2v_-'^2}(v_+-t''v_+')(v_--t''v_-')\times(v_+-t''v_-')(v_--t''v_+'). \quad \text{(A2)}$$

This seems to indicate that the choice of the parameters s, t, s'', t'', v_+, v_-, v_+', v_-', and w'' (the latter or ω'' must be retained as convenient integration variables) or s, t, s'', t'', w, w', and w'' (to retain "symmetry" between w's) has some definite advantages, and we shall therefore in fact try to present most of our analysis in terms of these variables. However, lengthy expressions in these parameters are sometimes still unavoidable, as best exemplified by the following.

Because of the importance of the variable w'', as well as of the quadratic expression Q, we would like to factorize the latter as follows:

$$Q=A(w''-\tilde{w}_+'')(w''-\tilde{w}_-''), \quad \text{(A3)}$$

where

$$\tilde{w}_\pm''=\frac{-B\pm(B^2-4AC)^{1/2}}{2A} \quad \text{(A4)}$$

and also write

$$\xi=\frac{2\sqrt{A}}{(B^2-4AC)^{1/2}}\times\left\{w''+\frac{B}{2A}\pm[(w''-\tilde{w}_+'')(w''-\tilde{w}_-'')]^{1/2}\right\}. \quad \text{(A5)}$$

Unlike the expressions for $\sqrt{A}$ and $(B^2-4AC)^{1/2}$, the expression for $B/2A$ which enters (A4) and (A5) is now indeed rather complicated and reads

$$\frac{B}{2A}=-\tfrac{1}{2}(\tilde{w}_+''-\tilde{w}_-'')$$
$$=\frac{s''t''}{(t''-s'')^2}\frac{1}{(stv_+'v_-')^{1/2}}\left\{\left[(s+t)-\frac{1}{2s''t''}(ts+t''s'')(s''+t'')\right](v_+'+v_-')+\frac{v_+'v_-'}{v_+v_-}[(ts+t''s'')-\tfrac{1}{2}(s+t)(s''+t'')](v_++v_-)\right\}, \quad \text{(A6)}$$

where, in contrast with (A1) and (A2), no further useful simplification seems to be possible. In discussing the pertinent analytic properties of ξ, it would therefore be desirable never to be forced to use the explicit algebraic form (A6) in any direct mathematical manipulation. To see that this can indeed be achieved, we should at this point be more specific and realize that analytic properties of ξ of interest to us (i.e., those required by the discussions of the main text), pertain only to the behavior of ξ at, or in the vicinity of

(a) points corresponding to the ends of the integration interval in z, i.e., corresponding to $z=\pm1$ [corresponding in turn to the solid and dashed lines of Fig. 4 in the case of the Wick equation and to the surfaces given by (106) and (107) in the case of the original BS equation],

(b) interfaces given by Eqs. (108) and (109) in the case of the original BS equation, and

(c) in conjunction with (b), points corresponding to $w''=\tilde{w}_\pm''$, where $\tilde{w}_\pm''$ are given by (A4), with special emphasis on the *relative signs* of the proper branches of $\sqrt{A}$, $\sqrt{Q}$, and $(B^2-4AC)^{1/2}$, since the latter determine whether the upper or the lower semicircle should be traversed by the point representing $\xi^{(1)}$ in Figs. 16, 17, 25, and 26, as w'' varies between $\tilde{w}_-''$ and $\tilde{w}_+''$.

A more detailed analysis of (a) now follows.

(a) *The values of* ξ *at* $z=\pm1$. It is first of all to be realized that, independently of the type of propagator used for the exchange particle, the values of $(1-z^2-y'^2-z''^2+2zy'z'')^{1/2}$ for $z=\pm1$ are $i(y'\mp z'')$, respectively (emphasis on signs by virtue of the proper determination of the analytic branch of this square-root expression), since the left-hand side of (54) is then simply

$$\int_0^{2\pi}d\chi'\frac{1}{\pm z''-y'}=\frac{2\pi}{\pm z''-y'} \quad \text{(A7)}$$

(compare notations used in Fig. 2). Consequently, according to (63) we have for $z=\pm1$

$$\sqrt{Q}=i\sin\psi\sin\psi'\sin\psi''(y'-\epsilon_1z''), \quad \text{(A8)}$$

where we have introduced the abbreviation

$$\epsilon_1=\begin{cases}+1 & \text{for } z=+1,\\ -1 & \text{for } z=-1.\end{cases} \quad \text{(A9)}$$

To compute the values of w'' corresponding to $z=\epsilon_1$ we first realize that, according to (36) and (34), we have either

$$v''_+ = tv'_- \,;\; v''_- = sv'_+ \tag{A10}$$

or

$$v''_+ = sv'_- \,;\; v''_- = tv'_+ \tag{A11}$$

for $z=-1$ and, correspondingly, either

$$v''_+ = tv'_+ \,;\; v''_- = sv'_- \tag{A12}$$

or

$$v''_+ = sv'_+ \,;\; v''_- = tv'_- \tag{A13}$$

for $z=+1$, which, using the abbreviation (A9), can be written summarily as either

$$v''_+ = tv'_{\epsilon_1};\; v''_- = sv'_{-\epsilon_1} \tag{A14}$$

or

$$v''_+ = sv'_{\epsilon_1};\; v''_- = tv'_{-\epsilon_1}, \tag{A15}$$

respectively. The fact that to each value ϵ_1 there correspond two different points in the (v''_+, v''_-) plane calls for an introduction of another ϵ symbol,

$$\epsilon_2 = \begin{cases} +1 \text{ if (A14) obtains ,} \\ -1 \text{ if (A15) obtains ,} \end{cases} \tag{A16}$$

so that, using both ϵ_1 and ϵ_2, we can write because of (7), (32), (33), and (30)

$$\psi'' = \epsilon_1\psi' - \epsilon_2(\varphi'' - \varphi + \tau) \quad \text{for } z=\epsilon_1 \tag{A17}$$

or, because of (19) and (39),

$$w''_{\epsilon_1} = w'x + \epsilon_1\epsilon_2(1-w'^2)(1-x^2)^{1/2} \tag{A18}$$

and

$$(1-w''^2_{\epsilon_1})^{1/2} = \epsilon_1 x(1-w'^2)^{1/2} - \epsilon_2 w'(1-x^2)^{1/2}, \tag{A19}$$

where $w''_\pm$ are from now on understood as values of w'' corresponding to ± 1, respectively, and where $(1-x^2)^{1/2}$ and $(1-w^2)^{1/2}$, etc. are understood as equal to $\sin(\varphi''-\varphi'+\tau)$ and $\sin\psi$, etc., respectively (emphasis is on signs of the proper analytic branches again). In (A18) and (A19) we have temporarily reverted to the x and w variables in order to facilitate the computation of the expression

$$\sqrt{A}\left(w'' + \frac{B}{2A}\right) + \sqrt{Q}\ , \tag{A20}$$

[compare (59)] for $z=\pm 1$, without invoking the complicated formula (A6). Because of (64), (A18), (60), (61), and (41), we now have

$$\sqrt{A}\left(w''_{\epsilon_1} + \frac{B}{2A}\right) = \frac{i}{(1-x''^2)^{1/2}}\{[w'x+\epsilon_1\epsilon_2(1-w'^2)^{1/2}(1-x^2)^{1/2}](1-x''^2) - (x-x'x'')w' - (x'-x''x)w\}\ . \tag{A21}$$

Similarly, because of (43) and (44) the right-hand side of (A8) becomes

$$i[-\cos\psi\sin(\psi'-\epsilon_1\psi'') + x'\sin\psi' - \epsilon_1 x''\sin\psi'']\ , \tag{A22}$$

which, because of (A17), (A18), and (39) (i.e., identifying ψ'' with $\tilde{\psi}''_\pm$) and finally likewise reverting to the x and w parameters, can be rewritten as

$$i[(x'-x''x)(1-w'^2)^{1/2} - \epsilon_1\epsilon_2(w-w'x'')(1-x^2)^{1/2}]. \tag{A23}$$

Now, probably the most important purely algebraical finding, which allows many computations of the main text to be performed explicitly, is the fact that the sum of (A21) and (A23), which represents the expression (A20) for $z=\pm 1$, can be factorized to give

$$[\sqrt{A}(w''+B/2A)+\sqrt{Q}\,]_{z=\epsilon_1} = -\frac{i}{(1-x''^2)^{1/2}}[w-w'x''-(1-w'^2)^{1/2}(1-x''^2)^{1/2}] \times[x'-xx''+\epsilon_1\epsilon_2(1-x^2)^{1/2}(1-x''^2)^{1/2}]\ . \tag{A24}$$

In a way similar to (A2) we furthermore find that, because of (41), (7), and (33),

$$w-w'x''-(1-w'^2)^{1/2}(1-x''^2)^{1/2} = \cos\psi - \cos(\psi'-\varphi+\varphi'-\tau'') = \frac{-1}{2t''v'_+(v_+v_-)^{1/2}}(v_+-t''v'_+)(v_--t''v'_+) \tag{A25}$$

and also, by (39), (40), and (41),

$$x'-xx''+(1-x^2)^{1/2}(1-x''^2)^{1/2} = \frac{1}{2(sts''t'')^{1/2}}(t-s'')(s-t'')\ , \tag{A26}$$

while

$$x'-xx''-(1-x^2)^{1/2}(1-x''^2)^{1/2} = \frac{1}{2(sts''t'')^{1/2}}(t-t'')(s-s'')\ , \tag{A27}$$

where we have again returned to the v, t, and s parameters. Combining the last results with the formula for

$$B^2-4AC = \frac{1}{4sts''^2t''^4v'^2_+v'^2_-}(t-t'') \times(s-s'')(t-s'')(s-t'')\delta_+\delta_-\ , \tag{A28}$$

stemming from (26), (A1), and (A2) and where we have also used the abbreviations $\delta_\pm$ given by (127), we finally have

$$(\xi)_{z=\epsilon_1}=\frac{2}{s''-t''}\left(s''t''\frac{v_-'\delta_+}{v_+'\delta_-}\right)^{1/2}\times\left[\frac{(t-s'')(s-t'')}{(t-t'')(s-s'')}\right]^{1/2}\text{ for }\epsilon_1\epsilon_2=1 \qquad (A29)$$

and

$$(\xi)_{z=\epsilon_1}=\frac{2}{s''-t''}\left(s''t''\frac{v_-'\delta_+}{v_+'\delta_-}\right)^{1/2}\times\left[\frac{(t-t'')(s-s'')}{(t-s'')(s-t'')}\right]^{1/2}\text{ for }\epsilon_1\epsilon_2=-1\ , \qquad (A30)$$

so that, denoting the expressions on the right-hand sides of (A30) and (A29) by ξ_+ and ξ_-, respectively (i.e., identifying $\pm$ with $-\epsilon_1\epsilon_2$), we have also derived Eq. (90).

In this connection it should be noted for the sake of general reference that although the ϵ symbols have been avoided in the main text, we have also the following correspondences:

(i) $\epsilon_1\epsilon_2=1$ corresponds to curves $\alpha=\rho''/\rho'$ and $\alpha=1/\rho'\rho''$ of Fig. 4 (Wick-rotated BS equation) and to lines in all (v_+'', v_-'') diagrams of Sec. V (the original BS equation), where either v_+'' or v_-'' is set equal to tv_+', whereas

(ii) $\epsilon_1\epsilon_2=-1$ corresponds to curves $\alpha=\rho'/\rho''$ and $\alpha=\rho'\rho''$ of Fig. 4 and to lines in all (v_+'', v_-'') diagrams where either v_+'' or v_-'' is set equal to tv_-'.

Equations (A29) and (A30) contain in principle the full answer to (a) above. Before discussing (b) and (c), however, we are at this point in the best position to provide a rationale motivating the introduction of $\xi^{(1)}$ defined by (122) or (123). We want to avoid square-root expressions like those present in (A29) and (A30), at least at the ends of the z interval, and also we want to make the values of $\xi^{(1)}$ *real* there.[42] For arbitrary w'' it then follows that

$$\xi^{(1)}(w'')=\frac{s''-t''}{2}\left(\frac{v_+'\delta_-}{v_-'\delta_+s''t''}\right)^{1/2}\times[(t-t'')(t-s'')(s-t'')(s-s'')]^{1/2}\xi(w'')\ . \qquad (A31)$$

Concurrently with introducing $\xi^{(1)}$ it is convenient to go over to the variable ω'' defined by (20), in preference to w'' [as defined by (19)]. The motivation for doing so derives from a principle of maximal rationality of the algebraic expressions involved, similar to that which prompted us to change the normalization of ξ, but is this time primarily aimed at the maximum avoidance of expressions like $[v_+''v_-'']^{1/2}$, $(st)^{1/2}$, etc., which even in the case of the original BS equation could become imaginary. Using (A31) in conjunction with (A28), we thus arrive at Eqs. (124)–(126), where, needless to say, $\bar\omega_\pm''$ correspond to $\bar w_\pm''$ defined by (A4), while $\omega_\pm''$, defined by (119) and/or (120) correspond to $w_\pm''$ defined by (A18). As a particularly important consequence of the latter change of variables, note finally that the right-hand side of (125)—which derives from (A6)—while still remaining a rather lengthy expression in the t, s, and ω parameters, is now at least *rational* and therefore *real* in the case of the original BS equation. This in turn implies that, since ω'' is always real in that case, $\xi^{(1)}$ is real for $w''=\bar w_\pm''$, which enormously facilitates the discussions in terms of Figs. 16, 17, 25, and 26.

This brings us in a natural way to a brief discussion of problems posed by (b) and (c) above which are best considered jointly.

(b), (c) *Behavior of* $\xi^{(1)}$ *for* $\bar\omega_+''\geq\omega''\geq\bar\omega_-''$, *in particular at* $\omega''=1+stv_+'v_-'$ *(interface)*. A detailed investigation of the behavior of the phase P defined by (144) as a function of the t and s parameters, on which the presently considered properties of $\xi^{(1)}$ have a particular bearing, is actually given in Sec. V and need not be repeated here. However, for the sake of reference, here is a brief summary of the "recipe" of how to construct P for given values of t and s, as well as that of the main results of this analysis:

(i) The phase P is determined jointly by the over-all direction of progress (from left to right or vice versa) of the point representing $\xi^{(1)}(\omega'')$ between the (always real) end points $(t-s'')(s-t'')$ and $(t-t'')(s-s'')$, together with the information of whether the lower or the upper semicircle (compare again Figs. 16, 17, 25, and 26) has to be traversed, should the passage involve regions where $[(\omega''-\bar\omega_+'')(\omega''-\bar\omega_-'')]^{1/2}$ is imaginary.

(ii) The direction of progress depends on the signs of $(t-s'')(s-t'')$ and $(t-t'')(s-s'')$ and on the proper determination of ϵ_2 defined by (A16) [i.e., on whether (A14) or (A15) obtains in the actual—or "physical"—domain of integration, viz., subdivision of the domain D_2 into V_i's discussed at the beginning of Sec. V], while

(iii) whether the upper or the lower semicircle has to be traversed depends on whether the expression

$$-\frac{(t''-s'')^2t''v_-'}{2\delta_-}[(\omega''-\bar\omega_+'')(\omega''-\bar\omega_-'')]^{1/2} \qquad (A32)$$

[compare (124)] for $\min(\bar\omega_\pm'')<\omega''<\max(\bar\omega_\pm'')$ is positive imaginary or negative imaginary, respectively.

(iv) The proper branch of $[(\omega''-\bar\omega_+'')(\omega''-\bar\omega_-'')]^{1/2}$

in (A32) is in turn determined by invoking (128) [which is a direct consequence of (63) expressed in different parameters] and thus finally links the problem of determining P to the particular type of the propagator used for the *exchange* particle, the latter determining the proper branch of $(1-z^2-y'^2-z''^2+2zy'z'')^{1/2}$.

(v) Closer analysis along these lines reveals that for certain type of propagators—notably Feynman—an important mathematical "phenomenon" takes place to the effect that whenever interface crossing (see Sec. V for its definition) occurs within the interval $\min(\tilde{\omega}''_\pm)<\omega''<\max(\tilde{\omega}''_\pm)$, this is accompanied by a discontinuous flip of sign of (A32), giving rise to the appearance of the shaded areas of Fig. 18, where the phase P becomes ω-dependent.

To conclude this appendix, it bears mentioning that in the case of the *Wick rotated BS equation*, the variable w'', as defined by (19) [though no longer ω'' defined by (20)] remains real because $i\psi''$ is real [compare (76)]. On the other hand, we have for the quantities defined by (A4)

$$\tilde{\omega}''_+=\tilde{\omega}''^*_-, \qquad \text{(A33)}$$

since A and B remain real, while

$$B^2-4AC\leq 0\ , \qquad \text{(A34)}$$

the latter following from the fact that the expressions (A1) and (A2) and entering (62) remain upon closer analysis always real and <0 and >0, respectively. Although w'' itself is not used as an integration variable in the case of the Wick-rotated BS equation, we must therefore conclude that Q defined by (57) remains in this instance always *positive-definite* (or at most equal to zero) and therefore—no matter what integration variables are subsequently used—the integration domains never contain the area of the troublesome branch cut defined by $[(\omega''-\tilde{\omega}''_+)(\omega''-\tilde{\omega}''_-)]^{1/2}$, which of course is yet another way of restating that the difficulties connected with (b) and (c) above are entirely absent in this case.

APPENDIX B: PROPAGATORS OF THE SCATTERED PARTICLES

The problem considered here is that of finding the positions of the integration contours in $v''_\pm$ in relation to the positions of the singularities of $[(v''_+v''_--u_+^2)(v''_+v''_--u_-^2)]^{-1}$ (or in other words to learn a set of rules of how to bypass these singularities while performing the integrations with respect to $v''_\pm$) in integrals of the general type

$$\int\int\frac{dv''_+\,dv''_-}{(v''_+v''_--u_+^2)(v''_+v''_--u_-^2)}(\cdots) \qquad \text{(B1)}$$

[compare (10) and (15)], consistent with a particular type of propagator (Feynman, retarded, advanced, etc.) used to represent the *scattered* particles A and B. So formulated, the problem should be considered as complementary to, yet distinct from, that considered at great length in Sec. V, at the beginning of Sec. VI, and in Appendix A [especially parts (b) and (c)], where our main concern was to perform integrations consistently with a particular type of the propagator used for the *exchange* particle. The findings of this appendix are of practical importance in Sec. VI, since they enable us to define uniquely the integration contours in t and s in integrals of the general type

$$\int\int\frac{dt\,ds}{(v'_+v'_-ts-u_+^2)(v'_+v'_-ts-u_-^2)}(\cdots)\ , \qquad \text{(B2)}$$

where the integration (121) is presumed already performed and therefore $v''_\pm$ are related to t and s [i.e., at the ends of the integration interval in (121) for $\omega''=\omega''_\pm$] by (A14) and (A15).

As is well known, using Feynman propagators is, e.g., tantamount to assigning small negative imaginary parts to m_A and m_B in expressions like (48) and then performing the p''_0 integration strictly along the real axis. Likewise, it can easily be shown that using the retarded (advanced) propagators to represent the particles A and B is equivalent to keeping m_A and m_B this time strictly real, but assigning small positive (negative) imaginary parts to both E_+ and E_- defined by (14) and (4) [compare (2)], which according to these two relations is in turn simply equivalent to assigning a (single) small positive (negative) imaginary part to the over-all energy E. The effect of retarded (advanced) propagators is then achieved by again integrating with respect to p''_0 strictly along the real axis.

Now, according to the definition of $v''_\pm$ given by (5), which we want to strictly retain [i.e., *never* to consider it only valid in some limit $i\epsilon\to 0$; the same applies to the definition of $u_\pm^2$ given by (16)], we can easily see that in all the three cases (i.e., for Feynman, retarded, or advanced propagators) the trajectories described by (or geometrical loci of) $v''_\pm$ as $p''_0\pm|\vec{p}''|$ vary from $-\infty$ to $+\infty$ through real values are *circles*, since ic, with c given by (6), is no longer strictly real. To be more precise, both v''_+ and v''_- describe in fact *the same* circle, but remain *independent*, since $p''_0\pm|\vec{p}''|$ are independent real variables. As we do not propose to consider propagators other than the above three types, this is the most general situation we are going to deal with in this appendix.

At this point much clarity is gained by consistently using an exaggerated picture of the above common geometrical locus of both v_+'' and v_-'' as that of a circle of *finite* size (and always containing the point zero in its interior), even though it degenerates of course into the real axis in the limiting case of real $d = ic$. This is done to emphasize the very important fact of "*two-dimensionality*" of the resulting geometrical locus of the *product* $v_+'' v_-''$, since the problem under investigation is now easily seen to be essentially that of finding the positions of the points representing $u_\pm^2$ in relation to the latter two-dimensional domain; the situation becomes especially involved if either (or both) of the points $u_\pm^2$ are found to lie *inside* this domain (which *does* actually happen in the case of *Feynman* propagators; see below).

The problem is now best discussed not in terms of the geometrical locus of points representing $v_+'' v_-''$ itself, but rather of its square-root mapping, or, in other words, in terms of the geometrical locus of points representing $(v_+'' v_-'')^{1/2}$, since the shape of the latter is especially simple and is given by the following theorem.[43]

Theorem. Let the geometrical locus of points (in the analytic plane) representing each of the two independent complex variables v_+'' and v_-'' be (the circumference of) a circle C, containing the point zero in its interior (Fig. 28). The geometrical locus of points representing $Z = (v_+'' v_-'')^{1/2}$ is then the joint area of the two crescent-shaped figures (Fig. 29), consisting of points belonging to the interior of one and only one of the following: (i) the circle C itself, (ii) the circle C' obtained from C by reflection in the origin.

The problem of locating the points $u_\pm^2$ in relation to the geometrical locus of $Z' = v_+'' v_-''$ is thus finally seen as equivalent to that of finding the positions of *four* points, $-u_-$, $+u_-$, $-u_+$, and $+u_+$ (this accidentally provides one of the motivations to define the left-hand side of (16) as $u_\pm^2$ and not $u_\pm$) in relation to the above-mentioned crescent-shaped areas of Fig. 29.

Now, to determine the position of an arbitrary point D in relation to the latter domain it simply

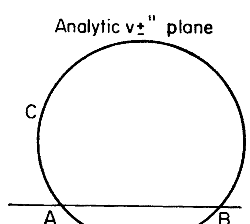

FIG. 28. An exaggerated picture of the geometrical locus of points representing $v_\pm''$, if d given by (18) has a small imaginary part.

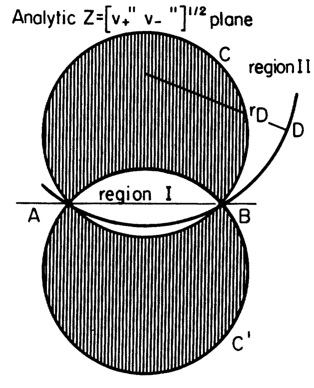

FIG. 29. Geometrical locus of points representing $Z = (v_+'' v_-'')^{1/2}$ in its analytic plane (shaded area), as v_+'' and v_-'' describe (independently) the circle C of Fig. 28.

suffices to compare the magnitudes of the radii of the following two circles: (1) C itself and (2) the circle through the points of intersection A, B, between C and C' and the given point D (see Fig. 29). Denoting the former by r and the latter by r_D, we have either

$$r_D > r\,, \tag{B3}$$

in which case the point D lies *outside* the geometrical locus of $(v_+'' v_-'')^{1/2}$, i.e., in one of the two regions I and II depicted in Fig. 29, or

$$r_D < r\,, \tag{B4}$$

in which case it must lie *inside* this geometrical locus. To apply this criterion to the points representing u_+, u_-, $-u_+$, and $-u_-$, we first see that, according to Eqs. (5), (6), and (18), the circle C intersects the real axis and therefore also the circle C' at the points -1 and $+1$, respectively, so that

$$Z_A = Z_B = 1\,. \tag{B5}$$

We see also that its radius is given by

$$r = \frac{|d|}{|\mathrm{Im}(d)|} \tag{B6}$$

[with d given by (18) and c by (6)]. The relations (B5) and (B6) apply to all three types of propagators considered here. Furthermore, because of (B5) the radius of a circle through A, B, and D is given by

$$r_D = \frac{1}{2|Y_D|}\,|Z_D^2 - 1|\,, \tag{B7}$$

where Z_D is a complex number representing the point D, and X_D and Y_D are its real and imaginary parts, respectively. Identifying Z_D^2 with $u_\pm^2$ given by (16) and also using (14) and (17) we therefore have for the appropriate radii through the points

$u_\pm$ (or $-u_\pm$)

$$r_\pm = \frac{1}{|\mathrm{Im}[m_\pm/(E_\pm \mp d)]|}\left|\frac{d}{E_\pm \mp d}\right| . \qquad (B8)$$

Thus the criterion (B3), (B4) applied to the points u_+ and u_- (or to the points $-u_+$ and $-u_-$, the situation being now entirely symmetric with respect to $Z \to -Z$) finally reads

$$\left|\mathrm{Im}\left(\frac{m_\pm}{E_\pm \mp d}\right)\right| |E_\pm \mp d| \lessgtr |\mathrm{Im}(d)|, \qquad (B9)$$

which, employing the useful relation

$$m_\pm^2 = E_\pm^2 - d^2 \qquad (B10)$$

[equivalent to (14) because of (17) and (18)], can also be rewritten as

$$|\mathrm{Im}(m_\pm)\mathrm{Re}(E_\pm \mp d) - \mathrm{Re}(m_\pm)\mathrm{Im}(E_\pm \mp d)| \gtrless |\mathrm{Im}(d)|\,|E_\pm \mp d| . \qquad (B11)$$

With help of (B11) we can now examine the cases of Feynman and retarded (advanced) propagators separately—beginning with the latter as the simpler of the two—as follows:

1. Retarded (advanced) propagators

As mentioned before, this case is characterized by setting

$$\mathrm{Im}(m_\pm) = 0 , \qquad (B12)$$

which combined with (B10) gives

$$\mathrm{Im}(E_\pm) = \frac{\mathrm{Re}(d)\,\mathrm{Im}(d)}{\mathrm{Re}(E_\pm)} , \qquad (B13)$$

so that, considering $\mathrm{Re}(E_\pm)$ and $\mathrm{Re}(d)$ as given, $\mathrm{Im}(E_\pm)$ are determined jointly by a single (small) $\mathrm{Im}(d)$. Substituting the above in (B11), dividing by $|\mathrm{Im}(d)|$, and multiplying by $|\mathrm{Re}(E_\pm)|$ we get

$$|\mathrm{Re}(m_\pm)\mathrm{Re}(E_\pm \mp d) \lessgtr |\mathrm{Re}(E_\pm)|\,|E_\pm \mp d| , \qquad (B14)$$

with the *upper* inequality sign obviously applying, since for sufficiently small $\mathrm{Im}(d)$ (B14) must become

$$|m_\pm| < |E_\pm| \qquad (B15)$$

[compare (B10) with real finite d in the bound-state region]. Consequently, all the four points u_+, u_-, $-u_+$, and $-u_-$ lie *outside* the geometrical locus of $(v_+'' v_-'')^{1/2}$, with the additional very important circumstance that, while u_- and $-u_-$ lie in the "internal" region I, the points u_+ and $-u_+$ must lie in the "external" region II (see Fig. 29), since

$$|u_-^2| < 1 \qquad (B16)$$

and

$$|u_+^2| > 1 \qquad (B17)$$

[i.e., for positive d; compare (16), (18) of the main text]. In terms of integrals of the general type (B1), these findings imply that the integration contour in v_+'' (v_-'') will always encircle the point u_-^2/v_-'' (u_-^2/v_+''), while the point u_+^2/v_-'' (u_+^2/v_+'') will always lie outside this integration contour, irrespective of the value of the other integration parameter v_-'' (v_+''). In the limiting case of real d, i.e., when this integration contour "straightens up" and becomes coincidental with the real axis, this in turn implies that for retarded propagators the point u_-^2/v_-'' (u_-^2/v_+'') should be considered as lying always slightly *below* the real axis and the point u_+^2/v_-'' (u_+^2/v_+'') always slightly *above* this axis, with the situation exactly reversed for the advanced propagators. (The center of C has to be considered as lying above and below the real axis for the two respective cases.) Making the identifications (A14) or (A15), this finally leads to the Eq. (180) of the main text.

2. Feynman propagators

This case is characterized by

$$\mathrm{Im}(E_\pm) = 0 , \qquad (B18)$$

but, according to (B10),

$$\mathrm{Im}(m_\pm) = -\frac{\mathrm{Re}(d)}{\mathrm{Re}(m_\pm)}\mathrm{Im}(d) , \qquad (B19)$$

so that again it suffices to deal with a single small $\mathrm{Im}(d)$ in order to assign proper signs to $\mathrm{Im}(m_+)$ and $\mathrm{Im}(m_-)$. In addition to (B19) which stems from equating the imaginary parts of (B10), we need this time also information stemming from the real part of (B10) which reads

$$[\mathrm{Re}(m_\pm)]^2 - [\mathrm{Im}(m_\pm)]^2 = [\mathrm{Re}(E_\pm)]^2 - [\mathrm{Re}(d)]^2 + [\mathrm{Im}(d)]^2 . \qquad (B20)$$

Substituting (B18) and (B19) in (B11) we have

$$\left|\frac{\mathrm{Re}(d)}{\mathrm{Re}(m_\pm)}\mathrm{Re}(E_\pm \mp d) \pm \mathrm{Re}(m_\pm)\right| \gtrless |E_\pm \mp d| , \qquad (B21)$$

which after multiplying by $\mathrm{Re}(m_\pm)$ and using (B20) becomes

$$|\mathrm{Re}(E_\pm)\mathrm{Re}(E_\pm \mp d) + [\mathrm{Im}(d)]^2 + [\mathrm{Im}(m_\pm)]^2| \gtrless |E_\pm \mp d|\,|\mathrm{Re}(m_\pm)| . \qquad (B22)$$

Because inequality (B15) must again obtain for vanishing $\mathrm{Im}(d)$, the *lower* inequality sign must this time apply in (B21), so that, in the case of Feynman propagators, all the points u_+, u_-, $-u_+$, and $-u_-$ lie *inside* the geometrical locus of $(v_+'' v_-'')^{1/2}$. Consequently, depending on the value of v''_- (v''_+), some integration contours with respect to v''_+ (v''_-) in integrals of the general type (B1) *will* contain in their interiors the point $u_+{}^2/v''_-$ or $u_-{}^2/v''_-$ $(u_+{}^2/v''_+$ or $u''_-/v''_+)$, while others will *not*. Closer (and rather cumbersome) analysis reveals that in fact, as the "other" integration variable v''_- (v''_+) itself describes the circle C, the point $u_\pm{}^2/v''_-$ $(u_\pm{}^2/v''_+)$ remains inside the integration contour in v''_+ (v''_-) between the points 1 and $u_\pm^{\mp 2}$ [according to (16) it so happens that the points $u_\pm^{\mp 2}$ themselves lie on the circle C if—as assumed—$E_\pm$ are real] and outside elsewhere. In the limit $\mathrm{Im}(d) \to 0$ and making again the identification (A14) or (A15), the kernel K defined by (178) will therefore no longer be given by a relatively simple formula (181), but will contain step functions $\theta(tv'_\pm - u_-{}^2)$ and $\theta(t\,v'_\pm - 1/u_+{}^2)$, as well as $\theta(t\,v'_+ - 1)$ and $\theta(tv'_- - 1)$.

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[1]G. C. Wick, Phys. Rev. 96, 1124 (1954).

[2]R. E. Cutkosky, Phys. Rev. 96, 1135 (1954).

[3]See, e.g., J. Schwinger, J. Math. Phys. 5, 1606 (1964) for perhaps the best and most concise presentation on the subject.

[4]K. Nishijima, Prog. Theor. Phys. 14, 203 (1955).

[5]S. Okubo and D. Feldman, Phys. Rev. 117, 279 (1960).

[6]Reference is here made specifically to Eq. (3.16) of Ref. 4 and Eq. (19a) of Ref. 5. Equation (3.16) of Nishijima actually separates, but only in the case of the final four-momenta on the mass shell, i.e., when Eq. (3.17) is satisfied.

[7]I-Ming Tang, Ph.D. thesis, Univ. of Cincinnati, 1969 (unpublished).

[8]Tang's results (Ref. 7) are rederived by a different method in Sec. IV. The original method need not be reproduced here, partly because it makes use of the partial-wave expansion, totally avoided in the present work, and partly because the new method derives (essentially) the Wick-Cutkosky equations *as a result of* and therefore *concurrently with* Tang's boundary-value problem rather than vice versa.

[9]H. S. Green, Nuovo Cimento 5, 866 (1957); S. N. Biswas, *ibid.* 7, 577 (1958). See also H. S. Green and S. N. Biswas, Phys. Rev. 171, 1511 (1968).

[10]It seems almost needless to add that precisely the same difficulty precludes us from trying—in terms of the Green-Biswas or related parameters—to follow either the argument of Wick himself (Ref. 1) or that of N. Kemmer and A. Salam [Proc. R. Soc. A230, 266 (1955)] in their well-known attempts to prove the validity of the Wick rotation.

[11]Corroborating evidence to very much the same effect seems to be provided by C. Fronsdal and Y. C. Yang [IAEA Trieste Report No. IC/68/70, 1968 (unpublished)], who attempt to show that the "accidental O(4) symmetry" (similar to that of the H atom) of the Wick-rotated BS equation and constituting the starting point of the Cutkosky method is absent in the original equation.

[12]The Feynman propagators for the *scattered* particles A and B are likewise replaced by advanced or retarded propagators. Though also needed for the exact solubility *and* changing the mathematical structure of the BS equation, this step seems to have a much less profound effect on the analytic structure of the solution [an integro-differential equation of the type (175), i.e., in *one* variable, still obtains]. Besides it could well be argued (see Ref. 39) that, for physical reasons, the retarded propagators actually *should* be used in preference to other types as far as scattered particles are concerned.

[13]Compare C. Fronsdal and L. E. Lundberg, Phys. Rev. D 1, 3247 (1970), where similar results were obtained based on the same assumption. The exactly soluble model discussed by these authors differs, however, from that of Sec. VI in that at least one of the propagators representing *scattered* particles is replaced by—in their notation—$\delta(p^2 - m_1{}^2)\epsilon(p_0)$, which according e.g. to the classification of Jauch and Rohrlich (Ref. 38) is a combination of "closed integration path" Δ_+ and Δ_- propagators, instead of the retarded or advanced propagators used in our model. Compare Eq. (2.5) with Eq. (7.1) of Ref. 13 and see our footnote 39.

[14]Compare F. L. Scarf, Phys. Rev. 100, 912 (1955); D. A. Geffen and F. L. Scarf, *ibid.* 101, 1829 (1956); R. E. Cutkosky and G. C. Wick, *ibid.* 101, 1830 (1956).

[15]Compare e.g., Biswas, Ref. 9, Eqs. (3.2), (3.3).

[16]In the course of the necessary changes of variables, the essence of this problem is finally discussed in terms of a two-dimensional (t, s) space (see below), where the crucial test is whether or not the operator $\partial^2/\partial t\,\partial s$ produces a two-dimensional δ function when applied to the phase defined by (144). Compare also Eqs. (156) and (167).

[17]Note the departure from the convention about cyclic permutations of primes for quantities defined in Sec. II C. If this convention were followed here, we would have instead of (29)

$$z'' = \frac{-\cos\psi'\cos\psi + \cos(\varphi' - \varphi + \tau'')}{\sin\psi\,\sin\psi'} .$$

[18]Together with the bifocal coordinates of Green and Biswas, these variables constitute the most important tools of our new method, but they are introduced here—to the best of the present author's knowledge—for the first time. However, see footnote 26.

[19]The question as to whether or not the residual dependence on the ω parameters can then be also made to disappear from our equations in in fact the most crucial in discussing the original BS equation *vis à vis* the Wick-rotation equation.

[20]No such "operator" could be properly defined, of course, only expressions like Eq. (46) as a whole having a well-defined meaning.

[21]No "proof" of the validity of the Wick rotation is of course required here, since the very purpose of the present work is in part, to examine this point *a posteriori* by comparing the analytical structures of Eqs. (69 and (70) (see below). See also footnote 10.

[22]The notation χ'' is preferred instead of the usual φ, to avoid confusion with the Green-Biswas parameter φ.

[23]Frequent change and simultaneous use of different notations seems unavoidable, not only in order to preserve the conciseness of the formulas, but also to maintain maximal clarity as to the relevant analytic properties of the mathematical expressions momentarily under discussion. In addition, it should be borne in mind that parameters best suited in a particular instance to emphasize certain analytic aspects may at the same time not represent the "best" integration variables, and vice versa.

[24]The dependence on the final-state parameters, e.g., φ' and ψ', is of course suppressed, since they, as well as the total energy E, are constant parameters of the problem.

[25]We prefer here not to invoke shorthands such as

$$\left(\frac{\partial^2}{\partial\alpha_1''^2}+\frac{\partial^2}{\partial\alpha_2''^2}\right)\ln\left[(\alpha_1-\alpha_1'')^2+(\alpha_2-\alpha_2'')^2\right]^{1/2} = 2\pi\delta(\alpha_1-\alpha_1'')\delta(\alpha_2-\alpha_2''),$$

since (100) is no longer true in general if α_1'', α_2'' are complex and/or the real integration domain D_1 is replaced by a more general two-dimensional continuum embedded in the four-dimensional space $(\mathrm{Re}(\alpha_1), \mathrm{Im}(\alpha_1), \mathrm{Re}(\alpha_2), \mathrm{Im}(\alpha_2))$, as is indeed the case in the Sec. V.

[26]The results (101), (103), the last condition [(i)], and most significantly the ansatz (80), (81), which means that the dependence of Φ on the cosine of the scattering angle z and ψ should be only through the parameter α, were—as already mentioned in the Introduction—borne out in detail in a synthetic way in a closed-form partial-wave summation by Tang (Ref. 7). In addition, the choice of parameters employed by Tang to convert the original sums into contour integrals strongly suggested to the present author the possibility of an over-all usefulness of the parameters t and s defined in Sec. II E (see footnote 18).

[27]$v_\pm''$ remain "good" integration variables even if E, m_A, and m_B are complex with *finite* imaginary parts. See Appendix B for details.

[28]Unless solid models of the "subdomains" V_i are made, this form of graphical representation has—for our purposes—a distinct advantage, from the points of view of both clarity and conciseness, over either three-dimensional pictures (as in Fig. 14) of V's or representations by means of projections onto the (v_+'', v_-''), (v_-'', t), and (t, v_+'') planes.

[29]The above sign convention is motivated by the fact that for symmetric $\ln\xi$ the interfaces become completely "invisible," "melting" into one integration domain D_2' depicted in Fig. 14. They are in fact truly "invisible" in this sense for the exactly soluble model considered in Sec. VI.

[30]The fact that $\Lambda \to -\Lambda$ under the substitution $v_+'' \rightleftarrows v_-''$ actually *follows* from (48). A relation resulting from formally changing the signs of $|\vec{p}|$ and z'' in (48) can first be considered as a *defining relation* of

$$(p_0, -|\vec{p}|\,|\Lambda(E,-z'')|p_0', |\vec{p}'|)$$

(i.e., as that of Λ'' for "negative $|\vec{p}|$") in terms of the values of this function from the "positive range of $|\vec{p}|$," because the integration on the right always extends from $|\vec{p}''| = 0$ to $|\vec{p}''| = +\infty$. It is then a matter of very simple algebra to actually show that

$$(p_0, |\vec{p}|\,|\Lambda(E,-z'')|p_0', |\vec{p}'|) = -(p_0, |\vec{p}|\,|\Lambda(E,z'')|p_0', |\vec{p}'|).$$

However, this formal device will be resorted to *as seldom as possible* in order to avoid confusion and to prevent us from unwittingly covering the same integration area twice.

[31]See Sec. VI for changes brought about by the use of other types of propagators.

[32]The opposite assumption, $v_+'v_-' < 0$, would change neither our qualitative results in general nor the analytic structure of the most important phase P defined by (144) in particular. However, repeating all the arguments of the present section under the assumption that $v_+'v_-' < 0$ would entail lengthy though trivial changes in our "geometry" and therefore will not be presented here.

[33]The "old" ξ was so normalized that the reversal of the sign of $\sqrt{Q}$ implies $\xi \to 1/\xi$.

[34]In other words, Fig. 18 seems *a priori* to suggest that the "truth" lies somewhere between a solution similar to that of (69) (i.e., of the Wick-Cutkosky type) and that of the exactly soluble model of Sec. VI. The analysis which follows seems to indicate, however, that only the *latter* approach can actually be pursued in a consistent way (see later).

[35]Chosen as positive in Fig. 18 for the explicit purpose of obtaining the closest correspondence with the results of Sec. IV.

[36]See later (and Appendix B) for the detailed determination of how the integration path should be chosen to avoid the singular points at $s = u_\pm^2/v_+'v_-'t$, stemming from the type of propagators used to represent the *scattered* particles A and B (as distinguished from the types of propagators used to represent the *exchange* particle, which actually constitute our primary concern at this point).

[37]Here belongs first of all the fact that this propagator does still satisfy the *inhomogeneous* Klein-Gordon equation (with a four-dimensional Dirac delta function on its right-hand side) when written in position space. It is even possible to construct a quantized free field (involving "ghosts," however), representing particles of mass zero, in such a way that its "contraction symbol" [in the usage of the Wick ordering theorem; see e.g., J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., (1955)] can be made exactly equal to this "Coulomb" propagator [see e.g., M. Günther, Phys. Rev. 125, 1061 (1961), where retarded and advanced propagators were so obtained in connection with a relativistic Lee model].

[38]Compare Jauch and Rohrlich, Ref. 37, p. 421, Eq. (A1-7).

[39]There are reasons to believe that the replacement of the Feynman propagators by either retarded or advanced propagators for the *scattered* particles constitutes, however, at least a much lesser "sin" against "reality" than the replacement of the Feynman propagator by a "Coulomb" propagator for the *exchange* particle (see footnote 27). It could even be argued that while the latter should probably always be equated to the "contraction symbol" (in the usage of the Wick theorem) of the field "transmitting" the interaction—and thus harder to explain away when changed from the

usual Feynman type—the former should more properly be either retarded or advanced rather than Feynman propagators. In fact, if the BS equation is consistently derived from field theory [as done some time ago by the present author; see M. Günther, Phys. Rev. 88, 1411 (1952); 94, 1347 (1954)], as contrasted with the usual derivation by partial summation of the Feynman graphs ("ladders") where all internal lines are represented by Feynman propagators, the appearance of retarded or advanced propagators to represent the *scattered* particles—while the *exchange* particle propagators *retain* the meaning of "contraction symbols"—becomes *a logical consequence of the formalism* [compare Eqs. (69), (66) and (42) of the first and the second of the above papers, respectively]. It should be observed parathentically that the above-mentioned field-theoretical derivation (or rather definition) of the BS equation (i.e., that based on retarded propagators) as contrasted with the now "standard" one (i.e., based on all Feynman propagators), does allow a clear-cut physical interpretation of its wave function in the ordinary usage of nonrelativistic quantum theory (see the first of the above papers, especially the defining equations of Secs. I and II), contrary to the rather widespread belief that such an interpretation is hard to achieve, frequently mentioned in the literature as one of the shortcomings of the BS approach.

[40]The latter is defined, e.g., as that accessible from the original value of E at which (182) was solved via a semicircle above (or below) the threshold at $E = m_A + m_B$.

[41]The expression $e^{il(\gamma-\pi)}$ stems from the last two terms in (181), and the "preceding" 1 of (198) and (199) from the first two terms in (182). Tracing the origin of these expressions still further back, we thus come to the conclusion that they stem initially from the appearance of $\xi_+^{(1)}$ and $\xi_-^{(1)}$ given by (122) and (123), respectively. Now, it is rather hard to visualize a theory in which $\xi_+^{(1)}$ and $\xi_-^{(1)}$ would *not* play a highly symmetric role.

[42]A corresponding rationale behind the original "normalization" (59) was provided in footnote 33. This would have as its consequence that, drawing diagrams like those depicted in Figs. 16, 17, 25, and 26 "in ξ," the radii of the semicircles would be equal to 1, but their end points could become complex.

[43]As the proof is somewhat lengthy, at least in the version known to the author, it is omitted here. The general idea, however, is to first keep one of the variables (e.g., v_-'') constant, so that if v_+'' describes the circle C, $v_+''v_-''$ will likewise describe a circle $K(v_-'')$. The boundary of the geometrical locus of points representing $Z' = (v_+''v_-'')$ is then obtained by forming an envelope of the family of circles $K(v_-'')$, as v_-'' describes the circle C. This is finally followed by the discovery that the so-obtained (and rather complicated) boundary curve ("cardioid") maps precisely into the pair of circles C and C', as depicted in Fig. 29 by means of $Z = (Z')^{1/2}$.