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# Spectral-function sum rules in asymptotically free theories*

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We show that the necessary and sufficient condition to derive Weinberg's second spectral-function sum rule within the framework of a Lagrangian theory invariant under a local non-Abelian gauge group $G$ and the global chiral $SU(2)\otimes SU(2)$ group is that $G$ should commute with $SU(2)\otimes SU(2)$. The $SU(3)\otimes SU(3)$ spectral-function sum rules for currents and sum rules involving spectral functions of scalar and pseudoscalar densities are also discussed.

## I. INTRODUCTION

The discovery of asymptotic freedom in non-Abelian gauge theories by Gross and Wilczek[1] and by Politzer[2] has already led to important results in understanding Bjorken scaling in electroproduction. The success of current algebra, on the other hand, suggests the relevance of chiral symmetries to strong interactions. The simplest synthesis of these considerations is to assume that the strong-interaction Lagrangian is locally invariant under some non-Abelian gauge group $G$, and also globally invariant (or approximately invariant) under the chiral $SU(2)\otimes SU(2)$ group. It has been suggested in the literature that the gauge group should commute with the chiral group. The motivation for this suggestion comes from the fact that if one attempts to break the gauge symmetry by the Higgs mechanism (to avoid massless gluons), one also seems to lose[1,2] asymptotic freedom. It has been conjectured by Weinberg[3] and by Gross and Wilczek[4] that the gauge group $G$ may not be broken at all, but a certain "shielding mechanism" may be at work due to the rather serious nature of the infrared-divergence problem associated with the non-Abelian symmetry, whereby only those particle states that transform as singlets under $G$ can be observed (which would include all the observed hadrons, if $G$ commutes with isospin and charge), but the massless gluons (and also quarks) which are not singlets under $G$ are unobservable. This is an attractive idea, but whether it works or not has yet to be demonstrated. It has also been shown by Weinberg[5] that strong interactions generated by a non-Abelian gauge symmetry can be incorporated into the unified theory of weak and electromagnetic interactions in a manner which naturally conserves parity, strangeness, etc. Among other conditions, this synthesis requires that $G$ should commute with $G_W$, the weak and electromagnetic gauge symmetry group, which contains charge as a generator.

In the present paper, we wish to study more directly the relationship (or lack of it) of the gauge group $G$ and the chiral symmetry. The extra information comes from considerations of the Weinberg sum rules.[6] The first sum rule is a statement about Schwinger terms and follows from current algebra[7] without any constraints on $G$. However, it is well known that the second Weinberg sum rule is model-dependent. The main result of our paper is to show that the necessary and sufficient condition under which the second sum rule can be derived is that $G$ should commute with $SU(2)\otimes SU(2)$. It should be pointed out that the second sum rule plays a crucial role in the calculations of Das *et al.*[8] in proving that the mass difference between $\pi^{\pm}$ and $\pi^0$ is finite in the

$SU(2)\otimes SU(2)$ limit. For Weinberg-Salam-type unified theories of weak and electromagnetic interactions, although one does not need the second sum rule[9] for convergence of the mass difference, the numerical value of the mass difference would be hard to understand if the second sum rule were false.

A general approach to Weinberg sum rules was suggested[10] some time ago through the concept of asymptotic symmetry. The notion of asymptotic symmetry has, however, been somewhat vague. This is because it appears to be "intuitively obvious," so that one would, for instance, expect that the second sum rule is good not only for the $SU(2)\otimes SU(2)$ group, but also for groups like $SU(3)\otimes SU(3)$ and SU(3), even when strong interactions are not invariant under these symmetries. The intuitive argument is of course naive, and indeed one knows that the second Weinberg sum rule (in the pole-dominated form) is contradicted[11] by experimental data for the SU(3) case. We hope to provide clarification of the notion of asymptotic symmetry. We show in particular that Weinberg's second sum rule, corresponding to some group $H$, which commutes with the non-Abelian gauge group $G$, can only be derived if strong interactions are invariant under the global symmetry $H$. If the symmetry $H$ is realized in the normal fashion with vacuum invariance [like SU(3)], this result is trivial. The nontrivial case arises if $H$ is a Nambu-Goldstone symmetry, like $SU(2)\otimes SU(2)$ or $SU(3)\otimes SU(3)$.

We shall work within the framework of renormalizable field theory. The Lagrangian will be chosen to be invariant under a local gauge group $G$ and also under a global group $H$. If $G$ is non-Abelian, the theory will be asymptotically free, and if $H$ is identified with the chiral $SU(2)\otimes SU(2)$ group, realized in the Nambu-Goldstone way with pions as Goldstone particles, the current-algebra results will be incorporated. The question whether $G$ commutes with $H$ or contains $H$ will be left open to start with. For purposes of clarity, however, we shall adopt a specific field-theory model, and write the Lagrangian in terms of the quark and gauge fields and their interaction

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \bar{\psi}\gamma_\mu D_\mu\psi - \bar{\psi}M\psi\ , \tag{1}$$

where

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - C_{abc}B^b_\mu B^c_\nu\ ,$$

$$D_\mu = \partial_\mu + i\sigma^a B^a_\mu\ .$$

$M$ is the quark mass matrix, $C_{abc}$ are the structure constants of the non-Abelian group $G$, and $\sigma^a$ are the matrices representing the $a$th generator of $G$ on the quark fields $\psi$. If $G$ commutes with $H$, the gauge fields $B^a_\mu$ will be $H$ singlets, and it is convenient to think in terms of the "colored" $\mathcal{P}$, $\mathfrak{N}$, $\lambda$ quarks, with $G$ being the "color" SU(3) group. In this case, if $H$ is identified with the chiral SU(2) $\otimes$SU(2) group, the Lagrangian (1) will be invariant under $H$ only if the $\mathcal{P}$ and $\mathfrak{N}$ quarks are massless. Note that the mass matrix is already color-degenerate due to invariance under the group $G$. On the other hand, if $G\supset H$, and $H$ is, say, $SU(2)\otimes SU(2)$, one requires both vector and axial-vector gauge bosons appropriately coupled and carrying, in particular, isotopic spin.

Our investigation employs the techniques of current algebra and the short-distance expansion of Wilson.[12] In fact, some time ago Wilson[12] used the short-distance expansion in a non-Lagrangian framework to study the Weinberg sum rules. We find that his results can be reproduced by field-theory models like (1), as long as $G$ commutes with $SU(2)\otimes SU(2)$. Our analysis is closely related to his, but for completeness and because of some obvious differences we shall present our arguments in some detail.

## II. WEINBERG SUM RULES FOR CHIRAL SU(2)⊗SU(2)

In this section we identify the symmetry group $H$ with $SU(2)\otimes SU(2)$ realized in the Nambu-Goldstone manner. To derive Weinberg sum rules, we start by considering the following combination of matrix elements:

$$M_{\mu\nu}(q,k) \equiv (2k_0V)\,i\int d^4x\, e^{iq\cdot x}[\langle\pi^+(k)|\,TV^3_\mu(x)\,V^3_\nu(0)\,|\pi^+(k)\rangle - \langle\pi^0(k)|\,T\,V^3_\mu(x)\,V^3_\nu(0)\,|\pi^0(k)\rangle]\ , \tag{2}$$

where $V^3_\mu(x)$ is the neutral isovector component of vector current. The matrix element $M_{\mu\nu}$ appears in the $\pi^\pm$-$\pi^0$ mass difference calculations, and the reason for studying it here is that, using current-algebra techniques, it is well known from the work of Das *et al.*[8] that (2) is related to the difference between the propagator functions for vector and axial-vector currents. Indeed, using the partially-conserved-axial-vector-current (PCAC) hypothesis, and the $SU(2)\otimes SU(2)$ current algebra, we obtain in the soft-pion limit $k\to 0$

$$M_{\mu\nu}(q,k\to 0) = -\frac{4}{f_\pi^{\,2}}\,[V^{33}_{\mu\nu}(q^2) - A^{33}_{\mu\nu}(q^2)]\ . \tag{3}$$

In deriving (3), we have made the usual assumptions that the Schwinger terms are $c$ numbers and the $\sigma$ terms have no $I=2$ component. $V^{ij}_{\mu\nu}(q^2)$ is the vector-current propagator function

$$V^{ij}_{\mu\nu}(q^2)=i\int d^4x\, e^{iq\cdot x}\langle 0|T\,V^i_\mu(x)\,V^j_\nu(0)|0\rangle \qquad (4)$$

and $A^{ij}_{\mu\nu}(q^2)$ is the axial-vector current propagator defined in the same way as $V^{ij}_{\mu\nu}(q^2)$ with $V$ replaced by $A$. $f_\pi$ stands for the $\pi$ decay constant.

We now discuss Eq. (3) for large spacelike 4-momentum $q$ ($q^2>0$). The matrix element $M_{\mu\nu}$ is then dominated by the short-distance singularities of its integrand, and we shall use Wilson's expansion[12] to study these singularities. First note that the operator product $V^3_\mu(x)\,V^3_\nu(0)$ has only $I=0,2$ components, and since the $I=0$ component makes no contribution to the difference of $\pi^+$ and $\pi^0$ matrix elements, only $[V^3_\mu(x)\,V^3_\nu(0)]_{I=2}$ contributes to Eq. (2). Since isospin is a good quantum number, in the Wilson expansion only $I=2$ local operators need be considered. To search for operators responsible for the leading singularities at short distances, we may consider the following subcases:

(i) The non-Abelian gauge group $G$ commutes with SU(2) ⊗ SU(2). In this case, the gauge fields are isospin singlets and cannot contribute to $[V^3_\mu(x)\,V^3_\nu(0)]_{I=2}$ by themselves. Since bilinear quark operators cannot carry $I=2$, it is easy to see that the most singular contribution will arise from operators with four quark fields and hence must have dimension six. Since the theory is asymptotically free, it follows[4] that in the deep spacelike region for $q$, $M_{\mu\nu}$ goes to zero as $(\ln q^2)^a/q^4$. The log term arises from the exponential function in the asymptotic solution[4] of the Wilson coefficient, and the specific value of $a$ is of no importance to us in the present work. It follows then from Eq. (3) that

$$\lim_{q^2\to\infty}\,[V^{33}_{\mu\nu}(q^2)-A^{33}_{\mu\nu}(q^2)]=0\;, \qquad (5)$$

$$\lim_{q^2\to\infty}\,q^2[V^{33}_{\mu\nu}(q^2)-A^{33}_{\mu\nu}(q^2)]=0\;. \qquad (6)$$

Equations (5) and (6) are identical to those obtained by Das, Mathur, and Okubo[10] from considerations of asymptotic SU(2) ⊗ SU(2) symmetry. We may now use the standard Lehmann-Källén representation

$$V_{\mu\nu}(q^2)=\delta_{\mu\nu}\int\frac{\rho^{ij}_1(m^2,V)}{q^2+m^2}\,dm^2 +q_\mu q_\nu\int\frac{\rho^{ij}_2(m^2,V)}{m^2(q^2+m^2)}\,dm^2 -\delta_{\mu4}\delta_{\nu4}\int\frac{\rho^{ij}_2(m^2,V)}{m^2}\,dm^2 \qquad (7)$$

and a similar representation for $A^{ij}_{\mu\nu}(q^2)$, where $\rho_1$ is the spin-1 spectral function and $\rho_2=\rho_1+\rho_0$, with $\rho_0$ as the spin-0 spectral function. Equations (5) and (6) then lead to the usual Weinberg sum rules[6]:

$$\int\frac{\rho^{33}_2(m^2,V)}{m^2}\,dm^2=\int\frac{\rho^{33}_2(m^2,A)}{m^2}\,dm^2\;, \qquad (8)$$

$$\int\rho^{33}_1(m^2,V)\,dm^2=\int\rho^{33}_1(m^2,A)\,dm^2\,. \qquad (9)$$

Note that Eq. (6), which leads to Weinberg's second sum rule, also seems to imply

$$\int\rho^{33}_2(m^2,V)\,dm^2=\int\rho^{33}_2(m^2,A)\,dm^2\;.$$

However, for conserved isovector currents $\rho^{33}_0(m^2,V)=0$, whereas for isovector axial-vector currents $\int\rho^{33}_0(m^2,A)\,dm^2=0$, since the only nonvanishing contribution to this integral could have come from a pion pole, but this contribution is proportional to the pion mass, which vanishes in the SU(2) ⊗ SU(2) limit that we are considering. Thus there is no extra sum rule. Note also that in the first sum rule (8) we can of course replace $\rho^{33}_2(m^2,V)$ by $\rho^{33}_1(m^2,V)$ on the left-hand side, but cannot replace $\rho^{33}_2(m^2,A)$ by $\rho^{33}_1(m^2,A)$ on the right-hand side since

$$\int\frac{\rho^{33}_0(m^2,A)}{m^2}\,dm^2$$

receives a nonvanishing contribution from the (massless) pion pole.

(ii) The non-Abelian gauge group contains SU(2) or SU(2) ⊗ SU(2) as a subgroup.[13] In this case the gauge fields would carry isospin, so that in the short-distance expansion of $[V^3_\mu(x)\,V^3_\nu(0)]_{I=2}$ the most important contribution will come from the $I=2$ components of gauge-invariant operators like $\delta_{\mu\nu}F^a_{\rho\sigma}(0)\,F^a_{\rho\sigma}(0)$ or $F^a_{\mu\rho}(0)\,F^a_{\rho\nu}(0)$. These operators have dimension four, so that for large spacelike $q$, $M_{\mu\nu}$ would now behave as $1/q^2$ times some power of $\ln q^2$, for the asymptotically free theories. This asymptotic behavior leads to Eq. (5), but not (6), so one can derive only the first Weinberg sum rule (8) and not the second.

From the arguments outlined in the cases (i) and (ii), it follows that the necessary and sufficient condition to derive Weinberg's second sum rule for a Lagrangian field theory invariant under a non-Abelian gauge group $G$ and under the chiral SU(2) ⊗ SU(2) group is that $G$ should commute with the chiral group.

A few remarks are in order at this stage.

(a) It is easy to see that Weinberg's first sum rule follows directly from Eq. (3) if we multiply it by $q_\mu$ and observe the Ward identity

$$q_\mu M_{\mu\nu}=0 \ . \tag{10}$$

This follows from (2) by current conservation and the assumption that the Schwinger term is a $c$ number (it is actually sufficient to assume there are no $I=2$ Schwinger terms). Now from Eqs. (10) and (3) and the Lehmann-Källén representation for $V_{\mu\nu}$ and $A_{\mu\nu}$ we directly obtain Eq. (8). This is not surprising, since it is well known that the first sum rule which states that the vector and axial-vector Schwinger terms are equal can be obtained[7] algebraically from current commutators.

(b) The Lehmann-Källén representation in the form (7) assumes no subtractions. However, it is easy to see from the renormalization-group equations that $V_{\mu\nu}(q^2)$ and $A_{\mu\nu}(q^2)$ behave as $q^2$ for large $q$, so that subtractions are needed in general. A prescription for performing the subtractions has been discussed by Wilson.[14] For our purposes, suffice it to say that these subtraction terms must be identical for $V^{33}_{\mu\nu}(q^2)$ and $A^{33}_{\mu\nu}(q^2)$ in the SU(2) $\otimes$ SU(2) limit.

(c) It is of interest to see how the above analysis is modified if the theory is not asymptotically free. Suppose as an example that $G$ contains an Abelian subgroup. If the gauge fields do not carry isospin, in the limit $q\to\infty$, the most important contribution to $M_{\mu\nu}$ will come from a local operator quadrilinear in $\psi$, as discussed before. However, such an operator will now have anomalous dimensions, and $M_{\mu\nu}(q)$ will behave asymptotically as $(q^2)^\gamma/q^4$ times some power of $\ln q^2$, where $\gamma$ is the anomalous dimension. The second sum rule will then follow only if $\gamma<1$.

### III. WEINBERG SUM RULES FOR CHIRAL SU(3)$\otimes$SU(3)

Before we discuss the generalization of Weinberg sum rules to the SU(3) $\otimes$ SU(3) currents, it is instructive to reconsider the derivation of Sec. II in an alternative way. Consider the matrix element

$$T^{ij}_{\mu\nu}(q,k)\equiv(2k_0V)^{1/2}\,i \times\int d^4x\,e^{iq\cdot x}\langle 0|\,T\,V^i_\mu(x)A^j_\nu(0)\,|\pi^0(k)\rangle \ . \tag{11}$$

Using PCAC and current algebra, one obtains in the soft-pion limit

$$T^{ij}_{\mu\nu}(q,k\to 0)=\frac{\sqrt{2}}{f_\pi}\left[f_{3ik}A^{kj}_{\mu\nu}(q)+f_{3jk}V^{ik}_{\mu\nu}(q)\right]. \tag{12}$$

In particular for $i=1+i2$ and $j=1-i2$, we obtain using isospin invariance

$$T^{1+i2,1-i2}_{\mu\nu}(q,k\to 0)=\frac{2\sqrt{2}\,i}{f_\pi}\left[V^{33}_{\mu\nu}(q^2)-A^{33}_{\mu\nu}(q^2)\right]. \tag{13}$$

As before, we now discuss Eq. (13) for large spacelike $q$. We shall henceforth assume that the gauge group $G$ commutes with SU(2) $\otimes$ SU(2). We now seek the Wilson expansion for the operator product $V^{1+i2}_\mu(x)A^{1-i2}_\nu(0)$, and from isospin invariance we now expand this product in terms of local operators corresponding to $I=1$. Thus, in contrast with the case in Sec. II, contributions from operators bilinear in the quark field can no longer be dropped on the basis of isospin arguments. There is of course no contradiction, as the following argument shows. Note first that Lorentz invariance together with parity and isospin conservation shows that, in the limit $k\to 0$, the operator with the least dimension that contributes to the left-hand side of Eq. (13) is given by $\delta_{\mu\nu}(\bar{q}\gamma_5 q)_{I=1}$, where $q$ stands for the quark proton or neutron field. Note that only the pseudoscalar density can contribute in the $k\to 0$ limit. However, $\bar{q}\gamma_5 q$ is proportional to $\partial_\mu(\bar{q}\gamma_\mu\gamma_5 q)$, whose matrix element between a pion and vacuum state must be proportional to $M_\pi^{\,2}$, vanishing in the chiral limit. It is easy to verify that in the Wilson expansion the most important nonvanishing contribution comes from operators quadrilinear in the quark field with dimension six. For asymptotically free theories, as before, Weinberg's second sum rule for SU(2) $\otimes$ SU(2) currents follows immediately.

The above procedure can be easily generalized for the SU(3) $\otimes$ SU(3) currents. Before we do this, however, we would like to generalize the argument that leads to vanishing matrix element for the pseudoscalar density $\bar{q}\gamma_5 q$, as discussed above. This conclusion follows, in fact, from $\gamma_5$ bookkeeping. Under the chiral SU(2) transformation on proton or neutron quarks $q=\binom{\mathcal{P}}{\mathfrak{N}}$,

$$q\to\gamma_5 q\,,\quad \bar{q}\to-\bar{q}\gamma_5\,, \tag{14}$$

note that the isovector currents $V^i_\mu$ and $A^i_\mu$ are even (go into themselves), but the isovector pseudoscalar density is odd. If the theory is invariant under chiral SU(2) transformations, it is clear that in the Wilson expansion of the operator product like $V^{1+i2}_\mu(x)A^{1-i2}_\nu(0)$, pseudoscalar density terms like $P^3\equiv\bar{q}\gamma_5\frac{1}{2}\lambda_3 q$ cannot contribute. However, if the theory is not SU(2) $\otimes$ SU(2)-invariant, because of mass terms, the pseudoscalar density terms can contribute, but are multiplied by an odd power of $m$, the proton or neutron quark mass, since in this case the theory will be invariant under the $\gamma_5$ transformation (14) together with mass reversal $m\to -m$.

For the SU(3)⊗SU(3) octets of vector and axial-vector currents, one can easily see that the strangeness-preserving currents $V_\mu^{1,2,3,8}$ and $A_\mu^{1,2,3,8}$ are even under the SU(2) $\gamma_5$ transformation (14). Furthermore, for strangeness-carrying currents the $V-A$ combinations $V_\mu^{4+i5}+A_\mu^{4+i5}$, $V_\mu^{6+i7}+A_\mu^{6+i7}$ together with their Hermitian conjugates are even, whereas the $V+A$ currents $V_\mu^{4+i5}-A_\mu^{4+i5}$, $V_\mu^{6+i7}-A_\mu^{6+i7}$ and Hermitian conjugates are odd under the transformation (14). If SU(2)⊗SU(2) is a good symmetry, the property of evenness or oddness must be preserved in the Wilson expansion of operator products of currents. It is now easy to show that if the theory is SU(2)⊗SU(2)-invariant, but not SU(3)⊗SU(3)-invariant, the second Weinberg sum rule for strangeness-carrying vector and axial-vector currents cannot be derived. Consider the case when the indices $i$ and $j$ in Eqs. (11) and (12) refer to strangeness-carrying currents. It is clear that in order to derive the second sum rule the matrix element $T_{\mu\nu}$ must vanish sufficiently rapidly as $q\to\infty$ so that the left-hand side of the formula (12) does not receive contribution from the pseudoscalar density operator $P_3$ in the Wilson expansion of the product $V_\mu^i(x)A_\nu^j(0)$. Since $P_3$ is an odd operator, the way to guarantee this is to start with the product of currents which is even. However, it is easy to verify that for even combinations like

$$[V_\mu^{4+i5}(x)+A_\mu^{4+i5}(x)][V_\nu^{4-i5}(0)+A_\nu^{4-i5}(0)]$$

or

$$[V_\mu^{4+i5}(x)-A_\mu^{4+i5}(x)][V_\nu^{4-i5}(0)-A_\nu^{4-i5}(0)]$$

the right-hand side of Eq. (12) vanishes. On the other hand, this right-hand side is proportional to the desired difference between the propagator functions for strangeness-carrying vector and axial-vector currents only if we start with odd combinations like

$$[V_\mu^{4+i5}(x)+A_\mu^{4+i5}(x)][V_\mu^{4-i5}(0)-A_\nu^{4-i5}(0)]$$

in the definition (11). In this case, however, operator terms like $P_3$ will contribute in the short-distance expansion, and one cannot derive the second sum rule for strangeness-carrying currents. Note, however, that the first sum rule

$$\int \frac{\rho_2^{44}(m^2, V)}{m^2}dm^2 = \int \frac{\rho_2^{44}(m^2, A)}{m^2}dm^2 \qquad (15)$$

does follow.

It is of interest to note that the second SU(3)⊗SU(3) sum rule can be derived if the Lagrangian is SU(3)⊗SU(3)-invariant. In this case the $\mathcal{P}$, $\mathfrak{N}$, $\lambda$ quarks are massless and the theory is invariant under the $\gamma_5$ transformation $\mathcal{P}\to\gamma_5\mathcal{P}$, $\mathfrak{N}\to\gamma_5\mathfrak{N}$, and $\lambda\to\gamma_5\lambda$. It is easy to see that under this extended $\gamma_5$ transformation $V_\mu^i$ and $A_\mu^i$ $(i=1,\ldots 8)$ are all even, but $P^3$ is odd. Thus, if we choose $i=4+i5$, $j=4-i5$ in Eqs. (11) and (12), the asymptotic behavior of $T_{\mu\nu}$ will be determined by operators of dimension six or more, and both SU(3)⊗SU(3) sum rules can be derived. Alternatively, one could start with matrix elements of suitable currents sandwiched between a kaon and vacuum states, and use kaon PCAC to derive the two SU(3)⊗SU(3) sum rules. Clearly the validity of the second SU(3)⊗SU(3) sum rule then depends on how good SU(3)⊗SU(3) symmetry is, and, at least, one expects that it is not as well satisfied as the second SU(2)⊗SU(2) sum rule.

## IV. SUM RULES FOR SCALAR AND PSEUDOSCALAR DENSITY SPECTRAL FUNCTIONS

In this section we apply our technique to study the spectral-function sum rules for scalar and pseudoscalar densities

$$S^\alpha(x)=\bar\psi(x)\tfrac{1}{2}\lambda_\alpha\psi(x),$$
$$P^\alpha(x)=i\bar\psi(x)\gamma_5\tfrac{1}{2}\lambda_\alpha\psi(x), \qquad (16)$$

where $\alpha=0,1,\ldots,8$. Note that the nonets $S^\alpha$ and $P^\alpha$ transform as the $(3,\bar 3)\oplus(\bar 3,3)$ representation of SU(3)⊗SU(3) and satisfy the following commutation relations:

$$[V_0^i(x), S^\alpha(0)]_{x_0=0}=if_{i\alpha\beta}S^\beta(0)\delta^3(x),$$
$$[V_0^i(x), P^\alpha(0)]_{x_0=0}=if_{i\alpha\beta}P^\beta(0)\delta^{(3)}(x),$$
$$[A_0^i(x), S^\alpha(0)]_{x_0=0}=id_{i\alpha\beta}P^\beta(0)\delta^3(x), \qquad (17)$$
$$[A_0^i(x), P^\alpha(0)]_{x_0=0}=-id_{i\alpha\beta}S^\beta(0)\delta^3(x),$$

where $i=1,\ldots,8$ and $\alpha,\beta$ can take the values $0,1,\ldots,8$. Define the matrix element

$$T^{\alpha\beta}(q,k)=(2k_0V)^{1/2}i \times\int d^4x\, e^{iq\cdot x}\langle 0|\,TS^\alpha(x)P^\beta(0)\,|\pi^0(k)\rangle. \qquad (18)$$

Using PCAC and the commutation relations (17), one obtains in the soft-pion limit

$$T^{\alpha\beta}(q,k\to 0)=\frac{\sqrt{2}}{f_\pi}[d_{3\alpha\gamma}P^{\gamma\beta}(q^2)-d_{3\beta\gamma}S^{\alpha\gamma}(q^2)], \qquad (19)$$

where $S^{\alpha\beta}$ is the propagator function for scalar densities

$$S^{\alpha\beta}(q^2)=i\int d^4x\, e^{iq\cdot x}\langle 0|\,TS^\alpha(x)S^\beta(0)\,|0\rangle \qquad (20)$$

and $P^{\alpha\beta}$ is the corresponding function for pseudoscalar densities.

If $SU(2)\otimes SU(2)$ is a good symmetry, it is easy to verify that for strangeness-preserving densities $S^0-\sqrt{2}S^8$ and $P^0-\sqrt{2}P^8$ are even under the $\gamma_5$ transformation (14), whereas $\sqrt{2}S^0+S^8$, $\sqrt{2}P^0+P^8$, $S^{1,2,3}$, and $P^{1,2,3}$ are odd. For strangeness-carrying densities, $S^{4+i5}+iP^{4+i5}$, $S^{6+i7}+iP^{6+i7}$ (and Hermitian conjugates) are even, while $S^{4+i5}-iP^{4+i5}$, $S^{6+i7}-iP^{6+i7}$ (and Hermitian conjugates) are odd.

As before we choose an even operator product in Eq. (18), so that in the Wilson expansion the operator term proportional to $P^3$ does not contribute. For strangeness-preserving densities, operator products even under the $\gamma_5$ transformation (14) and odd under $G$ conjugation are $[\sqrt{2}S^0(x)+S^8(x)]P^3(0)$ or $[\sqrt{2}P^0(x)+P^8(x)]S^3(0)$. Defining

$$P_-(x)\equiv\frac{\sqrt{2}P^0(x)+P^8(x)}{\sqrt{3}},$$

with a similar definition of $S_-(x)$, we obtain the following asymptotic relations:

$$\lim_{q^2\to\infty}[P^{33}(q^2)-S^{--}(q^2)]=\lim_{q^2\to\infty}[S^{33}(q^2)-P^{--}(q^2)]=0, \quad (21)$$

$$\lim_{q^2\to\infty}q^2[P^{33}(q^2)-S^{--}(q^2)]=\lim_{q^2\to\infty}q^2[S^{33}(q^2)-P^{--}(q^2)]=0. \quad (22)$$

Equation (21) implies that although the spectral representations for $S^{\alpha\beta}$ and $P^{\alpha\beta}$ individually may require subtractions, no subtractions are needed for the combinations that appear in Eqs. (21). Equation (22) leads to the spectral-function sum rules

$$\int\rho^{--}(m^2,P)dm^2=\int\rho^{33}(m^2,S)dm^2, \quad (23)$$

$$\int\rho^{--}(m^2,S)dm^2=\int\rho^{33}(m^2,P)dm^2. \quad (24)$$

Note that these are the only sum rules that can be derived on the assumption that the theory is $SU(2)\otimes SU(2)$-invariant and the gauge bosons do not carry isospin. As in the case of vector and axial-vector current densities, it is easy to check that in the $SU(2)\otimes SU(2)$ limit no sum rule for strangeness-carrying scalar and pseudoscalar densities can be derived. All one can obtain in this case are results similar to Eqs. (21), which bear only on the question of subtractions in the spectral representation. However, if we assume that the theory is invariant under the higher symmetry group $SU(3)\otimes SU(3)$, additional sum rules can be obtained.[15] Since the experimental information on scalar mesons is somewhat sketchy, it is not very useful to discuss the saturation of the sum rules (23) at the present time.


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[15]Recently R. N. Mohaptra [this issue, Phys. Rev. D 9, 2355 (1974)] has obtained several sum rules using the Bjorken expansion technique. In particular, he has obtained the relation (23) except that on the right-hand side he has the pseudoscalar spectral function rather than the scalar function. It is easy to check from our arguments that one can obtain this result or equivalently the sum rule $\int\rho^{33}(m^2,P)dm^2=\int\rho^{33}(m^2,S)dm^2$ if the theory is $U(2)\otimes U(2)$- or $SU(3)\otimes SU(3)$-invariant realized in an appropriate Nambu-Goldstone manner.