(1973); H. D. Politzer, *ibid.* 30, 1346 (1973); G. 't Hooft (unpublished).

[5]S. Weinberg, Phys. Rev. D 8, 605 (1973); R. N. Mohapatra and J. C. Pati, *ibid.* 8, 4212 (1973); S. L. Glashow, R. Jackiw, and S.-S. Shei, Phys. Rev. 187, 1916 (1969).

[6]V. S. Mathur and S. Okubo, Phys. Rev. D 1, 2046 (1970).

[7]S. Weinberg, Phys. Rev. Lett. 18, 507 (1967); T. Das, V. S. Mathur, and S. Okubo, *ibid.* 18, 761 (1967).

[8]The idea of the heavy pion already exists in the literature: S.D. Drell, Phys. Rev. D 7, 2190 (1973); I. Bars and M. B. Halpern, *ibid.* 7, 3043 (1973); C. Michael, Phys. Rev. 166, 1826 (1967); H. Pagels and A. Zepeda, Phys. Rev. D 5, 3262 (1972).

[9]We follow the notation of Ref. 6 and define currents as $V_\mu^i = \frac{1}{2} i \bar{q}\gamma_\mu \lambda^i q$, $A_\mu^i = \frac{1}{2} i \bar{q}\gamma_\mu\gamma_5\lambda^i q$, $U^i = \frac{1}{2}\bar{q}\lambda^i q$, $\mathcal{V}^i = \frac{1}{2} i\bar{q}\gamma_5\lambda^i q$, where $i = 0, \ldots, 8$. μ denotes the Lorentz index.

[10]In models of the type discussed in Ref. 2, $\mathcal{H}_0$ is invariant under the $U(4)_L \otimes U(4)_R$ group and there is an extra $\epsilon_{15}U_{15}$ term in $\mathcal{H}_{\text{mass}}$. But, since we are concerned only with the $U(3)_L \otimes U(3)_R$ subspace of this, the $\epsilon_{15}U_{15}$ term is a singlet and is absorbed into the $\epsilon_0 U_0$ term.

[11]Note that in deriving the last two lines of Eq. (18) the $U(3)_L \otimes U(3)_R$ symmetry of $\mathcal{H}_0 + \mathcal{H}_1$ is crucial. For example, in a theory with broken $SU(3)_L \otimes SU(3)_R$ symmetry, one will have instead of the last two lines the following result:

$$I_{-2,-2} = \gamma(1-2a)(1-2b) + \tfrac{2}{3}\langle 0|S|0\rangle,$$

$$I_{-1,-1} = I_{33} + \tfrac{1}{3}\langle 0|S|0\rangle,$$

where

$$S = \left[Q_5^0, \left[Q_5^0, \int d^3x (\mathcal{H}_{\text{tot}} - \mathcal{H}_{\text{mass}})\right]\right].$$

[12]V. S. Mathur, S. Okubo, and J. Subbarao, Phys. Rev. D 1, 2058 (1970).

[13]M. Gell-Mann, in *Proceedings of the Third Hawaii Topical Conference on Particle Physics*, edited by S. F. Tuan (Western Periodicals, North Hollywood, California, 1970).

[14]Mohapatra and Pati, Ref. 5.

[15]R. Brandt and G. Preparata, Ann. Phys. (N.Y.) 61, 119 (1970).

[16]K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett. 16, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966). It is worth pointing out that the KSRF relation also gets modified if the heavy pion is important and becomes $G_\rho^2 = f_\pi^2 m_\rho^2 (1+\epsilon)$, where $\epsilon = (f_{\pi'}/f_\pi)(G_{\rho\pi'\pi}/G_{\rho\pi\pi})$.

[17]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[18]Particle Data Group, Rev. Mod. Phys. 45, S1 (1973).

[19]H. Pagels, Phys. Rev. 179, 1337 (1969).

[20]S. L. Adler, Phys. Rev. 177, 2426 (1969); S. Okubo, *ibid.* 179, 1629 (1969).

[21]K. Wilson, Phys. Rev. 179, 1499 (1969).

[22]S. Weinberg, Phys. Rev. D 8, 605 (1973).

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Proof of the Weinberg sum rules in the Bars-Halpern-Yoshimura model

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Both sets of Weinberg spectral-function sum rules are proved in the context of the Bars-Halpern-Yoshimura model. We first discuss how to obtain the appropriate weak currents of the hadrons in such a gauge model, and then obtain the results by proving the equal-time commutation relations $[V_0^A, V_i^B] = [A_0^A, A_i^B]$, and $[\partial_0 V_i^A - \partial_i V_0^A, V_j^B] = [\partial_0 A_i^A - \partial_i A_0^A, A_j^B]$; the proof allows the spectral-function integrals involved to be different for the separate I-spin multiplets.

I. INTRODUCTION

Recently, models have been proposed for a unified theory of strong, weak, and electromagnetic interactions,[1,2] based on the ideas of local gauge invariance and the Higgs-Kibble mechanism.[3] In one of these,[1] the strong spin-1 gauge bosons are identified as the usual low-lying nonets of spin-1 mesons,[4] i.e., the ρ, A_1, etc., and (some of) the spin-0 mesons as the corresponding pseudoscalar and scalar particles, i.e., π, π_N, etc.

In such a model, it seems natural to check whether the Weinberg spectral-function sum rules[5] for the weak currents of the hadrons can be proved, since there is no convincing proof to date.[6] The very interesting algebra-of-fields approach employed by Lee, Weinberg, and Zumino[7] has the basic drawback of being based on a nonrenormalizable model. Accordingly, in the present paper, we use the renormalizable gauge model of Bars, Halpern, and Yoshimura as a convenient vehicle in which to prove the equality of the two sets of time-space equal-time commutators for the vector and axial-vector currents from which Lee *et al.*

derived the two sum rules. It should be remarked that the sum rules are proved in their less restrictive form, i.e., there is no reason for the spectral-function integrals to be equal for different values of the I-spin, and the loosening of this constraint is welcome, since certain undesirable consequences for K_{l3} parameters are thereby avoided.[6]

The format of the paper is as follows. Section II contains a brief summary of the Bars-Halpern-Yoshimura (BHY) model, while Sec. III has a more detailed discussion of the U-gauge fields and their couplings, since it is important to be able to identify certain fields with known physical particles. In Sec. IV we define the weak currents of the hadrons (in the sense of Gell-Mann's current-algebra currents),[8] and then demonstrate, using the canonical commutation relations of the fields, how these currents do satisfy the usual current-algebra relations, and also those equal-time commutators (ETCs) including both time and space components, from which both Weinberg spectral-function sum rules can be deduced.[7] Finally, Sec. V contains some concluding remarks, including a summary of general conditions, obtained by Bars, Halpern, and Lane,[9] under which both sum rules must be valid in a large class of renormalizable models.

II. BARS-HALPERN-YOSHIMURA MODEL

In this section, we give a brief summary of the above model and introduce our own notation for later convenience; the original paper[1] contains a detailed explanation of the choice of groups, representations, etc.

The gauge group of the strong interactions is chiral $U(3)_L \otimes U(3)_R$, with three independent sets of 18 real fields X, Y, and Σ (in a 3×3 complex matrix notation) respectively transforming according to the $(3, 1)$, $(1, 3)$, and $(3, \bar{3})$ representations; i.e., the X couple to the "left-hand" gauge mesons L^a_μ, the Y to the "right-hand" R^a_μ, and the Σ to both (a runs from 0 to 8). Under parity transformations, $X \longleftrightarrow Y$, $\Sigma \rightarrow \Sigma^+$, and $L_\mu \longleftrightarrow R_\mu$. In addition, there is a further global chiral $U(3)'_L \otimes U(3)'_R$ group under which X and Y transform as $(\bar{3}, 1)$ and $(1, \bar{3})$, respectively, with the remaining fields all being singlets. The set of hadrons is completed by two sets of quarks, q_L and q_R, which transform as triplets under the appropriate gauge groups, and are neutral under the primed group. The vacuum expectation values of the spin-0 fields have the form

$$\langle X\rangle = \langle Y\rangle = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \equiv \mathcal{A}, \quad \langle\Sigma\rangle = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} \equiv C .$$

This ensures that the vacuum is invariant under parity, I-spin, and hypercharge transformations.

To introduce the weak and electromagnetic interactions in the manner of Weinberg, we have four new gauge fields W^a_μ ($a = 1, 2, 3$) and B_μ belonging to an $SU(2)''_L \otimes U(1)''$ group which has to be embedded in the hadronic primed group. However, in order to eliminate all strangeness-changing neutral processes to first order, BHY found it necessary to enlarge this primed group to $U(4)'_L \otimes U(4)'_R$, so that the matrices X and Y now acquire an extra column each. Furthermore, since the weak bosons couple both to strange and nonstrange hadrons, the $SU(2)''_L$ group has to undergo the Cabibbo rotation relative to $U(4)'_L$, so that it is a rotated W_μ matrix, $\overline{W}_\mu \equiv R W^a_\mu T^a R^{-1}$, which couples to the hadrons X: Here, the sum on a includes the first three generators T^a of the $U(4)'_L$ group, and R represents the Cabibbo rotation:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the generator of the $U(1)'$ group is fixed by the properties of the electric charge generator Q, and its relation to the generators of the primed and unprimed groups.

The leptons are introduced into representations of the rotated primed group, as is the "Weinberg scalar" field $\phi \rightarrow \overline{\phi} = \rho T^0 + i\omega^a \overline{T}^a$, where a again runs from 1 to 3. The vacuum expectation value of ϕ, $\langle\phi\rangle = \eta T_0$, is responsible for the major part of the W_μ mass and for the lepton masses (just as the quark masses are proportional to $\langle\Sigma\rangle$). Also, there is a gauge-invariant coupling of $\overline{\phi}$ to the hadrons, which induces mixing between hadrons of different I spins, hypercharge, and parity; clearly, the effective coupling is "weak," and must vanish when the electromagnetic and weak interactions are switched off. Details of the leptonic part of $\mathcal{L}$ can be found in BHY,[1] but will be omitted here as we are primarily concerned with the hadrons.

III. U-GAUGE FIELDS

Before making the appropriate gauge transformations, we shall simplify matters somewhat by introducing a vector notation for all of the fields, rather than a matrix one. For example, the elements of $X \equiv (1/\sqrt{2})\lambda_i(\chi + i\alpha)_i$ (summing over all 16 generators of U_4, but with four omissions because χ is a 3×4 matrix, not a 4×4 one), are rewritten in the form of a 32-component column vector, with the first 16 elements being the

16 χ_i; $\chi = \binom{\chi}{\alpha}$ in block form. Then, under the full group,

$$X_i \to \exp(\theta_L \cdot T_L + \theta_L' \cdot T_R)_{ij} X_j \,, \tag{1a}$$

where

$$2(T_L^A)_{ij} \equiv \begin{pmatrix} f_{ij}^A & -d_{i,j-16}^A \\ d_{i-16,j}^A & f_{i-16,j-16}^A \end{pmatrix}, \qquad 2T_R \equiv \begin{pmatrix} f & d \\ -d & f \end{pmatrix}, \tag{2}$$

and $f_{ij}^A = f^{iAj}$, $d_{ij}^A = d^{iAj}$, with f^{iAj}, d^{iAj} the usual SU(4) structure constants, extended in the normal way to include i, A, or $j = 16$, viz.,

$$f^{iAj} = 0 \text{ if any index} = 16 \,, \; d^{iA16} = \frac{1}{\sqrt{2}} \delta^{iA}, \text{ etc.}$$

Also, $\theta_L \cdot T_L$ denotes the sum

$$\sum_{a=1}^{16} (\theta_L)_a (T_L)_a \,.$$

Similarly, $Y = \binom{\psi}{\beta}$, $\Sigma = \binom{\zeta}{\gamma}$, $\phi = \binom{\rho}{\omega}$, where ρ and ω have only one and three nonzero components, respectively. Then

$$Y \to \exp(\theta_R \cdot T_L + \theta_R' \cdot T_R) Y \,, \tag{1b}$$

$$\Sigma \to \exp(\theta_L \cdot T_L + \theta_R \cdot T_R) \Sigma \,, \tag{1c}$$

$$\phi \to \exp(\theta_L' \cdot T_L + \theta_R' \cdot T_R) \phi \,, \tag{1d}$$

where for ϕ the sums over θ_L' and θ_R' involve only four generators altogether.

The covariant derivatives of these fields are then

$$\Delta_\mu X = (\partial_\mu + hL_\mu \cdot T_L + gW_\mu \cdot \overline{T}_R + g' B_\mu T_R^B) X \,, \tag{3a}$$

$$\Delta_\mu Y = (\partial_\mu + hR_\mu \cdot T_L + g' B_\mu T_R^B) Y \,, \tag{3b}$$

$$\Delta_\mu \Sigma = (\partial_\mu + hL_\mu \cdot T_L + hR_\mu \cdot T_R) \Sigma \,, \tag{3c}$$

$$\Delta_\mu \phi = (\partial_\mu + gW_\mu \cdot T_L + g' B_\mu T_{L+R}^B) \phi \,, \tag{3d}$$

where h is the strong coupling constant, g and g' are the electromagnetic-weak constants, and T^B is the generator of the U(1)$''$ group.

Now, in order to go to the U gauge, we must find the different combinations of these spin-0 fields which constitute the Goldstone bosons corresponding to each generator of a local gauge symmetry, and then perform the appropriate gauge transformation which makes these Goldstone fields vanish. We shall then be left with a set of massive spin-1 mesons (together with the photon) and only massive spin-0 mesons (assuming that there are no pseudo-Goldstone bosons around).

To begin with, the Goldstone bosons for the generators of U(3)$_L$ are g_L^A, where

$$N_L g_L^A = \langle \tilde{X} \rangle T_L^A X + \langle \tilde{\Sigma} \rangle T_L^A \Sigma = \tfrac{1}{2}[\tilde{a}(f^A \chi - d^A \alpha) + \tilde{C}(f^A \zeta - d^A \gamma)] \equiv 0 \,, \tag{4a}$$

where N_L is simply the normalization factor, $\langle X \rangle = \binom{a}{0}$, $\langle \Sigma \rangle = \binom{c}{0}$, and the tilde denotes the transpose. Similarly,

$$N_R g_R^A = \tfrac{1}{2}[\tilde{a}(f^A \psi - d^A \beta) + \tilde{C}(f^A \zeta + d^A \gamma)] \equiv 0 \,, \tag{4b}$$

$$N_W \overline{g}_W^A = \tfrac{1}{2}[\tilde{a}(\overline{f}^A \chi + \overline{d}^A \alpha) + \tilde{\eta}(f^A \rho - d^A \omega)] \equiv 0 \,, \tag{4c}$$

$$N_B g_B = \langle \tilde{X} \rangle T_R^B X + \langle \tilde{Y} \rangle T_R^B Y + \langle \tilde{\phi} \rangle (T_L + T_R)^B \phi \equiv 0 \,, \tag{4d}$$

where the matrices $\overline{f}^A, \overline{d}^A$ in Eq. (4c) are the Cabibbo-rotated f and d, and appear operating on the X fields since the W_μ^A couple via $\overline{T}_R$, not T_R, to X; of course, the normal generators f, d operate on the ϕ term. In addition, in Eq. (4c), A takes the values 1, 2, 3 only, whereas in Eqs. (4a) and (4b), A takes the values 1, 2, ..., 8, 16, corresponding to the U(3) subgroup of the U(4) group in which X has been embedded.

As a particularly simple example, consider the case of g_L^3 and g_R^3:

$$g_L^3 \equiv 0 \to a\alpha_3 + c\gamma_3 \equiv 0 \,, \tag{5a}$$

$$g_R^3 \equiv 0 \to a\beta_3 - c\gamma_3 \equiv 0 \,. \tag{5b}$$

In the absence of the conditions for g_W^A and g_B to vanish (i.e., when the electromagnetic and weak interactions of the hadrons are turned off), we can eliminate α_3 and β_3 in favor of γ_3 throughout the Lagrangian $\mathcal{L}$, and find that the appropriate kinetic term is $\frac{1}{2}(1 + 2c^2/a^2)(\partial_\mu \gamma_3)^2$. Hence, if we define $\pi_3 \equiv (1 + 2c^2/a^2)^{1/2} \gamma_3 \equiv N\gamma_3$, then π_3 has the normal kinetic term for a pseudoscalar field, and we can relate it to a physical particle if its other properties permit: In the present case, this would be with the π^0 meson, since π_3 would be massless if $\mathcal{L}$ were invariant under the global $\mathrm{U}(4)_L' \otimes \mathrm{U}(4)_R'$ symmetry. Also, under the SU(2) subgroup of the diagonal $\mathrm{U}^4(3) \equiv \mathrm{U}(3)_L \oplus \mathrm{U}(3)_R \oplus \mathrm{U}(3)_L' \oplus \mathrm{U}(3)_R'$, the vacuum is invariant, so that we associate I spin with this group, and π_3 is part of an I triplet. In similar fashion, we can eliminate all nine scalar and nine pseudoscalar Goldstone bosons, associated with the strong gauge group, from the model.

It should be noticed that since only a_8, a_{15}, and a_{16} are nonzero, the U(3) Goldstone bosons g_L^A and g_R^A do not involve any of the additional χ_i, α_i, etc. ($i = 9, \ldots, 15$) which constitute the fourth columns of the X and Y matrices, and which were added when the primed groups were enlarged from U(3) to U(4) in order to accommodate the weak and electromagnetic interactions. This reflects the fact, mentioned in Appendix A of BHY, that these fourth columns have no important role to play in the purely hadronic system. Therefore, under any strong gauge transformation on the fields, the elements of these columns just mix among themselves, each covariant time derivative corresponds to the appropriate conjugate momentum, and no extra problems arise in identifying such covariant derivatives when the current commutation relations are being evaluated.

If we now turn to the conditions $g_W^A \equiv 0$ and $g_B \equiv 0$, we find immediately that interaction terms are introduced into $\mathcal{L}$ which violate I-spin, hypercharge, and parity conservation. For example, since $\bar{f}^1 = f^1\cos\theta - f^4\sin\theta$, we find

$$\bar{g}_W^1 \sim a\cos\theta\alpha_1 - \sin\theta[\tfrac{1}{2}(a-b)\chi_5 + \tfrac{1}{2}(a+b)\alpha_4] - \eta\omega_1 \equiv 0\,. \tag{6}$$

Now, $\alpha_1 = -(c/aN)\pi_1$ [using Eqs. (5)], and χ_5 and α_4 are constituent parts of both K_4 and κ_4: Since the potential part of $\mathcal{L}$ contains several terms in which $\bar{\varphi}$ couples to the regular hadrons in X, Y, and Σ, then the replacement of ω_1 in such terms by the above combination of α_1, α_4, and χ_5 causes the aforementioned symmetry violations. In fact, such terms are of order $1/\eta^2 \sim \alpha/M_W{}^2$, so there is no need to be concerned (the last-mentioned α is, of course, the fine-structure constant, and not the field).

One interesting effect which arises from this is the direct coupling of some spin-0 mesons to the leptons. The Lagrangian contains a term $\mathrm{Tr}(\bar{l}_D\varphi l_S G + \mathrm{H.c.})$, where G is a numerical matrix, l_D, l_S are the particular matrix representations of the leptons described in BHY (called ψ_D and ψ_S there), and φ is also in matrix form above. The vacuum expectation value η of φ then obviously gives a mass term m_l to the leptons, while the $\omega_{1,2}$ terms lead to a direct coupling of spin-0 hadrons to a lepton-antilepton pair, with a coupling constant which is easily shown to be $(m_l/\eta^2)\times(c/N)\cos\theta$ for π. In the current-current model for weak interactions, the effective coupling is $2\sqrt{2}(G/\sqrt{2})f_\pi m_l\cos\theta$ (where f_π is the coupling of the π field to the axial-vector current); in gauge models involving W_μ bosons, we have the usual relation $G/\sqrt{2} = g^2/8M_W{}^2$. Here, $M_W{}^2 = \frac{1}{2}g^2(a^2+\eta^2) \simeq \frac{1}{2}g^2\eta^2$ since the condition $\eta \gg a$ is necessary for the W_μ mass to be very much larger than the masses of strong-gauge mesons, i.e., the ρ, A_1, etc. Thus we can identify the pion decay constant (approximately) with $\sqrt{2}\,c/N$.

The kinetic part of $\mathcal{L}$ is thus

$$\begin{aligned}\mathcal{L}_0 = &-\tfrac{1}{4}\{[F_{\mu\nu}^A(L)]^2 + [F_{\mu\nu}^A(R)]^2 + [F_{\mu\nu}^A(W)]^2 + [F_{\mu\nu}(B)]^2\}\\ &+\tfrac{1}{2}[(\Delta_\mu X_i)^2 + (\Delta_\mu Y_i)^2 + (\Delta_\mu \Sigma_i)^2 + (\Delta_\mu \varphi_i)^2]\\ &- i\bar{q}\gamma^\mu\Delta_\mu q - i\bar{l}\gamma^\mu\Delta_\mu l\,,\end{aligned} \tag{7}$$

where, for example,

$$F_{\mu\nu}^A(L) \equiv \partial_\mu L_\nu^A - \partial_\nu L_\mu^A + hf^{ABC}L_\mu^B L_\nu^C\,,$$

and

$$\Delta_\mu q \equiv (\partial_\mu + ihL_\mu^A\tfrac{1}{2}\lambda^A)q\,.$$

For the "physical" U gauge for the system, all of the Goldstone boson fields are set equal to zero, the fields χ_i, etc. are expressed (through these conditions) in terms of the physical hadron fields such as π, and all of the spin-1 mesons (except for the photon) are massive, with three independent degrees of freedom each.

IV. HADRONIC WEAK CURRENTS

In this section we define a set of currents for the hadrons which bear the closest resemblance to the vector and axial-vector currents which Gell-Mann chose to satisfy his current algebra. At that time, the Lagrangian for the weak interaction was written in one of two forms,

$$\mathcal{L}_W = \frac{G}{\sqrt{2}}\mathcal{J}_\mu^\dagger\mathcal{J}^\mu \text{ or } \mathcal{L}_W = g(\mathcal{J}^\mu W_\mu^\dagger + \mathrm{H.c.})\,.$$

In either case, the current $\mathcal{J}_\mu$ was considered to be the sum of a purely leptonic part and a purely hadronic part; it was (the time component of) this hadronic part, along with the Hermitian-conjugate current and the electromagnetic current of the hadrons, which was postulated to generate an SU(2)⊗ U(1) algebra.

To find the appropriate current in a proper field-theory model, we must first evaluate the quantity $(1/g)(\delta\mathcal{L}/\delta W_\mu^A)$, which clearly interacts linearly with W_μ^A in $\mathcal{L}$. The purely hadronic part is then isolated by turning off all weak and electromagnetic interactions completely, which means not only letting $g\to 0$ and $\eta\to\infty$, etc. (which decouples W_μ and φ from the hadrons), but also ignoring the conditions $g_W^A \equiv 0 \equiv g_B$, since their expression in terms of both hadron and φ fields assumes an interaction between these same fields.

Following this procedure, we find that the total current $\bar{\mathcal{J}}_\mu^A$ coupling to W_μ^A is

$$\begin{aligned}\bar{\mathcal{J}}_\mu^A = &\Delta_\mu X_i(\bar{T}_R^A)_{ij}X_j + \Delta_\mu\varphi_i(T_L^A)_{ij}\varphi_j\\ &- i\bar{l}\tfrac{1}{2}i\lambda^A\gamma_\mu l - F_{\mu\nu}^B(W)f^{BAC}W^{\nu c}\,.\end{aligned} \tag{8}$$

However, the purely hadronic part of this, J_μ^A (which has now been Cabibbo-rotated back) is

$$\begin{aligned}J_\mu^A &= (\Delta_\mu X)^T T_R^A X\\ &= \tfrac{1}{2}[(\Delta_\mu S')^T T_R^A S' + (\Delta_\mu D')^T T_R^A D']\\ &\quad + \tfrac{1}{2}[(\Delta_\mu S')^T T_R^A D' + (\Delta_\mu D')^T T_R^A S']\\ &\equiv \tfrac{1}{2}(V_\mu^A + A_\mu^A)\,,\end{aligned} \tag{9}$$

where

$$S' \equiv \frac{1}{\sqrt{2}}(X+Y) = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi+\psi\\ \alpha+\beta\end{pmatrix},$$

$$D' \equiv \frac{1}{\sqrt{2}}(X-Y) = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi-\psi\\ \alpha-\beta\end{pmatrix}.$$

Before examining the commutation relations of these currents, consider those terms including only the derivative of a spin-0 field, and for simplicity take $A=1$:

$$\begin{aligned}\mathcal{J}^1_\mu &= (1/\sqrt{2})\{a\cos\theta\partial_\mu\alpha_1 \\ &\quad -\sin\theta[\tfrac{1}{2}(a-b)\partial_\mu\chi_5+\tfrac{1}{2}(a+b)\partial_\mu\alpha_4] \\ &\quad -\eta\partial_\mu\omega_1\}+\cdots \\ &\sim \partial_\mu g^1_W+\cdots = 0+\cdots ,\end{aligned} \tag{10}$$

i.e., there is no spin-0 pole in the current. Of course, this is not surprising, since it is the removal of such terms as $W^1_\mu\partial^\mu g^1_W$ in $\mathcal{L}$ which defines a gauge transformation from the R gauge to the U gauge. However, we cannot set $g^1_W=0$ when there are no weak interactions, so that ω_1 is a separate field from the α's and χ's. However, ω does not couple directly to the strong gauge bosons at all (nor to the spin-0 hadrons in this particular limit), so it does not contribute to the hadronic current. The corresponding term in J^1_μ is

$$(1/\sqrt{2})\partial_\mu(a\alpha_1) = -\tfrac{1}{2}\frac{\sqrt{2}c}{N}\partial_\mu\pi_1,$$

and we see that the axial-vector current A^1_μ has a pion-pole term, with coefficient $-\sqrt{2}c/N \simeq -f_\pi$, as shown above. Thus, although these currents are not so useful now for looking at the weak interaction of the hadrons, it is rather gratifying to know that the pion pole still couples to the axial-vector current with a residue (almost) equal to f_π.

Now, in order to evaluate the ETCs for the currents by using canonical commutation relations for the fields, we must work in a U gauge, since all of the fields there have regular properties (e.g., no indefinite-metric fields as we find in an R gauge). We say "a U gauge" and not "the physical U gauge" as previously described, since it is a rather tedious algebraic problem to express all time derivatives in terms of conjugate momenta for the physical fields. However, we perform another gauge transformation on the hadron fields so that they have the following simple form:

$$X\to\begin{pmatrix}x\\0\end{pmatrix},\quad y\to\begin{pmatrix}y\\0\end{pmatrix},\quad \Sigma\to\begin{pmatrix}z\\u\end{pmatrix}.$$

Parity eigenstates are now $S\equiv(1/\sqrt{2})(x+y)$, z, (0^+), and $D\equiv(1/\sqrt{2})(x-y)$, u, (0^-). In this gauge, the spin-1 mesons are still massive, although $\mathcal{L}$ now contains terms of the form $L^\mu\partial_\mu x$. For convenience, we drop the terms in X and Y corresponding to the extra column in each matrix; the above gauge transformation just mixes these terms among themselves, and the removal of these terms does not affect the results which we are about to prove. Thus, in the above notation, the column vectors S, D, z, and u each contain nine components.

The covariant derivatives for the parity eigenstates S' and D' now become

$$\begin{aligned}\Delta_\mu S' &\to \partial_\mu\begin{pmatrix}S\\0\end{pmatrix}+\frac{h}{\sqrt{2}}T_L\left[v_\mu\begin{pmatrix}S\\0\end{pmatrix}+a_\mu\begin{pmatrix}D\\0\end{pmatrix}\right] \\ &= \begin{bmatrix}\partial_\mu S+\dfrac{h}{2\sqrt{2}}f\cdot(v_\mu S+a_\mu D)\\ \dfrac{h}{2\sqrt{2}}d\cdot(v_\mu S+a_\mu D)\end{bmatrix} \\ &\equiv \begin{pmatrix}\Delta_\mu S\\ \Delta_\mu \mathcal{S}\end{pmatrix},\end{aligned} \tag{11a}$$

$$\begin{aligned}\Delta_\mu D' &\to \begin{bmatrix}\partial_\mu D+\dfrac{h}{2\sqrt{2}}f\cdot(v_\mu D+a_\mu S)\\ \dfrac{h}{2\sqrt{2}}d\cdot(v_\mu D+a_\mu S)\end{bmatrix} \\ &\equiv \begin{pmatrix}\Delta_\mu D\\ \Delta_\mu \mathfrak{D}\end{pmatrix},\end{aligned} \tag{11b}$$

$$\begin{aligned}\Delta_\mu \Sigma &\to \begin{bmatrix}\partial_\mu z+\dfrac{h}{\sqrt{2}}(f\cdot v_\mu z-d\cdot a_\mu u)\\ \partial_\mu u+\dfrac{h}{\sqrt{2}}(d\cdot a_\mu z+f\cdot v_\mu u)\end{bmatrix} \\ &\equiv \begin{pmatrix}\Delta_\mu z\\ \Delta_\mu u\end{pmatrix},\end{aligned} \tag{11c}$$

where $v_\mu\equiv(1/\sqrt{2})(L_\mu+R_\mu)$ and $a_\mu\equiv(1/\sqrt{2})(L_\mu-R_\mu)$ are the 1^- and 1^+ fields.

Next, we write down the fields and their conjugate momenta:

$$\begin{aligned}&S_a:\ \Pi^a_S\equiv\Delta_0 S_a,\quad D_a:\ \Pi^a_D\equiv\Delta_0 D_a,\\ &z_a:\ \Pi^a_z\equiv\Delta_0 z_a,\quad u_a:\ \Pi^a_u\equiv\Delta_0 u_a,\\ &v^A_i:\ \Pi^{iA}_v\equiv -F^{0i}_A(v),\quad a^A_i:\ \Pi^{iA}_a\equiv -F^{0i}_A(a),\end{aligned} \tag{12}$$

where the index a runs from 1 to 9 in each case.

In order to eliminate $\Delta_0\mathcal{S}_a$ and $\Delta_0\mathfrak{D}_a$ from the currents, we use the field equations for v^A_0 and a^A_0:

$$\begin{aligned}-\partial^\mu F^A_{\mu 0}(v) &= -\partial_i\Pi^{iA}_v \\ &= \frac{h}{\sqrt{2}}[\Pi^{iB}_v f^A_{BC}v^C_i+\Pi^i_a f^A a_i] \\ &\quad+\frac{h}{2\sqrt{2}}[\tilde{\Pi}_S f^A S+\tilde{\Pi}_D f^A D+\Delta_0\mathcal{S}_a d^A_{ab}S_b \\ &\qquad +\Delta_0\mathfrak{D}_a d^A_{ab}D_b] \\ &\quad+\frac{h}{\sqrt{2}}[\tilde{\Pi}_z f^A z+\tilde{\Pi}_u f^A u+\tfrac{1}{2}i\tilde{\Pi}_q\lambda^A q]\end{aligned} \tag{13a}$$

and

$$-\partial_i \Pi_a^{iA} = \frac{h}{\sqrt{2}}[\tilde{\Pi}_v^i f^A a_i + \tilde{\Pi}_a^i f^A v_i] + \frac{h}{2\sqrt{2}}[\tilde{\Pi}_s f^A D + \tilde{\Pi}_D f^A S + \Delta_0 \mathcal{S}_a d_{ab}^A D_b + \Delta_0 \mathfrak{D}_a d_{ab}^A S_b] - \frac{h}{\sqrt{2}}[\tilde{\Pi}_z d^A u + \tilde{\Pi}_u d^A z - \tfrac{1}{2} i \tilde{\Pi}_q \lambda^A \gamma_5 q]. \tag{13b}$$

It is then straightforward to show that

$$V_0^A = \tilde{\Pi}_S f^A S + \tilde{\Pi}_D f^A D + \tilde{\Pi}_z f^A z + \tilde{\Pi}_u f^A u + \tilde{\Pi}_v^i f^A v_i + \tilde{\Pi}_a^i f^A a_i + \sqrt{2}\,\partial_i \Pi_v^{Ai} + \tilde{\Pi}_q \tfrac{1}{2} i \lambda^A q, \tag{14a}$$

$$A_0^A = \tilde{\Pi}_S f^A D + \tilde{\Pi}_D f^A S - \tilde{\Pi}_z d^A u + \tilde{\Pi}_u d^A z + \tilde{\Pi}_v^i f^A a_i + \tilde{\Pi}_a^i f^A v_i + \sqrt{2}\,\partial_i \Pi_a^{Ai} + \tilde{\Pi}_q \tfrac{1}{2} i \lambda^A \gamma_5 q. \tag{14b}$$

Also,

$$2V_i^A = (\partial_i \tilde{S}) f^A S - \frac{h}{2\sqrt{2}}(\tilde{S} v_i + \tilde{D} a_i) \cdot (f f^A - d d^A) S + (\partial_i \tilde{D}) f^A D - \frac{h}{2\sqrt{2}}(\tilde{D} v_i + \tilde{S} a_i) \cdot (f f^A - d d^A) D, \tag{15a}$$

and

$$2A_i^A = (\partial_i \tilde{S}) f^A D - \frac{h}{2\sqrt{2}}(\tilde{S} v_i + \tilde{D} a_i) \cdot (f f^A - d d^A) D + (\partial_i \tilde{D}) f^A S - \frac{h}{2\sqrt{2}}(\tilde{D} v_i + \tilde{S} a_i) \cdot (f f^A - d d^A) S. \tag{15b}$$

Finally, it can readily be shown that

$$\partial_0 J_i^A - \partial_i J_0^A = -h \tilde{X}(T_L \cdot \Pi_L^i) T_R^A X + 2(\Delta_i X)^T T_R^A \Delta_0 X. \tag{16}$$

From this, we find

$$2(\partial_0 V_i^A - \partial_i V_0^A) = (\Delta_i S)^T (f^A \Pi_S + d^A \Delta_0 \mathcal{S}) + (\Delta_i \mathcal{S})^T(-d^A \Pi_S + f^A \Delta_0 \mathcal{S}) + (\Delta_i D)^T (f^A \Pi_D + d^A \Delta_0 \mathfrak{D}) + (\Delta_i \mathfrak{D})^T(-d^A \Pi_D + f^A \Delta_0 \mathfrak{D}) + \tfrac{1}{4}(\tilde{S}\Pi_v^{Bi} + \tilde{D}\Pi_a^{Bi})(f^B f^A S + d^B d^A S) - \tfrac{1}{4}(\tilde{S}\Pi_a^{Bi} + \tilde{D}\Pi_v^{Bi})(f^B f^A D + d^B d^A S). \tag{17}$$

The corresponding expression for $(\partial_0 A_i^A - \partial_i A_0^A)$ is obtained by replacing each Π, and $\Delta_0 \mathcal{S}$, $\Delta \mathfrak{D}$ by its opposite parity partner.

We are now in a position to evaluate the current commutation relations. First of all, it is straightforward to show that the usual current-algebra relations are satisfied, viz.,

$$[V_0^A(x),\ V_0^B(y)]\,\delta(x_0 - y_0) = [A_0^A(x),\ A_0^B(y)]\delta(x_0 - y_0) = i f^{ABC} V_0^C(x)\delta^4(x-y),$$
$$[V_0^A(x),\ A_0^B(y)]\,\delta(x_0 - y_0) = i f^{ABC} A_0^C(x)\delta^4(x-y). \tag{18}$$

Next, we prove the relation

$$[V_0^A(x),\ V_i^B(y)]\,\delta(x_0 - y_0) = [A_0^A(x),\ A_i^B(y)]\,\delta(x_0 - y_0), \tag{19}$$

since the first spectral-function sum rule (SFSR) follows, as shown by Lee, Weinberg and Zumino, viz.,

$$\int d\mu^2 [\mu^{-2} \rho_{ab}^{(1)}(\mu^2) + \rho_{ab}^{(0)}(\mu^2)]_V = \int d\mu^2 [\mu^{-2} \rho_{ab}^{(1)}(\mu^2) + \rho_{ab}^{(0)}(\mu^2)]_A, \tag{20}$$

where, for example, $(\rho_{ab}^{(1)})_V$ and $(\rho_{ab}^{(0)})_V$ are the spin-1 and spin-0 spectral functions of the vector current V_μ:

$$\langle 0 | V_\mu^a(x) V_\nu^b(0) | 0 \rangle \equiv (2\pi)^{-3} \int d^4p\, \theta(p_0) e^{ip\cdot x} \left[\rho_{ab}^{(1)}(p^2)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \rho_{ab}^{(0)}(p^2) p_\mu p_\nu\right]. \tag{21}$$

Again, it is easy to show that both commutators in Eq. (19) have the value

$$i\delta^4(x-y) f^{ABC} V_i^C(x) - \tfrac{1}{4} i \partial_i(x)\delta^4(x-y) \times [\tilde{S}(f^A f^B - d^A d^B) S + \tilde{D}(f^A f^B - d^A d^B) D],$$

where all of the fields inside [ ] are functions of y.

Finally, we must show that

$$[\partial_0 V_i^A(x) - \partial_i V_0^A(x),\ V_j^B(y)]\,\delta(x_0 - y_0) = [\partial_0 A_i^A(x) - \partial_i A_0^A(x),\ A_j^B(y)]\,\delta(x_0 - y_0), \tag{22}$$

since the second SFSR follows, viz.,

$$\int d\mu^2\{[\rho^{(1)}_{ab}(\mu^2)]_V - [\rho^{(1)}_{ab}(\mu^2)]_A\} = 0\,. \tag{23}$$

Instead of writing down a long and not particularly instructive expression for the commutators, we shall point out how corresponding terms in each current produce the same result. The expression for $\partial_0 A_i - \partial_i A_0$ differs from that for $\partial_0 V_i - \partial_i V_0$ by the term by term replacement of Π_s for Π_D, Π_v for Π_a, $\Delta_0 \mathcal{S}$ for $\Delta_0 \mathfrak{D}$, etc., while the currents also differ in the corresponding replacement of fields. Thus, in the axial-vector commutator, the even- and odd-parity fields and momenta are removed in pairs, leaving exactly the same expression as in the vector commutator. For the case of the terms involving $\Delta_0 \mathcal{S}$ and $\Delta_0 \mathfrak{D}$, this is slightly involved in practice, as they have to be expressed in terms of all the other fields and momenta, but the principle still holds good. Thus, Eq. (22) is correct, and the second sum rule follows.

V. CONCLUDING REMARKS

One interesting feature of the above proofs is that there is no reason for the spectral-function integrals to be equal for different values of I spin, as happens in the proof by Lee *et al.*; the loss of this particular constraint is welcome, since it leads to some predictions involving the weak decay constants f_K and f_π, and the K_{l3} constant $f_+(0)$, which are not in too good agreement with experiment.[6] Thus, one of the main reasons for suspecting the accuracy of the sum rules has been removed, while the grounds for believing the validity of the formulas have been strengthened, since the model used here is renormalizable.

Another point requiring comment is that the above proof makes no reference to the potential part of $\mathcal{L}$, whereas one may expect symmetry-breaking terms to have an effect, e.g., a general condition for the validity of the second sum rule requires the equality of double ETCs of the space components of the currents with the total Hamiltonian.[6] In this case, we must find a reason for parts of these ETCs to vanish, i.e.,

$$[V_i^A(x), [V_i^B(0), H_V]] = [A_i^A(x), [A_i^B(0), H_V]]\,,$$

where H_V is the potential part of H, otherwise the result could not be independent of the potential. At first sight, this may seem obvious, since V_i, A_i, and H_V are constructed out of canonical fields and their space derivatives, but no conjugate momenta appear anywhere.

However, when dealing with multilinear combinations of fields at the same space-time point, great care must be exercised with respect to the field canonical commutation relations.[10] Okubo[11] has shown that inconsistencies arise in the current commutation relations when fermion fields are involved, for example (1) in the quark model, (2) when there is derivative coupling of spin-0 fields to fermions, (3) in σ-type models, with a possible exception when the fermions have no bare mass. In the present model, V_i and A_i do not involve fermions at all, there is no derivative coupling, and the fermions do have zero bare mass. Thus, the known inconsistent cases do not apply here, and it may well be true that the above ETCs vanish (and also that, in the direct proofs, there are no Schwinger terms apart from those explicitly indicated).

Actually, our direct proof of the sum rules ties in well with confirmation of their validity from a rather different approach. One of the uses to which both sum rules have been put in the past is in the calculation of the pion electromagnetic mass difference,[12] and recently, interest in similar mass-type calculations has been revived by various groups[13] in the context of gauge theories. In fact, after completing the present work, we received a paper by Bars, Halpern, and Lane[9] in which they show that both sum rules must be true if the pion, treated as a pseudo-Goldstone boson in a renormalizable model, is to have a finite mass. Their result, which applies to a large class of renormalizable models (including the one considered here), states that (1) renormalizability always implies the first sum rule and (2) if the currents under consideration describe a spontaneously broken (or conserved) symmetry, then the second sum rule is also satisfied for all components of the symmetry—and is independent of the constituents of the currents, bosons, or fermions. I should like to thank Professor I. Bars for bringing this to my attention.

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[1]I. Bars, M. B. Halpern, and M. Yoshimura, Phys. Rev. Lett. 29, 969 (1972); Phys. Rev. D 7, 1233 (1973).

[2]J. C. Pati and A. Salam, Phys. Rev. D 8, 1240 (1973).

[3]P. W. Higgs, Phys. Rev. 145, 1156 (1966); T. W. B. Kibble, *ibid.* 155, 1554 (1967).

[4]Particle Data Group, Rev. Mod. Phys. 45, S1 (1973).

[5]S. Weinberg, Phys. Rev. Lett. 18, 507 (1967).

[6]H. T. Nieh, Phys. Rev. 163, 1769 (1967).

[7]T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Lett. 18, 1029 (1967).

[8]M. Gell-Mann, Phys. 1, 63 (1964).

[9]I. Bars, M. B. Halpern, and K. Lane, Nucl. Phys. B65, 518 (1973).

[10]J. Schwinger, Phys. Rev. Lett. 3, 296 (1959).

[11]S. Okubo, Nuovo Cimento 44A, 1015 (1966).

[12]T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Lett. 18, 759 (1967).

[13]D. A. Dicus and V. S. Mathur, Phys. Rev. D 7, 525 (1973); I. Bars and K. Lane, *ibid.* 8, 1169 (1973); 8, 1252 (1973).

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Spectral-function sum rules in asymptotically free theories*

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We show that the necessary and sufficient condition to derive Weinberg's second spectral-function sum rule within the framework of a Lagrangian theory invariant under a local non-Abelian gauge group G and the global chiral $SU(2)\otimes SU(2)$ group is that G should commute with $SU(2)\otimes SU(2)$. The $SU(3)\otimes SU(3)$ spectral-function sum rules for currents and sum rules involving spectral functions of scalar and pseudoscalar densities are also discussed.

I. INTRODUCTION

The discovery of asymptotic freedom in non-Abelian gauge theories by Gross and Wilczek[1] and by Politzer[2] has already led to important results in understanding Bjorken scaling in electroproduction. The success of current algebra, on the other hand, suggests the relevance of chiral symmetries to strong interactions. The simplest synthesis of these considerations is to assume that the strong-interaction Lagrangian is locally invariant under some non-Abelian gauge group G, and also globally invariant (or approximately invariant) under the chiral $SU(2)\otimes SU(2)$ group. It has been suggested in the literature that the gauge group should commute with the chiral group. The motivation for this suggestion comes from the fact that if one attempts to break the gauge symmetry by the Higgs mechanism (to avoid massless gluons), one also seems to lose[1,2] asymptotic freedom. It has been conjectured by Weinberg[3] and by Gross and Wilczek[4] that the gauge group G may not be broken at all, but a certain "shielding mechanism" may be at work due to the rather serious nature of the infrared-divergence problem associated with the non-Abelian symmetry, whereby only those particle states that transform as singlets under G can be observed (which would include all the observed hadrons, if G commutes with isospin and charge), but the massless gluons (and also quarks) which are not singlets under G are unobservable. This is an attractive idea, but whether it works or not has yet to be demonstrated. It has also been shown by Weinberg[5] that strong interactions generated by a non-Abelian gauge symmetry can be incorporated into the unified theory of weak and electromagnetic interactions in a manner which naturally conserves parity, strangeness, etc. Among other conditions, this synthesis requires that G should commute with G_W, the weak and electromagnetic gauge symmetry group, which contains charge as a generator.

In the present paper, we wish to study more directly the relationship (or lack of it) of the gauge group G and the chiral symmetry. The extra information comes from considerations of the Weinberg sum rules.[6] The first sum rule is a statement about Schwinger terms and follows from current algebra[7] without any constraints on G. However, it is well known that the second Weinberg sum rule is model-dependent. The main result of our paper is to show that the necessary and sufficient condition under which the second sum rule can be derived is that G should commute with $SU(2)\otimes SU(2)$. It should be pointed out that the second sum rule plays a crucial role in the calculations of Das *et al.*[8] in proving that the mass difference between $\pi^\pm$ and π^0 is finite in the