PHYSICAL REVIEW D VOLUME 9, NUMBER 8 15 APRIL 1974

Causality, current algebra, and fixed poles

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(Received 11 July 1973)

We study the causality restrictions on the structure of the nonforward matrix element of the commutator of two conserved isovector currents. We give a fixed-pole interpretation of the noncausal contribution, derive a set of causality and current-algebra sum rules, and determine the structure of the equal-time commutator.

I. INTRODUCTION

Meyer and Suura[1] have studied the structure of the equal-time commutator of two conserved isovector currents with particular reference to causality. They were able to determine the form of the equal-time commutator between two single-particle states of equal momenta. Their work shows that although the commutator is causal, the invariant amplitudes occurring in its covariant decomposition need not, in general, be causal. In particular they were able to identify the noncausal part and deduce a number of causality sum rules. Their main tool is the causal Jost-Lehmann-Dyson (JLD) representation.[2]

It is our purpose in this paper to extend the work of Meyer and Suura to nonforward matrix elements, restricting ourselves to conserved currents and single-particle states of equal mass. We demonstrate (see Appendix A) that all the invariant amplitudes, except one (denoted by A_2^{ij}), are causal. The noncausal part $A_2^{ij,\text{n.c.}}$ in A_2^{ij} is explicitly identified. We verify that $A_2^{ij,\text{n.c.}}$ is annihilated by the operator acting on it so that the commutator remains causal. This work is accomplished in Sec. II. In Sec. III we interpret this noncausal part as a fixed pole in the angular momentum plane at $J=1$. The residue of this fixed pole is completely determined by current algebra. In the same section we discuss the connection of this work to that of Bronzan *et al.*[3] and Singh[4] who based their work on the Fubini–Dashen–Gell-Mann (FDG) sum rule.[5,6] Our work shows that current algebra does not necessarily require that the amplitude A_2^{ij} possesses a fixed pole at $J=1$.

In Sec. IV we discuss the sum rules that are implied by causality and a certain assumption on the asymptotic behavior of the JLD spectral functions. In particular we find two sum rules involving the antisymmetric amplitude $A_2^{[ij]}$ that are completely determined by current algebra. The structure of the equal-time commutator is then determined using these sum rules. We find that in the time-space equal-time commutator, the only Schwinger terms that arise are symmetric in the internal indices. This is not true of the space-space equal-time commutator, the expression for which is given in Appendix B. The existence of the equal-time commutator is seen to imply a certain scaling behavior [see Eq. (4.22)] for the absorptive functions which in turn determines the equal-time commutator in terms of the scaling limits. A particular assumption on scaling behavior[7] leads to a simple form for the time-space equal-time commutator that involves a single gradient Schwinger term.

II. CAUSALITY AND CURRENT ALGEBRA

Consider the matrix element $C_{\mu\nu}^{ij}$ of the commutator of two conserved vector currents between spinless single-particle states of momenta p_1 and p_2 $(p_1^2=p_2^2=1)$,

$$C_{\mu\nu}^{ij}(x)=\langle p_2|[J_\mu^i(\tfrac{1}{2}x),J_\nu^j(-\tfrac{1}{2}x)]|p_1\rangle\,. \tag{2.1}$$

The Fourier transform $W_{\mu\nu}^{ij}(Q)$ of $C_{\mu\nu}^{ij}$, defined by

$$W_{\mu\nu}^{ij}(Q)=\frac{1}{2\pi}\int e^{iQ\cdot x}C_{\mu\nu}^{ij}(x)d^4x, \tag{2.2}$$

may be written in the form

$$W_{\mu\nu}^{ij}=\sum_{k=1}^{5}L_{\mu\nu}^{(k)}A_k^{ij}\,, \tag{2.3}$$

where A_k^{ij} are invariant functions of

$$\nu=Q\cdot P,\quad t=\Delta^2,\quad Q^2,\quad \rho=\Delta\cdot Q,$$

with

$$P=\tfrac{1}{2}(p_1+p_2),\quad \Delta=p_1-p_2,$$

and the covariants $L_{\mu\nu}^{(k)}$ are given by

$$L_{\mu\nu}^{(1)}=(Q_\mu-\tfrac{1}{2}\Delta_\mu)(Q_\nu+\tfrac{1}{2}\Delta_\nu)-(Q^2-\tfrac{1}{4}t)g_{\mu\nu}\,,$$

$$L_{\mu\nu}^{(2)}=(Q^2-\tfrac{1}{4}t)P_\mu P_\nu+\nu^2 g_{\mu\nu}-\nu[P_\mu(Q_\nu+\tfrac{1}{2}\Delta_\nu)+(Q_\mu-\tfrac{1}{2}\Delta_\mu)P_\nu]\,,$$

$$L_{\mu\nu}^{(3)}=(Q^2-\tfrac{1}{4}t)P_\mu\Delta_\nu+\nu(\rho-\tfrac{1}{2}t)g_{\mu\nu}-(\rho-\tfrac{1}{2}t)P_\mu(Q_\nu+\tfrac{1}{2}\Delta_\nu)-\nu(Q_\mu-\tfrac{1}{2}\Delta_\mu)\Delta_\nu\,,$$

$$L^{(4)}_{\mu\nu} = (Q^2 - \tfrac{1}{4}t)\Delta_\mu P_\nu + \nu(\rho + \tfrac{1}{2}t)g_{\mu\nu} - (\rho + \tfrac{1}{2}t)(Q_\mu - \tfrac{1}{2}\Delta_\mu)P_\nu - \nu\Delta_\mu(Q_\nu + \tfrac{1}{2}\Delta_\nu), \tag{2.4}$$

$$L^{(5)}_{\mu\nu} = (Q^2 - \tfrac{1}{4}t)\Delta_\mu\Delta_\nu + (\rho^2 - \tfrac{1}{4}t^2)g_{\mu\nu} - (\rho + \tfrac{1}{2}t)(Q_\mu - \tfrac{1}{2}\Delta_\mu)\Delta_\nu - (\rho - \tfrac{1}{2}t)\Delta_\mu(Q_\nu + \tfrac{1}{2}\Delta_\nu).$$

Although the commutator (2.1) is causal, i.e., vanishes for $x^2 < 0$, the invariants A_k^{ij} need not, in general, be causal. However, any noncausal parts must be annihilated by the operators acting on them to give $C_{\mu\nu}^{ij}$. Denoting the Fourier transform of A_k^{ij} by $\tilde{A}_k^{ij}$, we show in Appendix A that if the noncausal parts in $\tilde{A}_k^{ij}$, $k = 1, 2, 3, 5$ are to vanish for $|\vec{x}| \to \infty$,[1,8] then these noncausal parts must vanish identically. The invariant amplitudes A_k^{ij}, $k = 1, 3, 4, 5$ are therefore causal.

The invariant A_2^{ij} may, however, possess a noncausal part, that vanishes as $|\vec{x}| \to \infty$, which is annihilated by the operator acting on it. It is, in fact, not difficult to show that such a noncausal part is necessary if the commutator

$$\langle p_2 | [J_0^i(\tfrac{1}{2}x), J_0^j(-\tfrac{1}{2}x)] | p_1\rangle \delta(x_0) \tag{2.5}$$

is nonzero. To do this, we use the fact that a causal function $A(Q)$ satisfies the sum rule[1,8,9]

$$\int_{-\infty}^{\infty} A(Q)dQ_0 = 0\,. \tag{2.6}$$

The Fourier transform E_{00} of (2.5) is

$$E_{00}^{ij} = \int_{-\infty}^{\infty} W_{00}^{ij}(Q)dQ_0\,. \tag{2.7}$$

Using (2.3), (2.4), and (2.6), taking into account that A_k^{ij}, $k \neq 2$, are causal, we obtain from (2.7)

$$E_{00}^{ij} = [(\vec{P}\cdot\vec{Q})^2 - (\tfrac{1}{4}t + \vec{Q}^2)P_0^{\,2}] \int_{-\infty}^{\infty} A_2^{ij,\text{n.c.}}(Q)dQ_0\,, \tag{2.8}$$

where $A_2^{ij,\text{n.c.}}$ denotes the noncausal part of A_2^{ij}. Thus $E_{00}^{ij} \neq 0$ implies that $A_2^{ij,\text{n.c.}} \neq 0$.

Assuming current algebra

$$E_{00}^{ij} = if^{ijk}F_k(t)P_0\,, \tag{2.9}$$

we have

$$\int_{-\infty}^{\infty} A_2^{ij,\text{n.c.}}dQ_0 = \frac{if^{ijk}F_k(t)P_0}{[(\vec{P}\cdot\vec{Q})^2 - (\tfrac{1}{4}t + \vec{Q}^2)P_0^{\,2}]}\,. \tag{2.10}$$

We note that Eq. (2.10) determines $A_2^{ij,\text{n.c.}}$ up to an additive arbitrary causal function, since such a function must satisfy (2.6). Thus the noncausal part $A_2^{ij,\text{n.c.}}$ is given by

$$A_2^{ij,\text{n.c.}} = \frac{if^{ijk}F_k(t)}{Q^2 - \tfrac{1}{4}t} \times [\epsilon(P_0 + Q_0)\delta(Q^2 - \tfrac{1}{4}t + 2\nu) + \epsilon(P_0 - Q_0)\delta(Q^2 - \tfrac{1}{4}t - 2\nu)]\,, \tag{2.11}$$

since this expression satisfies Eq. (2.10). One observes that (2.11) is a straightforward generalization of the noncausal function obtained in the forward case.[1]

The Fourier transform $\tilde{A}_2^{ij,\text{n.c.}}$ may explicitly be shown to be nonzero for $x^2 < 0$ and to tend to zero as $|\vec{x}| \to \infty$, for in the frame $\vec{P} = \vec{0}$,

$$\tilde{A}_2^{ij,\text{n.c.}} = -\frac{4\pi i f^{ijk}F_k(t)}{|\vec{x}|}\left[\frac{\cos P_0x_0}{P_0}\int_0^\infty \frac{r\sin(r|\vec{x}|)\cos[x_0(r^2 + P_0^{\,2} + \tfrac{1}{4}t)^{1/2}]}{(r^2 + \tfrac{1}{4}t)}dr + \sin P_0x_0 \int_0^\infty \frac{r\sin(r|\vec{x}|)\sin[x_0(r^2 + P_0^{\,2} + \tfrac{1}{4}t)^{1/2}]}{(r^2 + \tfrac{1}{4}t)(r^2 + P_0^2 + \tfrac{1}{4}t)^{1/2}}dr\right]. \tag{2.12}$$

For $x^2 < 0$ this gives[10]

$$\tilde{A}_2^{ij,\text{n.c.}} = 2if^{ijk}F_k(t)[(P\cdot x)^2 - x^2P^2]^{-1/2} \times \exp\{-i(-\tfrac{1}{4}t)^{1/2}[(P^2)^{-1}(P\cdot x)^2 - x^2]^{1/2}\}\,, \tag{2.13}$$

which is not identically vanishing and tends to zero as $|\vec{x}| \to \infty$ like $|\vec{x}|^{-1}$.

Next we verify that $\tilde{A}_2^{ij,\text{n.c.}}$, for $x^2 < 0$, as given by Eq. (2.13) is annihilated by the operator $\tilde{L}_{\mu\nu}^{(2)}$ so that the commutator $C_{\mu\nu}^{ij}$ remains causal. The operator $\tilde{L}_{\mu\nu}^{(2)}$ is given by

$$\tilde{L}_{\mu\nu}^{(2)} = -(\Box + \tfrac{1}{4}t)P_\mu P_\nu - (P\cdot\partial)^2 g_{\mu\nu} - iP\cdot\partial[P_\mu(i\partial_\nu + \tfrac{1}{2}\Delta_\nu) + (i\partial_\mu - \tfrac{1}{2}\Delta_\mu)P_\nu]\,. \tag{2.14}$$

In the frame $\vec{P} = \vec{0}$, this becomes

$$\tilde{L}_{\mu\nu}^{(2)} = P_0^{\,2}\{-(\Box + \tfrac{1}{4}t)g_{\mu 0}g_{\nu 0} - \partial_0^{\,2}g_{\mu\nu} - i\partial_0[g_{\mu 0}(i\partial_\nu + \tfrac{1}{2}\Delta_\nu) + i(\partial_\mu - \tfrac{1}{2}\Delta_\mu)g_{\nu 0}]\}\,, \tag{2.15}$$

whereas (2.13) gives

$$\tilde{A}_2^{ij,\text{n.c.}} = 2iP_0^{\,-1}f^{ijk}F_k(t)|\vec{x}|^{-1}\exp[-i(-\tfrac{1}{4}t)^{1/2}|\vec{x}|]\,. \tag{2.16}$$

Since (2.16) is independent of x_0, we have

$$\tilde{L}_{\mu\nu}^{(2)}\tilde{A}_2^{ij,\text{n.c.}} = 2iP_0 f^{ijk}F_k(t)g_{\mu 0}g_{\nu 0}(\nabla^2 - \tfrac{1}{4}t) \times\{|\vec{x}|^{-1}\exp[-i(-\tfrac{1}{4}t)^{1/2}|\vec{x}|]\} = 0\,. \tag{2.17}$$

Thus we have established the existence of the noncausal part $A_2^{ij,\text{n.c.}}$ related to the right-hand side of the time-time current algebra (2.9) by Eq. (2.10). The interpretation of this noncausal part as a fixed pole in the angular momentum plane at $J=1$ will be given in Sec. III.

III. FIXED-POLE INTERPRETATION

Denoting by T_k^{ij} the invariant amplitudes, the absorptive parts of which are given by A_k^{ij}, write

$$T_2^{ij}=T_2^{\{ij\}}+iT_2^{[ij]}, \tag{3.1}$$

where $T_2^{\{ij\}}$ and $T_2^{[ij]}$ are, respectively, the symmetric and antisymmetric parts of T_2^{ij}. We calculate the contribution of the noncausal part $A_2^{ij,\text{n.c.}}$ to $T_2^{[ij]}$:

$$T_2^{[ij],\text{n.c.}}=\frac{2\nu f^{ijk}F_k(t)}{Q^2-\frac{1}{4}t}\times\int_0^\infty\frac{1}{\nu'^2-\nu^2}\left[\epsilon(P^2+\nu')\delta(Q^2-\tfrac{1}{4}t+2\nu')+(\nu'\to-\nu')\right]d\nu',$$

i.e.,

$$T_2^{[ij],\text{n.c.}}=\frac{\nu f^{ijk}F_k(t)}{(Q^2-\frac{1}{4}t)[\frac{1}{4}(Q^2-\frac{1}{4}t)^2-\nu^2]}, \tag{3.2}$$

where we have set $\epsilon(P^2-\frac{1}{2}Q^2+\frac{1}{8}t)\equiv1$, since we are restricting our considerations to the case of currents with spacelike momenta. The asymptotic form of (3.2) as $\nu\to\infty$ at fixed Q^2 and t is

$$T_2{}^{[ij],\text{n.c.}}\sim\frac{-f^{ijk}F_k(t)}{(Q^2-\frac{1}{4}t)\nu}. \tag{3.3}$$

We recognize this as the asymptotic behavior of a fixed pole in the J plane at $J=1$. The residue of this contribution to the fixed pole is completely determined by the current-algebra term and is in fact equal to the residue of the fixed pole found by Bronzan *et al.*[3] and by Singh.[4,11]

Since the expression (2.11) gives $A_2^{ij,\text{n.c.}}$ for all values of ν, one may, in fact, explicitly calculate its partial-wave projection $F_J^{ij,\text{n.c.}}$ in order to check the presence of the fixed pole at $J=1$:

$$F_J^{ij,\text{n.c.}}(t,Q^2,\Delta\cdot Q)=\int\mathcal{C}^J(z)A_2^{ij,\text{n.c.}}(z,t,Q^2,\rho)dz, \tag{3.4}$$

where[3]

$$T_2^{ij,\text{n.c.}}=\sum_{J=2}^\infty(2J+1)P_J''(z)F_J^{ij,\text{n.c.}}(t,\ldots), \tag{3.5}$$

$$z=-2\left(1-\frac{4}{t}\right)^{-1/2}(\rho^2-Q^2t)^{-1/2}\nu\equiv g(t,\rho,Q^2)\nu, \tag{3.6}$$

and

$$\mathcal{C}^J(z)=[(2J-1)(2J+1)(2J+3)]^{-1}\times[(2J+3)Q_{J-2}(z)-2(2J+1)Q_J(z)+(2J-1)Q_{J+2}(z)]. \tag{3.7}$$

Inserting the expression (2.11) into (3.4), one obtains near $J=1$

$$F_J^{ij,\text{n.c.}}\sim\tfrac{2}{3}i\frac{g}{Q^2-\frac{1}{4}t}f^{ijk}\frac{F_k(t)}{J-1}, \tag{3.8}$$

exhibiting the presence of the fixed pole at $J=1$ in the partial-wave amplitude.

We observe that the expression (3.3), which we find for the contribution of the noncausal part to the fixed pole, coincides with the fixed pole term obtained by Bronzan *et al.*[3] for the whole amplitude $T_2^{[ij]}$. These authors derive their result by using the FDG sum rule whereas we only use the causality condition (2.6) together with the current-algebra relation (2.9). These relations by themselves are not sufficient to guarantee the validity of the FDG sum rule. In fact, if one assumes that (3.3) gives the complete contribution of the fixed pole to the amplitude $T_2^{[ij]}$, then one immediately obtains the FDG sum rule from (2.6) and (2.9), for such an assumption implies that the causal part is free of a fixed pole at $J=1$ and must therefore satisfy

$$\int A_2^{[ij],c}(\nu,t,Q^2,\rho)d\nu=0. \tag{3.9}$$

Since the noncausal part already gives

$$\int A_2^{[ij],\text{n.c.}}(\nu,t,Q^2,\rho)d\nu=\frac{f^{ijk}F_k(t)}{Q^2-\frac{1}{4}t}, \tag{3.10}$$

one has the FDG sum rule on addition. However, one is in general only entitled to the sum rule

$$\int A_2^{[ij]}(\nu,t,Q^2,\rho)dQ_0=\frac{f^{ijk}F_k(t)P_0}{[(\vec{P}\cdot\vec{Q})^2-(\frac{1}{4}t+\vec{Q}^2)P_0{}^2]} \tag{3.11}$$

as a consequence of causality and current algebra.

We remark that $A_2^{[ij],c}$ must possess a pole at $t=m_\rho{}^2$, since $A_2^{[ij],\text{n.c.}}$ has such a pole, whereas A_2^{ij} is free from it. According to the FDG sum rule $A_2^{[ij],c}$ has pure Regge behavior,

$$A_2^{[ij],c}\sim R^{[ij]}(t)\nu^{\alpha(t)-2}, \tag{3.12}$$

and satisfies the superconvergence relation (3.9). One then must require $R^{[ij]}(m_\rho{}^2)\neq0$ so that the divergence in the integral (3.9) as $\alpha(t)\to1$ is canceled by the divergence due to the pole at $t=m_\rho{}^2$ in the integrand. This fixed-pole–moving-pole collaboration at $t=m_\rho{}^2$ has previously been noted by Bronzan *et al.*[3]

One observes that since the integrand in

$$\int A_2^{[ij],c} dQ_0 = 0 \tag{3.13}$$

diverges as $t \to m_\rho^2$, the amplitude $\bar{A}_2^{[ij],c} \equiv A_2^{[ij],c}$ − (contribution of pole at $t = m_\rho^2$) must possess an asymptotic behavior as $Q_0 \to \infty$ that allows a divergence in its integral over Q_0 at $t = m_\rho^2$ which cancels this pole.

We finally remark that our considerations show that current algebra does *not* require that the amplitude $T_2^{[ij]}$ possesses a fixed pole at $J = 1$, since $T_2^{[ij],c}$ may asymptotically cancel the term (3.3). The assumption of the existence of this fixed pole in $T_2^{[ij]}$ with residue as in (3.3) is thus completely equivalent to the FDG sum rule, i.e., to the assumptions (besides current algebra) that lead to the FDG sum rule.

IV. CAUSALITY SUM RULES AND STRUCTURE OF THE EQUAL-TIME COMMUTATOR

Using the JLD representation for a causal function A,

$$A(Q) = \int d^4u\, ds\, \epsilon(Q_0 - u_0)\delta((Q-u)^2 - s)\psi(u,s), \tag{4.1}$$

one finds, on assuming the possibility of exchanging integrals, that

$$\int A\, dQ_0 = 0. \tag{4.2}$$

It has been shown by Meyer and Suura[1] that such a causality sum rule holds provided that

$$\lim_{s\to\infty} \psi(u,s) = 0. \tag{4.3}$$

Similarly, one has

$$\int Q_0 A(Q) dQ_0 = \int d^4u\, ds\, \psi(u,s) \tag{4.4}$$

and

$$\int Q_0^2 A(Q) dQ_0 = \int d^4u\, ds\, u_0 \psi(u,s), \tag{4.5}$$

provided that $\lim_{s\to\infty} s\psi(u,s) = 0$. Since the spectral function $\psi(u,s)$ is Lorentz-invariant, it is clear that the right-hand sides of (4.4) and (4.5) transform like a scalar and a time component of a Lorentz vector, respectively.[12] These equations may therefore be written in the form

$$\int Q_0 A(Q) dQ_0 = b, \tag{4.6}$$

$$\int Q_0^2 A(Q) dQ_0 = cP_0 + d\Delta_0, \tag{4.7}$$

where b, c, and d are invariants. We also note that $b = 0$ when $\psi(u,s)$ is odd in u, and that $c = d = 0$ when $\psi(u,s)$ is even in u. In fact, the amplitudes $A_k^{\{ij\}}$ and $A_k^{[ij]}$ are, respectively, represented by even and odd (in u) spectral functions $\psi(u,s)$.[1] Thus, one has the following causality sum rules for the amplitudes A_k^{ij}, $k = 1, 3, 4, 5$:

$$\int A_k^{ij}(Q) dQ_0 = 0, \tag{4.8}$$

$$\int Q_0 A_k^{ij}(Q) dQ_0 = b_k^{ij}(t), \tag{4.9}$$

$$\int Q_0^2 A_k^{ij}(Q) dQ_0 = c_k^{ij}(t) P_0 + d_k^{ij}(t)\Delta_0, \tag{4.10}$$

where

$$b_k^{[ij]}(t) = 0, \quad c_k^{\{ij\}}(t) = d_k^{\{ij\}}(t) = 0. \tag{4.11}$$

For $k = 2$, one has

$$\int A_2^{\{ij\}}(Q) dQ_0 = \int Q_0^2 A_2^{\{ij\}}(Q) dQ_0 = 0, \tag{4.12}$$

$$\int Q_0 A_2^{\{ij\}}(Q) dQ_0 = b_2^{\{ij\}}(t), \tag{4.13}$$

$$\int A_2^{[ij]}(Q) dQ_0 = \frac{f^{ijk} F_k(t) P_0}{[(\vec{P}\cdot\vec{Q})^2 - (\frac{1}{4}t + \vec{Q}^2) P_0^2]}, \tag{4.14}$$

$$\int Q_0 A_2^{[ij]}(Q) dQ_0 = \frac{f^{ijk} F_k(t)\, \vec{P}\cdot\vec{Q}}{[(\vec{P}\cdot\vec{Q})^2 - (\frac{1}{4}t + \vec{Q}^2) P_0^2]}, \tag{4.15}$$

$$\int Q_0^2 A_2^{[ij]}(Q) dQ_0 = \frac{(\vec{Q}^2 + \frac{1}{4}t) f^{ijk} F_k(t) P_0}{[(\vec{P}\cdot\vec{Q})^2 - (\frac{1}{4}t + \vec{Q}^2) P_0^2]} + c_2^{[ij]}(t) P_0 + d_2^{[ij]}(t)\Delta_0. \tag{4.16}$$

It is interesting to note that one may eliminate the current-algebra term from the sum rules (4.14) and (4.15), obtaining the sum rule

$$\int \nu A_2^{[ij]}(Q) dQ_0 = 0. \tag{4.17}$$

Using the above relations one may directly calculate the equal-time commutator E_{0r}^{ij} defined by

$$E_{0r}^{ij} = \int e^{iQ\cdot x} \delta(x_0) \langle p_2 | [J_0^i(\tfrac{1}{2}x), J_r^j(-\tfrac{1}{2}x)] | p_1 \rangle d^4x. \tag{4.18}$$

One finds that this is given by the expression

$$E_{0r}^{ij} = [if^{ijk}F_k(t) + (\tfrac{1}{2}P_0\Delta_0 + \vec{P}\cdot\vec{Q})b_2^{\{ij\}}(t) + (\tfrac{1}{2}\Delta_0^2 + \vec{\Delta}\cdot\vec{Q} - \tfrac{1}{2}t)b_4^{\{ij\}}(t)]\,P_r$$
$$+[\tfrac{1}{2}b_1^{\{ij\}}(t) - \tfrac{1}{2}P_0^2 b_2^{\{ij\}}(t) + \vec{P}\cdot\vec{Q}\,b_3^{\{ij\}}(t) - \tfrac{1}{2}P_0\Delta_0 b_4^{\{ij\}}(t) + (\vec{\Delta}\cdot\vec{Q} - \tfrac{1}{2}t)b_5^{\{ij\}}(t)]\Delta_r$$
$$+[b_1^{\{ij\}}(t) - P_0^2 b_2^{\{ij\}}(t) - \Delta_0 P_0(b_3^{\{ij\}}(t) + b_4^{\{ij\}}(t)) - \Delta_0^2 b_5^{\{ij\}}(t)]\,Q_r\,. \tag{4.19}$$

It is immediately seen that all the Schwinger terms that arise in this equal-time commutator are symmetric in the internal indices. This is not true of the space-space equal-time commutator E_{rs}^{ij}, the expression for which is given in Appendix B.

By the change of variable $Q_0 \to x$,

$$Q_0 = 2P_0 x + P_0^{-1}\vec{P}\cdot\vec{Q},$$

the integrals (4.9) defining the invariants $b_k^{\{ij\}}(t)$ may be written in the form

$$b_k^{\{ij\}}(t) = 4P_0^2\int xA_k^{\{ij\}}(2P_0^2x, t, 4P_0^2x^2 + 4\vec{P}\cdot\vec{Q}x + \mu, 2\lambda P_0^2 x + \gamma)dx, \tag{4.20}$$

where

$$\mu = P_0^{-2}(\vec{P}\cdot\vec{Q})^2 - \vec{Q}^2, \quad \lambda = P_0^{-1}\Delta_0, \qquad \gamma = \lambda\vec{P}\cdot\vec{Q} - \vec{\Delta}\cdot\vec{Q}. \tag{4.21}$$

Since $b_k^{\{ij\}}(t)$ are independent of the parameters P_0, μ, λ, and γ, we may evaluate them in any suitable limit of these variables.[13] Taking the limit $P_0 \to \infty$, we see that finite values for $b_k^{\{ij\}}(t)$ result only if

$$\nu A_k^{\{ij\}}(\nu, t, Q^2, \Delta\cdot Q) \to F_k^{\{ij\}}(t, \omega_1, \omega_2) \tag{4.22}$$

as $\nu \to \infty$ at fixed $Q^2/2\nu = \omega_1$, $\Delta\cdot Q/\nu = \omega_2$, and t. The behavior (4.22) does not preclude the identical vanishing of the right-hand side. In this limit, then, one obtains from (4.21) and (4.22)

$$b_k^{\{ij\}}(t) = 2\int F_k^{\{ij\}}(t, x, \lambda)dx. \tag{4.23}$$

The invariants $b_k^{\{ij\}}(t)$ may thus be determined by the scaling behavior of the absorptive functions. On the basis of the existence of certain lightlike restrictions, Georgelin *et al.*[7] postulate definite scaling behavior for the electromagnetic structure functions in the nonforward direction. Adopting their hypothesis, we postulate

$$\nu A_1^{\{ij\}}(\nu, t, Q^2, \Delta\cdot Q) \to F_1^{\{ij\}}(t, \omega_1, \omega_2)$$
$$\nu^2 A_k^{\{ij\}}(\nu, t, Q^2, \Delta\cdot Q) \to G_k^{\{ij\}}(t, \omega_1, \omega_2)\,, \tag{4.24}$$
$$k = 2, \ldots, 5\,.$$

Thus in Eq. (4.22) we have $F_k^{\{ij\}} = 0$ for $k = 2, \ldots, 5$, yielding

$$b_k^{\{ij\}}(t) = 0\,, \quad k = 2, \ldots, 5\,. \tag{4.25}$$

When substituted in (4.19), this gives

$$E_{0r}^{ij} = if^{ijk}F_k(t)P_r + b_1^{\{ij\}}(t)(Q_r + \tfrac{1}{2}\Delta_r)\,. \tag{4.26}$$

It thus appears that any experimental verification of (4.26) is also a test of the scaling hypothesis (4.24). One further notes that only a single derivative of a δ function appears in the commutator besides the canonical term.

ACKNOWLEDGMENTS

We wish to thank Professor V. de Alfaro for a discussion and Professor C. Fronsdal for reading the manuscript. We also thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, and the Swedish International Development Authority for making possible our associateships at the ICTP.

APPENDIX A: CAUSALITY OF THE INVARIANT AMPLITUDES

In this appendix we establish the causal character of $\tilde{A}_k^{ij}$, $k = 1, 3, 4, 5$. This is done by showing that any noncausal parts in these amplitudes which vanish as $|\vec{x}| \to \infty$ must vanish identically. One observes that, since $C_{\mu\nu}^{ij}$ is causal, the noncausal parts $\tilde{A}_k^{ij,\text{n.c.}}$ in $\tilde{A}_k^{ij}$ must satisfy

$$\tilde{L}_{\mu\nu}^{(k)}\tilde{A}_k^{ij,\text{n.c.}} = 0\,, \quad x^2 < 0; \forall k\,. \tag{A1}$$

Consider this equation for $k = 1$:

$$[(i\partial_\mu - \tfrac{1}{2}\Delta_\mu)(i\partial_\nu + \tfrac{1}{2}\Delta_\nu) + (\Box + \tfrac{1}{4}t)g_{\mu\nu}]\tilde{A}_1^{\text{n.c.}} = 0\,, \tag{A2}$$

where we have dropped the internal-symmetry indices. We work in the frame $\Delta_\mu = 0$, $\mu \neq 1$ (note that $\Delta^2 < 0$). For $\mu = 0$, $\nu = i \neq 1$, Eq. (A2) gives

$$\partial_0\partial_i\tilde{A}_1^{\text{n.c.}} = 0\,, \quad i = 2, 3\,. \tag{A3}$$

Suppose that $\partial_0\tilde{A}_1^{\text{n.c.}} \neq 0$, then from (A3), $\partial_0\tilde{A}_1^{\text{n.c.}} = f(x_0, x_1)$. For $\mu = 1$, $\nu = 0$, (A2) gives

$$(i\partial_1 - \tfrac{1}{2}\Delta_1)f(x_0, x_1) = 0; \tag{A4}$$

therefore,

$$f(x_0, x_1) = g(x_0)\exp(-\tfrac{1}{2}ix^1\Delta_1)\,. \tag{A5}$$

Integrating (A5) with respect to x_0, one obtains

$$\tilde{A}_1^{\text{n.c.}}=h(x_0)\exp(-\tfrac{1}{2}ix^1\Delta_1)+l(x_1,x_2,x_3)\,. \tag{A6}$$

Observe that the two terms on the right-hand side of (A6) must be independently invariant; therefore $h(x_0)$ is invariant. Since $\tilde{A}_1^{\text{n.c.}}$ is a function of x^2, $x\cdot\Delta$, Δ^2, and $x\cdot P$, it is clear that $h(x_0)$ is constant. Similarly, $l(x_1,x_2,x_3)=l(x^1\Delta_1)$. Thus $\partial_0\tilde{A}_1^{\text{n.c.}}=0$, contradicting our original supposition. Hence $\tilde{A}_1^{\text{n.c.}}$ is a function of x_1, x_2, and x_3, and is therefore a function of $x\cdot\Delta=x^1\Delta_1$ only.

Taking $\mu=\nu=0$ in (A2) one obtains

$$(\partial_1^{\,2}+\tfrac{1}{4}\Delta_1^{\,2})\tilde{A}_1^{\text{n.c.}}(x^1\Delta_1)=0\,, \tag{A7}$$

the solution of which is

$$\tilde{A}_1^{\text{n.c.}}=\alpha\exp[\tfrac{1}{2}ix\cdot\Delta]+\beta\exp[-\tfrac{1}{2}ix\cdot\Delta]\,, \tag{A8}$$

where α,β are arbitrary constants. The condition $\tilde{A}_1^{\text{n.c.}}\to 0$ as $|\vec{x}|\to\infty$ requires $\alpha=\beta=0$, so that $\tilde{A}_1^{\text{n.c.}}\equiv 0$.

We now consider Eq. (A1) for $k=3$:

$$\begin{aligned}&[-(\Box+\tfrac{1}{4}t)P_\mu\Delta_\nu+iP\cdot\partial(i\Delta\cdot\partial-\tfrac{1}{2}t)g_{\mu\nu}\\&-(i\Delta\cdot\partial-\tfrac{1}{2}t)P_\mu(i\partial_\nu+\tfrac{1}{2}\Delta_\nu)-iP\cdot\partial(i\partial_\mu-\tfrac{1}{2}\Delta_\mu)\Delta_\nu]\tilde{A}_3^{\text{n.c.}}\\&=0\,.\end{aligned}$$

In the same frame, $\Delta_\mu=(0,\Delta_1,0,0)$, the equations for $\mu=\nu=0$, $\mu=1$, $\nu=0$, and $\mu=0$, $\nu=i\neq 1$ give, after some algebra,

$$(i\partial_1+\tfrac{1}{2}\Delta_1)\partial_k\tilde{A}_3^{\text{n.c.}}=0\quad\forall\,k\,. \tag{A9}$$

If $\partial_k\tilde{A}_3^{\text{n.c.}}=0$, then $\tilde{A}_3^{\text{n.c.}}$ is a constant and must therefore vanish by the asymptotic requirement. Otherwise, we have

$$\partial_k\tilde{A}_3^{\text{n.c.}}=f_k(x_0,x_2,x_3)\exp(+i\tfrac{1}{2}\Delta^1x_1)+g_k(x_0,x_2,x_3)\,. \tag{A10}$$

Integrating with respect to x_1, one obtains

$$\begin{aligned}\tilde{A}_3^{\text{n.c.}}=&\frac{2i}{\Delta_1}f_1(x_0,x_2,x_3)\exp(i\tfrac{1}{2}\Delta^1x_1)\\&+g_1(x_0,x_2,x_3)x_1+h_1(x_0,x_2,x_3)\,.\end{aligned} \tag{A11}$$

Observing that the three terms on the right-hand side of (A11) must be independently invariant, we conclude that $\tilde{A}_3^{\text{n.c.}}$ is of the form

$$\tilde{A}_3^{\text{n.c.}}=\frac{c_1}{(-\Delta^2)^{1/2}}\exp(+\tfrac{1}{2}i\Delta\cdot x)+c_2\frac{\Delta\cdot x}{(-\Delta^2)^{1/2}}+c_3\,, \tag{A12}$$

where c_1, c_2, and c_3 are constant. Thus $\tilde{A}_3^{\text{n.c.}}$ vanishes, again by the asymptotic requirement.

The analysis leading to the vanishing of $\tilde{A}_4^{\text{n.c.}}$ is similar to the previous case. For $A_5^{\text{n.c.}}$ the operator $\tilde{L}_{\mu\nu}^{(5)}$ gives, in the same frame for $\mu=\nu=0$, the equation

$$(\partial_1^{\,2}+\tfrac{1}{4}\Delta_1^{\,2})\tilde{A}_5^{\text{n.c.}}=0\,, \tag{A13}$$

i.e.,

$$\begin{aligned}\tilde{A}_5^{\text{n.c.}}=&\alpha_1(x_0,x_2,x_3)\exp(-i\tfrac{1}{2}\Delta\cdot x)\\&+\alpha_2(x_0,x_2,x_3)\exp(i\tfrac{1}{2}\Delta\cdot x)\,.\end{aligned} \tag{A14}$$

The invariant character of $\tilde{A}_5^{\text{n.c.}}$ implies that (A14) reduces to the form (A8). This completes the proof of the causality of the invariants A_k, $k\neq 2$.

APPENDIX B: EXPRESSION FOR E_{rs}^{ij}

$$\begin{aligned}E_{rs}^{ij}\equiv&\int e^{iQ\cdot x}\delta(x_0)\langle p_2|[J_r^i(\tfrac{1}{2}x),J_s^j(-\tfrac{1}{2}x)]|p_1\rangle d^4x\\=&E_1^{ij}P_rP_s+E_2^{ij}P_rQ_s+E_3^{ij}P_r\Delta_s+E_4^{ij}Q_rP_s+E_5^{ij}Q_rQ_s+E_6^{ij}Q_r\Delta_s+E_7^{ij}\Delta_rP_s+E_8^{ij}\Delta_rQ_s+E_9^{ij}\Delta_r\Delta_s+E_{10}^{ij}g_{rs}\,,\end{aligned}$$

where

$$E_1^{ij}=c_2^{[ij]}(t)P_0+d_2^{[ij]}(t)\Delta_0\,,$$

$$E_2^{ij}=-b_3^{\{ij\}}\Delta_0\,,$$

$$E_3^{ij}=c_3^{[ij]}(t)P_0+d^{[ij]}(t)\Delta_0-\tfrac{1}{2}b_3^{\{ij\}}\Delta_0\,,$$

$$E_4^{ij}=-b_4^{\{ij\}}\Delta_0\,,$$

$$E_5^{ij}=0\,,$$

$$E_6^{ij}=-(b_3^{\{ij\}}+b_5^{\{ij\}})\Delta_0\,,$$

$$E_7^{ij}=c_4^{[ij]}(t)P_0+[d_4^{[ij]}(t)+\tfrac{1}{2}b_4^{\{ij\}}]\Delta_0\,,$$

$$E_8^{ij}=-b_4^{\{ij\}}P_0-b_5^{\{ij\}}\Delta_0\,,$$

$$E_9^{ij}=[c_5^{[ij]}(t)+\tfrac{1}{2}b_3^{\{ij\}}-\tfrac{1}{2}b_4^{\{ij\}}]P_0+d_5^{[ij]}(t)\Delta_0\,,$$

$$\begin{aligned}E_{10}^{ij}=&-f^{ijk}F_k(t)P_0-c_1^{[ij]}(t)P_0-d_1^{[ij]}(t)\Delta_0+c_2^{[ij]}(t)P_0^{\,3}+d_2^{[ij]}(t)P_0^{\,2}\Delta_0+c_3^{[ij]}(t)P_0^{\,2}\Delta_0+d_3^{[ij]}(t)P_0\Delta_0^{\,2}\\&-b_3^{\{ij\}}(t)[(-\vec{\Delta}\cdot\vec{Q}+\tfrac{1}{2}t)P_0+\vec{P}\cdot\vec{Q}\Delta_0]\\&+b_4^{\{ij\}}(t)[(-\vec{\Delta}\cdot\vec{Q}+\tfrac{1}{2}t)P_0-\vec{P}\cdot\vec{Q}\Delta_0]+c_5^{[ij]}(t)P_0\Delta_0^{\,2}+d_5^{[ij]}(t)\Delta_0^{\,3}-2b_5^{\{ij\}}\vec{\Delta}\cdot\vec{Q}\Delta_0\,.\end{aligned}$$

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[1]J. W. Meyer and H. Suura, Phys. Rev. 160, 1366 (1967).
[2]R. Jost and H. Lehmann, Nuovo Cimento 5, 1598 (1957); F. J. Dyson, Phys. Rev. 110, 1460 (1958).
[3]J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Lett. 18, 32 (1967); Phys. Rev. 157, 1448 (1967).
[4]V. Singh, Phys. Rev. Lett. 18, 36 (1967).
[5]S. Fubini, Nuovo Cimento 43A, 475 (1966).
[6]R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy, University of Miami, 1966*, edited by A. Perlmutter, J. Wojtaszek, E. C. G. Sudarshan, and B. Kurşunoğlu (W. H. Freeman, San Francisco, 1966).
[7]Y. Georgelin, J. Stern, and J. Jersak, Nucl. Phys. B27, 493 (1971).
[8]J. L. Gervais, Phys. Rev. Lett. 19, 50 (1967); Phys. Rev. 177, 2182 (1969).
[9]This sum rule holds only when a certain asymptotic condition—see Eq. (4.3)—is satisfied.
[10]*Table of Integral Transforms* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, p. 85.
[11]Our invariant amplitude A_2^{ij} differs from that of Refs. 3 and 4 by the factor $(Q^2-\frac{1}{4}t)$.
[12]B. Schroer and P. Stichel, Commun. Math. Phys. 3, 258 (1966).
[13]M. O. Taha, Nucl. Phys. B (to be published).

PHYSICAL REVIEW D VOLUME 9, NUMBER 8 15 APRIL 1974

Gauge theories of strong interactions, spectral-function sum rules, and chiral-symmetry breaking*

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(Received 26 November 1973)

Using the explicit form of the strong-interaction Hamiltonian provided by a certain class of gauge theories, and the technique of the Bjorken-Johnson-Low expansion, we derive sum rules relating the vector, axial-vector, scalar, and pseudoscalar spectral functions and the chiral-symmetry-breaking parameters. Consistency of the sum rules in the pseudoscalar- (PS) meson sector demands that there should be a heavy nonet of PS mesons. It is shown that the nature of chiral-symmetry breaking depends on the character of these heavy mesons. The two modes of symmetry breaking resulting from this are studied. Using the assumption of pole dominance for the spectral functions, we evaluate the chiral-symmetry-breaking parameters in one case. We also evaluate decay widths of ρ, ω, and ϕ mesons to lepton pairs.

I. INTRODUCTION

An interesting outcome of the recent investigations[1] into spontaneously broken gauge theories has been the suggestion that strong interactions may be mediated by a "color" octet of gluons, coupled to three quartets of quarks.[2] In such models, violations of parity and strangeness selection rules are computable and have been shown to be small[3] (of order G_F for $\Delta S=1$ and P-violating processes and of order $G_F{}^2$ for $\Delta S=2$ processes, in agreement with experiment). Further impetus to the study of such models has been provided by the recent observation that a subclass of such theories are asymptotically free[4] and may, therefore, provide an explanation of observed Bjorken scaling in deep-inelastic electroproduction. In this note, we investigate the structure of hadron symmetry breaking and the spectrum of low-lying hadron states in such models.

It has been pointed out[5] that the basic symmetry of strong interactions in gauge models of the type described in Ref. 2 is $U(3)_L\otimes U(3)_R$ (in the space of observed particles) broken by quark mass terms, which transform as $(3,3^*)\oplus(3^*,3)$ under this group. An analysis of the nature of symmetry breaking in theories with $U(3)_L\otimes U(3)_R$ symmetry structure has been done by Mathur and Okubo,[6] using spectral-function sum rules and the assumption of pole dominance. In view of the great relevance of such models in the context of gauge theories, we would like to reexamine their work using the spectral-function sum rules that can be *derived* in gauge models of the type mentioned above. For this purpose we study the general question of spectral-function sum rules in gauge theories. Such sum rules had been studied earlier by Weinberg[7] and by Das, Mathur, and Okubo,[7] but in both these