Fixed-angle behavior in 4-point dual models. I. Absolutely convergent series of beta functions*

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The large-s fixed-angle asymptotic behavior of scattering amplitudes of the form $T(s,t,u) = A(s,t) + A(s,u) + A(u,t)$, with $A(s,t) = \sum_{h=0}^{\infty} C_h B(h-\alpha(s),\ h-\alpha(t))$, is studied. We find that the asymptotic behavior is determined by the function $F(z) = \sum_{h=0}^{\infty} C_h z^h$. If $F(z)$ is entire the Veneziano fixed-angle behavior is preserved. If $F(z)$ is singular at $z=\lambda$, λ real and larger than one, a different, but still exponentially damped, fixed-angle behavior is produced around the forward and backward regions. If $\lambda = 1$, we find a behavior which depends critically on the growth of Im(S) as Re(S) $\to \infty$ [$S \equiv \alpha(s)$]. We find, in particular, that the models of Mandelstam and Frampton have this last type of fixed-angle behavior. The same is true of the model of Gervais and Neveu, unless their choice $\alpha_0 = -1$ is made. The model of Neveu and Schwarz, however, does have the usual Veneziano fixed-angle behavior. We finally relate our work to that of Ellis and Freund, and find some discrepancy, which we comment on.

I. INTRODUCTION

The explicit dual models, introduced by Veneziano,[1] have been generalized in many ways[2] so as to satisfy various physical requirements. Notable constraints on such models have been the freedom from ghosts, the ability to be factorized when generalized to N-point functions, and the preservation of those desirable aspects of duality and Regge behavior which first motivated the invention of such models. Frampton, together with the author[3] derived conditions on the coefficients of absolutely convergent symmetric Veneziano series of the form

$$A(s,\ t) = \sum_{l,h=0}^{\infty} C_{lh} \frac{\Gamma(l+h-\alpha(s))\Gamma((l+h-\alpha(t))}{l!\,\Gamma(l+2h-\alpha(s)-\alpha(t))} \tag{1.1}$$

in order that the duality and Regge behavior be preserved. However, we did not at the same time consider the conditions on the C_{lh} which would ensure the preservation of the good fixed-angle behavior as $s\to\infty$, which was stated in Veneziano's original paper. In a series of two papers, I propose to treat this problem completely.

The fixed-angle behavior discovered by Veneziano,

$$B(-\alpha(s),\ -\alpha(t)) \sim \exp\left\{-\alpha(s)\left[\frac{1-x}{2}\ln\left(\frac{2}{1-x}\right) + \frac{1+x}{2}\ln\left(\frac{2}{1+x}\right)\right]\right\}, \tag{1.2}$$

where $x=\cos\theta_t$, is a very desirable property of these models. Ellis and Freund[4] have stated evidence, both theoretical and experimental, that an asymptotic behavior of the precise form (1.2) is in fact true. However, the data on fixed-angle behavior are not yet very extensive, and their theoretical derivation does depend on a number of assumptions. Nevertheless, the qualitative behavior exhibited by (1.2), of strong damping in transverse momentum, is well verified. Furthermore, DeTar[5] and Segrè and Huang[5] have related this fixed-angle behavior to the strong transverse-momentum damping in inelastic processes.

In these two papers, our aim will be to determine under what conditions a Veneziano series of the form (1.1) satisfies one of the following as $s\to\infty$ at fixed angle:

(a) asymptotic behavior like (1.2);

(b) asymptotic behavior strongly damped in transverse momentum, but quantitatively different from (1.2); and

(c) asymptotic behavior which depends on the growth of Im(S) as Re(S) $\to\infty$ [$S\equiv\alpha(s)$]. Such a dependence on the growth of Im(S) is probably undesirable, but may provide some link between resonance widths and fixed-angle behavior.

This first paper treats only series of beta functions in the form given in Eq. (2.5). Most of the models currently in favor with dual theorists are of this form; notably those of Mandelstam, Gervais and Neveu, Frampton, and Neveu and Schwarz.[2] We can show that the model of Neveu and Schwarz falls into category (a), and that of Gervais and Neveu falls into category (a) if $\alpha = 1, -1, -3, \ldots,$ but otherwise into category (c). Both of the others are in the category (c).

An outline of this paper is as follows: Section II develops an integral representation, and Sec. III transforms this representation to a suitable form for our purposes. The analytic properties of the transformation are studied in Secs. IV and V, and in Sec. VI it is shown how the singularities of the transformation function are the cause of the asymptotic behavior (1.2). We show, in fact, that if

$$F(z)=\sum C_h z^h \tag{1.3}$$

(where the C_h are the coefficients of the beta function series) is an entire function, then the behavior (1.2) is preserved. In Sec. VII, we show that if $F(z)$ possesses singularities only for real z greater than 1, an acceptable asymptotic behavior still obtains. Section VIII shows that the various crossed-channel terms do not alter the conclusions above. Section IX summarizes the results. Section X is devoted to a study of borderline cases, when $F(z)$ is singular at $z=1$. The models of Mandelstam, Frampton, and Gervais and Neveu[2] are in this category, and these models give in general a fixed-angle behavior which depends on the rate at which $\mathrm{Im}(S)\to\infty$ as $\mathrm{Re}(S)\to\infty$. In particular, if the choice $\mathrm{Im}(S)=$constant is made, we get a power-law behavior, while if the choice $\mathrm{Im}(S)=\mu\mathrm{Re}(S)$ is made, an exponentially damped fixed-angle behavior is found. It is found that for small forward and backward angles, the usual Veneziano behavior is found while for larger angles, a slower exponential falloff obtains. The position of the changeover depends on the value of μ. It should be noted that a singularity of $F(z)$ at $z=1$ is an essential characteristic of these models, and it is just this singularity which alters the fixed-angle behavior.

II. AN INTEGRAL REPRESENTATION

Fixed-angle behavior is described by the constraint (for equal masses of scattered particles)

$$T=a+bS \quad [S=\alpha(s),\ \ T=\alpha(t)], \tag{2.1}$$

where

$$b=-\tfrac{1}{2}(1-\cos\theta) \tag{2.2}$$

so that

$$0\geq b\geq -1 \tag{2.3}$$

and

$$a=\alpha_0(1-b)-4\alpha' m^2 b, \tag{2.4}$$

An absolutely convergent symmetric four-point function with only beta-function terms can be written as,[3]

$$A(S,\ T)=\sum_{h=0}^{\infty} C_h \frac{\Gamma(h-S)\Gamma(h-T)}{\Gamma(2h-S-T)} \tag{2.5}$$

$$=\sum_{h=0}^{\infty} C_h \int_0^1 dx\, x^{-(S-h+1)}(1-x)^{-(T-h+1)} \tag{2.6}$$

$$=\sum_{h=0}^{\infty} C_h \int_0^1 dx[x(1-x)^b]^{-S-1}(1-x)^{b-a-1} \times [x(1-x)]^h\ . \tag{2.7}$$

We then interchange the order of integration and summation to get

$$A(S,\ T)=\int_0^1 dx[x(1-x)^b]^{-S-1}(1-x)^{b-a-1} \times F(x(1-x))\ , \tag{2.8}$$

with

$$F(z)=\sum_{h=0}^{\infty} C_h z^h\ . \tag{2.9}$$

In Ref. 3, Sec. II, essentially the same representation as (2.8) was derived, and the interchange of integration and summation shown to be possible if, as assumed here, the series (2.5) converges absolutely.

It was also shown in Ref. 3, Sec. V A, that Regge behavior as $S\to\infty$ at fixed T, and exponential decrease off the real axis as $S\to\infty$ at fixed U, necessarily required that $F(z)$ be analytic everywhere, except when z is real and

$$1\leq z<\infty\ .$$

We shall therefore impose this condition on $F(z)$ from now on.

III. TRANSFORMATION OF THE INTEGRAL REPRESENTATION

We note that the S dependence of (2.8) occurs only in the exponent of the function $w(x)$ defined by

$$w(x)=x(1-x)^b\ . \tag{3.1}$$

Notice also that, provided $b\neq 0$,

$$w(0)=1\ ,\qquad w(1)=\infty\ , \tag{3.2}$$

and $w(x)$ is a monotonically increasing function of x in the interval (0, 1). When $b=0$, the whole derivation we use fails. However, this corresponds to fixed T, which was treated in Ref. 3. We define also the inverse function $v(y)$, which satisfies

$$v(w(x))=x\ ,\qquad w(v(y))=y\ . \tag{3.3}$$

Finally note that

$$\frac{dv(y)}{dy}=\left[\frac{dw(x)}{dx}\right]^{-1}\Bigg|_{x=v(y)}$$

$$=(1-x)^{1-b}[1-(b+1)x]^{-1}|_{x=v(y)} \tag{3.4}$$

$$=v(y)[1-v(y)]\{y[1-(b+1)v(y)]\}^{-1}\ . \tag{3.5}$$

We thus make the substitution $y = w(x)$ in Eq. (2.8) to obtain

$$A(S,\ T) = \int_0^\infty dy\, y^{-S-2} v(y)[1 - v(y)]^{b-a} \times [1-(b+1)v(y)]^{-1} F_2(v(y)[1-v(y)]). \tag{3.6}$$

IV. SINGULARITIES OF $v(y)$

In order to study the asymptotic behavior as $S \to \infty$, with b fixed, it will be necessary to know where the singularities of $v(y)$ are. There is a well-known theorem which states that if

(i) $w(x)$ is analytic at $x = \beta$ and

(ii) $w'(\beta) \neq 0$, (4.1)

there is a unique function $v(y)$, analytic at $\alpha = w(\beta)$, which satisfies the conditions

$$w(v(y)) = y$$

and

$$v(\alpha) = \beta \,. \tag{4.2}$$

Furthermore

$$v'(\alpha) = 1/w'(\beta) \,.$$

Thus, the singularities of $v(y)$ can occur only when either (a) $w'(v(y)) = 0$ or (b) $v(y)$ is the value of a singularity of $w(x)$. Condition (a) occurs when

$$v(y) = 1/(b+1) \tag{4.3}$$

and thus, by (3.3) and (3.1), when

$$y = \left(\frac{b}{b+1}\right)^b \frac{1}{b+1} \,. \tag{4.4}$$

Condition (b) occurs when

$$v(y) = 1$$

or (4.5)

$$v(y) = \infty \,,$$

both of which correspond to

$$y = \infty \,. \tag{4.6}$$

Thus, we conclude that the only singularities of $v(y)$ occur at the points

$$y = \infty$$

and (4.7)

$$y = \left(\frac{b}{b+1}\right)^b \frac{1}{b+1} \,.$$

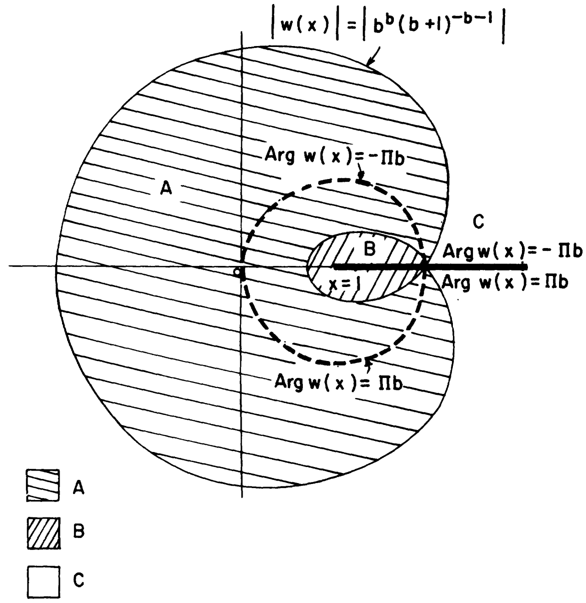

FIG. 1. Schematic graph of the various regions of $w(x) = x(1-x)^b$. A: Region $0 \le |w(x)| < |b^b(b+1)^{-b-1}|$ and $0 \le |\arg w(x)| \le \pi$. B: Region $|w(x)| > |b^b(b+1)^{-b-1}|$ and $0 \le |\arg w(x)| \le \pi|b|$. C: Region $|w(x)| > |b^b \times (b+1)^{-b-1}|$ and $\pi|b| \le |\arg w(x)| \le \pi$.

V. INTERPRETATION OF THE SINGULARITY AT $y = [1/(b+1)][b/(b+1)]^b$

Since b is nonintegral, there are many possible values of $[b/(b+1)]^b$, and the function $v(y)$ has several sheets. Furthermore, the function $w(x)$ also has several sheets. In Fig. 1 we show the curves of constant phase and modulus of the function $w(x)$, for $b = -\frac{1}{2}$.

There are three regions, A, B, C.

Region A. In this region

$$|w(x)| < |b^b(b+1)^{-b-1}| = |b|^{-|b|}(1-|b|)^{-1+|b|} > 1 \,. \tag{5.1}$$

Also, $\arg(w(x))$ has the range

$$-\pi \text{ to } +\pi \,. \tag{5.2}$$

Region B. In this region

$$|w(x)| > |b^b(1+b)^{-1-b}| \tag{5.3}$$

and x is finite. Further, $\mathrm{Arg}[w(x)]$ has the range

$$-\pi b \text{ to } +\pi b \,. \tag{5.4}$$

The value of $\arg[w(x)]$ is $-\pi b$ on the upper lip of the cut, and $+\pi b$ on the lower lip of the cut.

Region C. In this region

$$|w(x)| > |b^b(1+b)^{-b-1}| \tag{5.5}$$

and x may become infinite. Here $\arg[w(x)]$ has the range

$$\pi b \text{ to } \pi \,, \tag{5.6}$$

$$-\pi b \text{ to } -\pi \,. \tag{5.7}$$

Thus, the function $w(x)$ takes on all complex values in the x plane. We now consider the inverse function $v(y)$. The reader is asked to refer to Figs. 1 and 2. In the region

$$|y| < |b^b(1+b)^{-1-b}| \tag{5.8}$$

there are no singularities. Thus the mapping $y = w(x)$ maps the region A into the region A', defined by $y = w(x)$ with $w(x)$ satisfying (5.1) and (5.2). This is a circle with radius $|b^b(1+b)^{-1-b}|$.

The region B is defined by $y = w(x)$ satisfying (5.3) and (5.4). Thus the mapping $y = w(x)$ maps B into B', defined by

$$|y| > |b^b(1+b)^{-b-1}|, \tag{5.9}$$

$$0 < |\arg(y)| < \pi|b| \,. \tag{5.10}$$

In B there is only a singularity at $x = \infty$, but the mapping becomes double-valued along the cut from 1 to ∞. Thus a singularity in $v(y)$ arises when $x = 1/(b+1)$, as was shown previously. This gives two singularities in $v(y)$, occurring at

$$|y| = |b^b(b+1)^{-b-1}|$$

and (5.11)

$$\arg y = \pm \pi b \,.$$

The cut from ∞ to $x = (b+1)^{-1}$ then maps to two

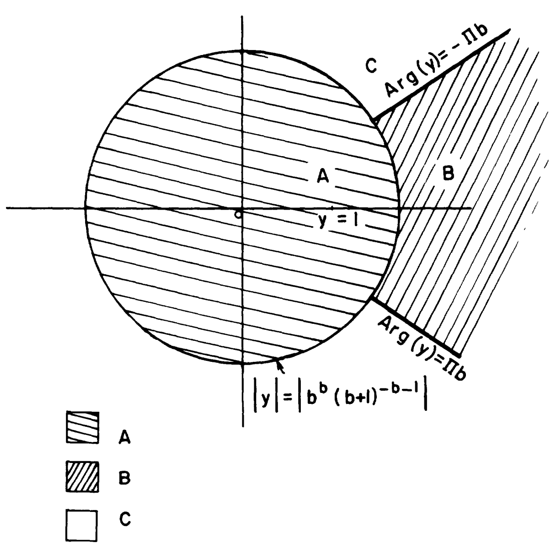

FIG. 2. Schematic graph of various regions of $v(y)$. The regions A, B, C correspond to the regions A, B, C in Fig. 1. The function $v(y)$ is analytic throughout the y plane, apart from the cuts shown along $\arg(y) = \pm\pi b$, with branch points at $|y| = |b^b(b+1)^{-b-1}|$.

cuts, from the positions (5.11) to ∞, with $\arg y = \pm \pi b$.

The region C is similarly shown to map to C', which satisfies

$$|y| > |b^b(b+1)^{-b-1}| \,, \qquad \pi > |\arg(y)| > \pi|b| \,. \tag{5.12}$$

Thus, the first sheet of $v(y)$ possesses only two singularities, both branch points at

$$|y| = |b|^{-|b|}(1-|b|)^{-1+|b|} \,, \qquad \arg y = \pm \pi b \,. \tag{5.13}$$

An approximate expression for $v(y)$, valid when y is close to the branch points, can be simply derived by approximating $w(x)$ by a quadratic function of $[x - 1/(b+1)]$. We can then solve for x in terms of $w(x)$, to give

$$v(y) \simeq \frac{1}{b+1} \pm \left[\frac{2b}{(b+1)^3}\right]^{1/2} \left[\frac{1}{b+1}\left(\frac{b}{b+1}\right)^b - y\right]^{1/2} . \tag{5.14}$$

VI. ASYMPTOTIC BEHAVIOR WHEN $F(x)$ POSSESSES NO SINGULARITIES

This section will show that if $F(x)$ has no singularities, the asymptotic behavior at fixed angle is that of the usual beta-function model.

We make the substitution

$$y = e^{-z} \tag{6.1}$$

so that

$$A(S, T) = \int_{-\infty}^{\infty} dz\, e^{(S+1)z} v(e^{-z})[1 - v(e^{-z})]^{b-a} \times [1 - (b+1)v(e^{-z})]^{-1} \times F_2(v(e^{-z})[1 - v(e^{-z})]) \tag{6.2}$$

$$= \int_{-\infty}^{\infty} e^{Sz}\Phi(z)dz \,. \tag{6.3}$$

The singularities of $\Phi(z)$ are at points where $v(e^{-z})$ is singular, i.e., at

$$z = \ln|b^b(1+b)^{-1-b}| \pm i\pi b + 2n\pi i \,, \tag{6.4}$$

and when

$$v(e^{-z}) = 1 \,,$$

i.e.,

$$e^{-z} = \infty \,, \tag{6.5}$$

i.e.,

$$z = -\infty \,.$$

However, singularities at infinity are irrelevant to this work.

We distort the integration contour, $(-\infty, \infty)$ to C', as in Fig. 3 when $\mathrm{Im}(S) > 0$. The asymptotic be-

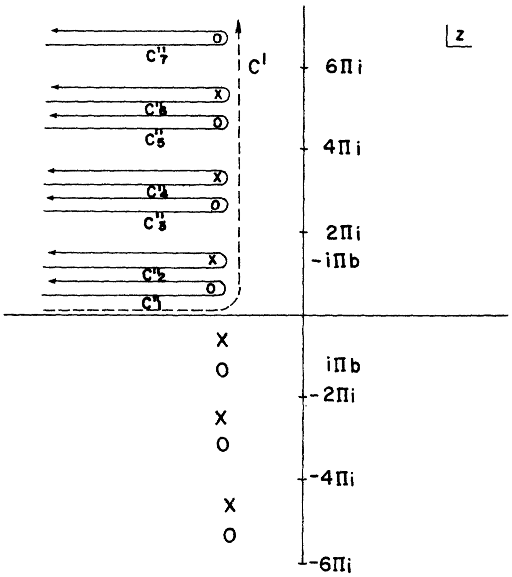

FIG. 3. Singularity structure of the integrand of Eq. (6.2). All singularities occur at $\mathrm{Re}(z) = -\ln|b^b(1+b)^{-1-b}|$. The crosses represent singularities at $\arg z = -i\pi b + 2n\pi i$; the circles represent singularities at $\arg z = i\pi b + 2n\pi i$.

havior as $\mathrm{Re}(S)\to\infty$ is dominated by the behavior of the least rapidly decreasing exponential, which occurs along the vertically distorted contour. Thus, the asymptotic behavior will be approximately of the form

$$|A(S,T)| \sim \exp\{-S[\ln|b^b(1+b)^{-1-b}| \pm i\pi b]\} \tag{6.6}$$

$$= \left[\left(\frac{b}{b+1}\right)^b \frac{1}{b+1}\right]^{-S} . \tag{6.7}$$

This form is the same as that in Veneziano's original paper.[2]

More careful calculation can give the more precise determination

$$A(S,T) \sim \frac{\sin\pi[-a-(b+1)S]}{\sin\pi S} |(b+1)/b|^{(1/2)+a} \times S^{-1/2} |b^b(b+1)^{-b-1}|^{-S} F(b/(b+1)^2) . \tag{6.8}$$

The only change between this and a careful evaluation of the asymptotic value of $A(S,T)$ (as given by Veneziano), using Stirling's approximation, is the factor $F(b/(b+1)^2)$. Clearly, when $F(z)=1$, we get the correct Stirling's approximation value for the beta function.

Equation (6.8) is derived as follows:

(i) Let $\mathrm{Im}(S)>0$. Then we distort the contour $(-\infty,+\infty)$ to the contour C' as shown in Fig. 3, and then to the contour C''', which is composed of the sum of all the contours C'' round individual positions of the singularities, which we have shown in Sec. V are branch points, of a structure given approximately by Eq. (5.14).

(ii) The integrals round the various contours C'' differ only by phases, which can be extracted, and summed over the various positions of the singularity. Doing this gives an integral around C_1'', multiplied by a factor

$$-\frac{\sin\pi[a+(b+1)S]}{\sin\pi S} . \tag{6.9}$$

This representation is, by analytic continuation, now valid for all values of $\mathrm{Im}(S)$.

(iii) We rewrite the integral around the contour C_1'' defined by $\mathrm{Im}(z) = -\pi b$ as an integral of the discontinuity of the cut, which turns out to be a Laplace transform.

(iv) Since the asymptotic behavior is determined to first order only by the behavior of the discontinuity of the integrand across the cut near the beginning of the cut, we approximate $v(y)$ by (5.14) in the integral, and carry out the integrations to get (6.8).

VII. ASYMPTOTIC BEHAVIOR WHEN $F(x)$ IS SINGULAR

In Sec. II we imposed the condition that $F(z)$ should be singular only for z real and

$$1 \le z < \infty . \tag{7.1}$$

Suppose then z has a singularity satisfying (7.1) at $z=\lambda$. Then $F(v(y)[1-v(y)])$ will be singular when $y=y_0$, defined by

$$v(y_0)[1-v(y_0)] = \lambda ,$$

i.e.,

$$v(y_0) = \tfrac{1}{2} \pm (\tfrac{1}{4}-\lambda)^{1/2} \tag{7.2}$$

$$= \tfrac{1}{2} + i\xi , \tag{7.3}$$

where

$$|\xi| \ge \tfrac{1}{2}\sqrt{3} . \tag{7.4}$$

Solving (7.3) for y_0, we find

$$y_0 = w(\tfrac{1}{2}+i\xi) = (\tfrac{1}{2}+i\xi)(\tfrac{1}{2}-i\xi)^b . \tag{7.5}$$

From the reasoning of Sec. VI, we can see that an exponential decrease of $A(S,T)$ as $S\to\infty$ at fixed angle can come about only if

$$|y_0| > 1 , \tag{7.6}$$

which would mean

$$(\tfrac{1}{4}+\xi^2)^{(1+b)/2} = \lambda^{(1+b)/2} > 1 . \tag{7.7}$$

Noting that $\tfrac{1}{2}(1+b)>0$, we see that (7.7) requires

$$|\xi| > \tfrac{1}{2}\sqrt{3} , \tag{7.8}$$

which is *not* the same as (7.4), which includes the

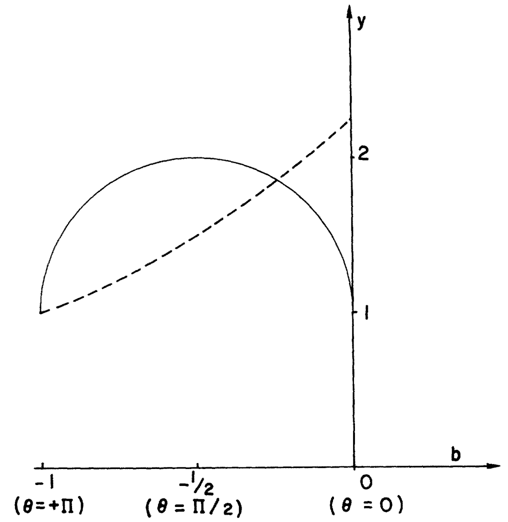

FIG. 4. Plots of $y = \lambda^{(1+b)/2}$ for $\lambda = 5$, as a function of b (dashed curve, and $y = |(b+1)^{-b-1}b^b|$ as a function of b (solid curve). The smaller function at any given value of b dominates the asymptotic behavior of $A(S,T)$ as $S \to \infty$ at fixed angle.

possibility of equality.

Translated into a statement about $F(z)$, we find that the requirement for asymptotic behavior at fixed angle, of the form $e^{-\kappa S}$, as $\mathrm{Re}(S) \to \infty$ is that $F(z)$ possess singularities only for z real, and $1 < z < \infty$.

It should be noted, however, that whenever $F(z)$ does possess singularities at $z = \lambda > 1$, although the requirement for exponential behavior at fixed angle is satisfied, the qualitative behavior is now different from that of a single beta function. It is clear that for any given angle, whichever is the smaller of $|(b+1)^{-(b+1)}b^b|$ and $\lambda^{(1+b)/2}$ will dominate the asymptotic behavior at fixed angle. Figure 4 shows the different behavior of the two functions as b varies. Notice that the tangents to $|(b+1)^{-(b+1)}b^b|$ at $b=0$ and $b=-1$ are vertical. Thus, the two curves will always intersect once in $0 \leq b \leq 1$. At b greater than the value where this intersection occurs, the fixed-angle behavior will be that of Eq. (6.8); at more negative b, the behavior will be

$$|A(S, T)| \sim (\lambda^{(1+b)/2})^{-S} . \tag{7.9}$$

Notice also that the symmetry of the fixed-angle behavior of an individual term under the transformation $b \to -b-1$ will now be destroyed.

VIII. ASYMPTOTIC BEHAVIOR OF THE TERMS $A(S,U)$ AND $A(U,T)$

The usual expression for the scattering amplitude will involve $A(S, U)$ and $A(U, T)$. Using

$$S + T + U = \kappa = 4m^2\alpha' + 3\alpha_0 \tag{8.1}$$

we see that

$$U = \kappa - a - (b+1)S ; \tag{8.2}$$

thus the term $A(S, U)$ can in fact be written

$$A(S, U) = A(S, \kappa - a - (b+1)S) . \tag{8.3}$$

Thus the substitutions

$$a \to \kappa - a \tag{8.4}$$

and

$$b \to -(b+1) \tag{8.5}$$

in (6.8) and (7.9) give the asymptotic expansion of $A(S, U)$. The usual behavior (6.8) is, up to constant factors, unaltered. Thus, in the case of nonsingular $F(z)$, $A(S, T)$ and $A(S, U)$ give the same fixed-angle behavior.

However, if $F(z)$ is singular, referring to Fig. 4, we can see that the dominant term near $b=0$ is now a term that behaves like $(\lambda^{b/2})^{-S}$. There may be a region of b around $b = -\frac{1}{2}$ where the behavior (6.8) dominates, but this happens only if

$$\lambda \geq 16 . \tag{8.6}$$

The term $A(U, T)$ does not have such an obvious treatment. A representation like (2.8) can be derived, and after substituting $x \to 1/x$, it becomes

$$A(U, T) = \exp\{i\pi[b(S-1) + b - a - 1]\} \times \int_1^\infty dx[x(1-x)^b]^{-S-1}x^{\kappa+1}(1-x)^{b-a-1} \times F((x-1)/x^2) , \tag{8.7}$$

where we choose $\arg(1-x) = -\pi$ in the integral, so that the integral is along the top of the cut from 1 to ∞. By now writing $y = v(x)$, we find

$$A(U, T) = \exp\{i\pi[b(S-1) + b - a - 1]\} \int_C dy\, y^{-S-2}v(y)[1 - v(y)]^{b-a}[1 - (b+1)v(y)]^{-1} \times v(y)^{\kappa+1}F([v(y) - 1]/[v(y)]^2) , \tag{8.8}$$

where the contour C runs around the cut along $\arg y = -\pi b$ (see Fig. 2). We see from this that when the substitution $y = e^{-z}$ is made, no distortion of contour will be necessary to obtain a representation suitable for an asymptotic expansion. Thus, the singularities of F are irrelevant to this case. The asymptotic expansion is then calculated to be

$$A(U,\ T) \sim (2\pi)^{1/2}(b+1)^{a-\kappa-1/2}(-b)^{-a-1/2}S^{-1/2} \times |(b+1)^{-b-1}b^{b}|^{-S}F(-b(b+1)) \qquad (8.9)$$

and will hold for all absolutely convergent beta-function series.

IX. SUMMARY OF ASYMPTOTIC EXPANSIONS AT FIXED ANGLE

When $F(z)$ has a singularity at $z=\lambda$, we find that

$$T(S,\ T,\ U) = A(S,\ T) + A(U,\ T) + A(S,\ U) \qquad (9.1)$$

has the following asymptotic expansion at fixed angles:

(a) If $\lambda \leqslant 16$, then for all angles

$$|T(S,\ T,\ U)| \sim \lambda^{[(1/2)-|(1/2)-b|]S}\ . \qquad (9.2)$$

Near the forward direction, this can be written

$$|T(S,\ T,\ U)| \sim \exp[S(\ln\lambda)\sin^2(\tfrac{1}{2}\theta)]\ , \qquad (9.3)$$

which for small angles is approximately

$$\sim \exp[-\tfrac{1}{4}\ln\lambda(\alpha' P_\perp)^2]\ . \qquad (9.4)$$

(b) If $\lambda > 16$, the asymptotic expansion (9.2) holds in the region where

$$\lambda^{-[(1/2)-|(1/2)-b|]} < |(b+1)^{-b-1}b^{b}|\ . \qquad (9.5)$$

This is a region which includes the points $b=-1$ and $b=0$, but excludes a central region of the form

$$|\tfrac{1}{2} - b| < \eta\ , \qquad (9.6)$$

where η is so defined that when $|\tfrac{1}{2}-b| = \eta$, (9.5) becomes an equality.

Thus, qualitatively, a singularity in $F(z)$ will change the fixed-angle asymptotic behavior near the forward and backward regions, but, provided $\lambda > 16$, the original behavior (6.8) will be preserved in a region round $\theta = 90°$.

X. APPLICATIONS

A. Applications to the models of Mandelstam, Gervais and Neveu, and Frampton[3]

There are several interesting models which can be written as sums of beta functions, and it is therefore relevant to investigate the fixed-angle behavior of these. The models under consideration all involve expressions of the form

$$A(S,\ T) = \int_0^1 dx\, x^{-S-1}(1-x)^{-T-1}[1-x(1-x)]^{\gamma/2} \times \psi(S,\ T,\ U,\ x)\ . \qquad (10.1)$$

In the models of Mandelstam and Gervais and Neveu,

$$\psi(S,\ T,\ U,\ x) = 1\ .$$

In Mandelstam's model,

$$\gamma = S + T + U + 1 = 4\alpha' m^2 + 3\alpha_0 + 1\ ; \qquad (10.2)$$

in that of Gervais and Neveu,

$$\gamma = 1 - \alpha_0\ , \qquad (10.3)$$

and it is preferred that

$$\alpha_0 = -1\ . \qquad (10.4)$$

In Frampton's ghost-free version of his model,

$$\gamma = S + T + U - 1 = 4\alpha' m^2 + 3\alpha_0 - 1\ , \qquad (10.5)$$

and

$$\psi(S,\ T,\ U,\ x) = [Sx + T(1-x) - Ux(1-x)] \times [1 - x(1-x)]^{-1}\ . \qquad (10.6)$$

The crucial point in all of these models is that

$$F(x(1-x)) = [1 - x(1-x)]^{\gamma/2}\ \text{(polynomial in } x)\ . \qquad (10.7)$$

The polynomial is unimportant; the crucial fact is that for all these models $F(z)$ is singular at $z=1$. [There is an exception if we make the preferred choice of Gervais and Neveu, given by (10.4), when $\gamma=2$, and the singularity disappears.] Thus, it is clear that in general, the fixed-angle behavior of the Veneziano model will be completely destroyed.

Consider a model in which $\phi(S,\ T,\ U,\ x) = 1$, and γ is arbitrary. The representation (3.6) becomes

$$A(S,\ T) = \int_0^\infty dy\, y^{-S-2}[1 - v(y)]^{b-a}[1 - (b+1)v(y)]^{-1} \times \{[v(y) - e^{-i\pi/3}][v(y) - e^{i\pi/3}]\}^{\gamma/2}\ . \qquad (10.8)$$

The final factor has branch points at

$$y = e^{\pm i\pi(by)/3}\ , \qquad (10.9)$$

and near these branch points, using the expression (3.5) for $v'(y)$, we can derive that

$$v(y) \simeq e^{\pm i\pi/3} + e^{\pm(i\pi/3)(b-1)}[1 - (b+1)e^{\pm i\pi/3}] \times [y - e^{(\pm i\pi/3)(b-1)}]\ . \qquad (10.10)$$

We may then follow the same methods as in Sec. VI to derive the asymptotic expansion

$$A(S,\ T) \sim \left[\left(\frac{2i\sin(\frac{1}{3}\pi)e^{-2\pi i(b-1)/3}}{1-(b+1)e^{i\pi/3}} \right)^{\gamma/2} \frac{e^{i\pi(1-b+a)/3}e^{i\pi(b-1)(S+1)/3}}{1-(b+1)e^{i\pi/3}} + \left(\frac{-2i\sin(\frac{1}{3}\pi)e^{2\pi i(b-1)/3}}{1-(b+1)e^{-i\pi/3}} \right)^{\gamma/2} \frac{e^{-i\pi(1-b+a)/3}e^{-i\pi(b-1)(S+1)/3}}{1-(b+1)e^{-i\pi/3}} \right] S^{-(\gamma/2)-1} \frac{\csc(\pi S)}{\Gamma(-\frac{1}{2}\gamma)} . \tag{10.11}$$

The interpretation of Eq. (10.11) depends very much on the behavior of $\mathrm{Im}(S)$ as $\mathrm{Re}(S) \to \infty$. If we assume that $\mathrm{Im}(S)$ is a large positive constant, (10.11) gives a power-law fixed-angle behavior, multiplied by some phases. If, however, more conventionally we take $S \to \infty$ along a ray, so that

$$\mathrm{Im}(S) = \mu \mathrm{Re}(S) , \tag{10.12}$$

and define b_0 by

$$|b_0^{b_0}(b_0+1)^{-b_0-1}| = e^{2\mu} . \tag{10.13}$$

Then,

(a) for $0 > b > b_0$, i.e., angles in the forward direction, we have

$$|A(S,\ T)| \sim |\exp[\mu\pi b\,\mathrm{Re}(S)]| \times |b^b(b+1)^{-b-1}|^{-\mathrm{Re}(S)}|S|^{-1/2} , \tag{10.14}$$

which, apart from the first slowly decreasing factor, is the same as the usual Veneziano form.

(b) for $-\frac{1}{2} < b < b_0$, i.e., angles from 90° up to that angle defined by $b = b_0$,

$$|A(S,\ T)| \sim \exp[-\tfrac{1}{3}\pi(b+2)\mu\,\mathrm{Re}(S)]|S|^{-(\gamma/2)-1} , \tag{10.15}$$

which is of the form of a slowly decreasing exponential multiplied by a power law.

It should be noted that the nature of the fixed-angle behavior, in particular the "break" between one behavior and the next, will depend strongly on μ, unlike nearly all other dual-model properties.

However, it should be remembered that $\gamma = 2$ corresponds to the Gervais-Neveu model, which leads to the vanishing of (10.11) because of the factor $[\Gamma(-\frac{1}{2}\gamma)]^{-1}$.

B. Application to the model of Neveu and Schwarz

This model is merely a sum of beta functions, multiplied by polynomials in S, T, and U. Thus, the fixed-angle behavior will be essentially that of a beta function.

XI. CONNECTION WITH THE RESULTS OF ELLIS AND FREUND[4]

These authors have recently derived the result that, under certain analyticity and consistency requirements, dual models give a *logarithmic scaling law,* of the form

$$-(\alpha' S)^{-1}\ln A(S,\ T) \sim f(\cos\theta) \tag{11.1}$$

as $|S| \to \infty$ at fixed angle, and that $f(x)$ is given by

$$f(x) = \frac{1-x}{2}\ln\left(\frac{2}{1-x}\right) + \frac{1+x}{2}\ln\left(\frac{2}{1+x}\right), \tag{11.2}$$

so that (11.1) corresponds exactly to the behavior expected from a single beta function. It is clear that all their assumptions cannot be correct in dual models expressible as series of beta functions. For example, taking $F(z)$ to be given by

$$F(z) = \frac{1}{z-\lambda} , \quad \lambda \text{ real, and } \lambda > 1 \tag{11.3}$$

$$T(S,\ T,\ U) \sim \frac{\exp[-\frac{1}{4}(1-x)S\ln\lambda]}{2(4\lambda-1)^{1/2}\sin\pi S} \Big((\tfrac{1}{2}-i\xi)^{-x-2-a}(\tfrac{1}{2}+i\xi)\exp\{i[-\tfrac{1}{2}(3+x)\tan^{-1}2\xi - \pi]S\} - (\tfrac{1}{2}+i\xi)^{-x-2-a}(\tfrac{1}{2}-i\xi)\exp\{-i[-\tfrac{1}{2}(3+x)\tan^{-1}2\xi - \pi]S\} \Big) + (x \to -x) , \tag{11.4}$$

where $x = \cos\theta$, and $\xi = |(\lambda - \frac{1}{4})^{1/2}|$. This expansion will be valid in a pair of regions; one including $x = -1$, and one including $x = 0$. (These regions overlap if $\lambda > 16$.) In a central region the asymptotic behavior is the usual Veneziano type.

Imagining firstly for simplicity that the limit $S \to \infty$ is taken with a *constant* imaginary part of S, namely, $\mathrm{Im}(S) > 0$, then the only significant term is $\exp[\frac{1}{4}(1-x)S\ln\lambda]$. Writing all this we see that (11.1) is valid if

$$f(x) = \min\left\{ \frac{1-x}{4}\ln\lambda,\ \frac{1-x}{2}\ln\frac{2}{1-x} + \frac{1+x}{2}\ln\frac{2}{1+x},\ \frac{1+x}{2}\ln\lambda \right\}, \tag{11.5}$$

that is,

$$f(x) = \tfrac{1}{4}(1-x)\ln\lambda \quad (\alpha \le x \le 1)$$
$$= \frac{1-x}{2}\ln\frac{2}{1-x} + \frac{1+x}{2}\ln\frac{2}{1+x} \quad (-\alpha \le x \le \alpha)$$
$$= \tfrac{1}{4}(1+x)\ln\lambda \quad (-1 \le x \le -\alpha)\ . \tag{11.6}$$

Here α and $-\alpha$ give the values of x where $f(x)$ changes from one functional form to another. When x approaches 1, the first two forms for $f(x)$ both approach zero. The second one gives rise to the Regge behavior, and of course satisfies the consistency condition of Ellis and Freund. However, the first form arises from $A(S, U)$, and gives not Regge behavior, but, instead, the type of term that decreases exponentially as $|S| \to \infty$, with $|\arg S| > 0$. A similar result occurs if we let $\mathrm{Im}(S) = \mu\, \mathrm{Re}(S)$ as $S \to \infty$. The condition of Ellis and Freund has not taken into account that, with an $f(x)$ like (11.5), the scaling behavior (11.1) can be obtained by a sum of exponentials, only one of which dominates at any angle.

XII. CONCLUSION

Proponents of dual models have taken great pains to construct factorizable ghost-free models with Regge behavior, with physically acceptable trajectories.[2] The aspect of fixed-angle behavior is, we believe, also very important and we feel that it is desirable that any proposed model satisfy the constraint of strong damping in transverse momentum, as does the original Veneziano model.

This paper has shown that the transverse-momentum behavior is by no means as simple in general dual models as in the original Veneziano form. In particular, there are many models, as mentioned in Sec. X, that have rather intricate fixed-angle behavior, which may possibly have physical consequences.

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