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Approximate measurement in quantum mechanics. II

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An approximate measurement procedure of the following type is considered: (i) An initial eigenstate of the object observable leads to a final statistical operator of the object plus apparatus describing a mixture of exact eigenstates of the apparatus observable; (ii) almost all the statistical weight of the mixture is assigned to eigenstates associated with one eigenvalue of the apparatus observable, which is uniquely determined by the initial value of the object observable. It is proved that each of a large class of initial states of the object leads to a final statistical operator which does not describe any mixture of exact eigenstates of the apparatus observable. The analysis also yields a proof of a theorem on measurement stated by Fine.

I. INTRODUCTION

In this paper the following formulation of an approximate measuring procedure is considered: If the initial state of the object is an eigenstate of the object observable with eigenvalue λ_m, then the final statistical state of the object plus apparatus can be described as a mixture of pure quantum states, all of which are exact eigenstates of the apparatus observable, and the total statistical weight in the mixture of those eigenstates associated with the eigenvalue μ_m is close to 1. (It is understood that $m \neq n$ implies both $\lambda_m \neq \lambda_n$ and $\mu_m \neq \mu_n$.) Hence, the value of the apparatus observable at the end of the interaction between the object and the apparatus is strongly correlated with the initial value of the object observable. This formulation of the procedure of measurement is more strictly in accordance with common sense than the formulation in a previous paper,[1] since a system in an exact eigenstate of the apparatus observable unequivocally *has* a sharp value of the apparatus observable, whereas it is not rigorously correct to speak of "having a sharp value" when the state is almost an eigenstate.[2]

Using the notation of Ref. 1, one can give the present formulation of approximate measurement in two conditions:

(a) $\{\underline{E}_m\}$ is a finite or denumerably infinite family of mutually orthogonal subspaces spanning $\mathcal{H}_1$, $\{\underline{F}_m\}$ is a family of mutually orthogonal subspaces of $\mathcal{H}_2$, and U is a unitary operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$;

(b) T is a statistical operator on $\mathcal{H}_2$ such that for every m and every $v \in \underline{E}_m$, $U(P_v \otimes T)U^{-1}$ can be expressed in the form $\sum_{n,r} a_{nr} P_{\chi_{nr}}$, where $\chi_{nr} \in \mathcal{H}_1 \otimes \underline{F}_n$, and the a_{nr} are non-negative real numbers summing to 1 such that

$$\sum_{\substack{r \\ n \neq m}} a_{nr} = \epsilon_m \ll 1 .$$

The theorem of Sec. II implies that if these two conditions are satisfied and if the number of subspaces $\underline{E}_m$ is greater than one, then there exist initial states of the object for which the final statistical state of the object plus apparatus is not expressible as a mixture of eigenstates of the apparatus observable.

II. A THEOREM ON MEASUREMENT

It will be convenient for proving the theorem of this section to use the Dirac bra and ket notation, in which $\langle \phi | \phi \rangle = 1$ implies that $| \phi \rangle \langle \phi |$ is the projection operator P_ϕ.

The theorem is the following:

Hypotheses.

(i) u_1, u_2 are normalized orthogonal vectors of $\mathcal{H}_1$, $\{\underline{F}_m\}$ is a family of mutually orthogonal subspaces of $\mathcal{H}_2$, U is a unitary operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and T is a statistical operator on $\mathcal{H}_2$;

(ii) there exist orthonormal sets $\{\xi^1_{nr}\}, \{\xi^2_{nr}\}$ such that

$$\xi^j_{nr} \in \mathcal{H}_1 \otimes \underline{F}_n \text{ for } j = 1, 2,$$

and

$$U(P_{u_j} \otimes T)U^{-1} = \sum_{n,r} b^j_{nr} \, | \xi^j_{nr} \rangle \langle \xi^j_{nr} | ;$$

and for some value of n,

$$\sum_r b^1_{nr} \neq \sum_r b^2_{nr} .$$

Conclusion. If u is defined as $g_1 u_1 + g_2 u_2$, with both g_1 and g_2 nonzero, then there exists no orthonormal set $\{\psi_{nr}\}$ with $\psi_{nr} \in \mathcal{H}_1 \otimes \underline{F}_n$ and no coefficients $\{b_{nr}\}$ such that

$$U(P_u \otimes T)U^{-1} = \sum_{n,r} b_{nr} | \psi_{nr} \rangle \langle \psi_{nr} | .$$

Proof. T can be written in the form

$$\sum_{i,s} a_i | \eta_{is} \rangle \langle \eta_{is} | ,$$

where the η_{is} are orthonormal vectors of $\mathcal{H}_2$, and where a_i is a positive N_i-fold degenerate eigenvalue of T for each i, so that $\sum_i N_i a_i = 1$. Hence, if $U(u_j \otimes \eta_{is})$ is abbreviated by χ^j_{is} for $j=1, 2$, then

$$U(P_{u_j} \otimes T)U^{-1} = \sum_{i,s} a_i | \chi^j_{is} \rangle \langle \chi^j_{is} | .$$

Since U is unitary, $\{\chi^j_{is}\}$ is a set of orthonormal vectors. The same statistical operator is thus expressed with respect to the two orthonormal sets $\{\chi^1_{is}\}$ and $\{\xi^1_{nr}\}$. Since the eigenvalues of a linear operator are invariant with respect to the choice of a basis, the coefficients $\{b^1_{nr}\}$ must be a permutation of the coefficients $\{a_i\}$ with proper multiplicities. The same is evidently also true for the coefficients $\{b^2_{nr}\}$ and $\{b_{nr}\}$. Consequently, the vectors ξ^j_{nr} can be relabeled $\bar{\xi}^j_{is}$ by appropriate permutation, and the vectors ψ_{nr} can be relabeled $\bar{\psi}_{is}$ in such a way that

$$U(P_{u_j} \otimes T)U^{-1} = \sum_{i,s} a_i | \bar{\xi}^j_{is} \rangle \langle \bar{\xi}^j_{is} | ,$$

and

$$U(P_u \otimes T)U^{-1} = \sum_{i,s} a_i | \bar{\psi}_{is} \rangle \langle \bar{\psi}_{is} | .$$

In order that the condition $\sum_r b^1_{nr} \neq \sum_r b^2_{nr}$ be satisfied for some value of n, there must be some value of i, say k, such that the number n_1 of $\{\bar{\xi}^1_{ks}\}$ belonging to $\mathcal{H}_1 \otimes \underline{F}_n$ in unequal to the number n_2 of $\{\bar{\xi}^2_{ks}\}$ belonging to $\mathcal{H}_1 \otimes \underline{F}_n$, and without loss of generality it may be assumed that $n_1 > n_2$. Then, by relabeling, we may write

$$\bar{\xi}^j_{ks} \in \mathcal{H}_1 \otimes \underline{F}_n , \quad s=1, \ldots, n_j$$
$$\in \mathcal{H}_1 \otimes \bigoplus_{m \neq n} \underline{F}_m , \quad s = n_j + 1, \ldots, N_k$$

for $j = 1, 2$.

Since each eigenvalue of the statistical operator $U(P_{u_j} \otimes T)U^{-1}$ is associated with an invariant subspace of the range of this operator

$$\chi^j_{kr} = \sum_{s=1}^{N_k} c^j_{rs} \bar{\xi}^j_{ks} , \quad j = 1, 2 . \tag{1}$$

The coefficients $\{c^j_{rs}\}$, with fixed $j=1,2$ but with r and s varying from 1 to N_k, constitute a unitary matrix, so that

$$\sum_{s=1}^{N_k} c^j_{rs} \bar{c}^j_{r's} = \delta_{rr'} , \quad j = 1, 2 . \tag{2}$$

The statistical operator $U(P_u \otimes T)U^{-1}$ can be expressed in terms of the set $\{\bar{\psi}_{is}\}$ and also in terms of the sets $\{\chi^j_{is}\}$, so that

$$\sum_{i,s} a_i | \bar{\psi}_{is} \rangle \langle \bar{\psi}_{is} |$$
$$= \sum_{i,s} a_i | g_1 \chi^1_{is} + g_2 \chi^2_{is} \rangle \langle g_1 \chi^1_{is} + g_2 \chi^2_{is} | .$$

If one considers only the terms associated with the eigenvalue a_k and makes use of Eqs. (1) and (2), one then obtains

$$\sum_{r=1}^{N_k} | \bar{\psi}_{kr} \rangle \langle \bar{\psi}_{kr} | = \sum_{r=1}^{N_k} \sum_{j=1}^{2} \sum_{j'=1}^{2} g_j \bar{g}_{j'}$$
$$\times \sum_{s=1}^{N_k} \sum_{s'=1}^{N_k} c^j_{rs} \bar{c}^{j'}_{rs'} | \bar{\xi}^j_{ks} \rangle \langle \bar{\xi}^{j'}_{ks'} | .$$

Those members of $\{\bar{\psi}_{ks}\}$ which belong to $\mathcal{H}_1 \otimes \underline{F}_n$ are linear combinations of members of $\{\bar{\xi}^1_{ks}\}$ and $\{\bar{\xi}^2_{ks'}\}$ with $s \leq n_1$ and $s' \leq n_2$, respectively; and similarly for those members of $\{\bar{\psi}_{ks}\}$ which belong to

$$\mathcal{H}_1 \otimes \bigoplus_{m \neq n} \underline{F}_m .$$

Consequently, if g_1 and g_2 are nonzero, then a necessary condition for the foregoing equation to hold is that

$$\sum_{r=1}^{N_k} c^1_{rs} \bar{c}^2_{rs'} = 0, \text{ for } s \leq n_1 , \ s' > n_2 . \tag{3}$$

But the N_k-tuples $\{c^1_{rs}\}$ (fixed s) and $\{c^2_{rs'}\}$ (fixed s'), $r = 1, \ldots, N_k$, can be considered as vectors in an N_k-dimensional complex vector space with an appropriate inner product. By Eqs. (2) and (3), the N_k-tuples such that $s \leq n_1$ and $s' > n_2$ constitute orthonormal vectors in this space. The number of them is

$$n_1 + (N_k - n_2) = N_k + (n_1 - n_2) > N_k .$$

But this is impossible, since the space is N_k-dimensional, and therefore the conclusion of the theorem follows.

III. DISCUSSION

In a procedure of measurement satisfying conditions (a) and (b) of Sec. I the hypotheses of the theorem are clearly satisfied whenever u_1 is chosen from one of the subspaces $\underline{E}_m$ and u_2 is chosen from another. Hence, if the number of the $\{\underline{E}_m\}$ is greater than 1, there exist initial states of the object leading to final statistical states of the object plus apparatus which cannot be expressed as mixtures of exact eigenstates of the apparatus observable. Consequently, the problem of measurement in quantum mechanics cannot be solved by imposing (a) and (b) as conditions of measurement.

Fine[3] has proposed a formulation of measurement more general than that discussed in Sec. I. Taking the operators O and A on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, to represent the object and apparatus observables, he gives the following two definitions (here slightly rewritten):

Definition 1. If Q is a self-adjoint operator on a Hilbert space $\mathcal{H}$ then the statistical operators W and W' are Q-distinguishable if and only if $\mathrm{Tr}(WP_q) \neq \mathrm{Tr}(W'P_q)$ for some projection operator P_q in the spectrum of Q.

Definition 2. If W_a is a statistical operator on $\mathcal{H}_2$, then a unitary operator U on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a W_a-measurement of O by means of A if and only if the O-distinguishability of W_o, W_o' implies the A-distinguishability of $U(W_o \otimes W_a)U^{-1}$ and $U(W_o' \otimes W_a)U^{-1}$. The procedure of measurement envisaged by Fine's second definition may give extremely little information regarding the initial eigenvalue of an object observable by means of a single interaction of the object with a measuring apparatus. For this reason, his conception of measurement is very different from those considered in Ref. 1 and in Sec. I of this paper. However, his conception of measurement is legitimate as a procedure for determining some statistical information (in general less than the statistical state) about an ensemble of objects by means of an arbitrarily large number of measurements using a certain type of apparatus.

Concerning his conception of measurement, Fine asserts, but does not give a complete proof, of this theorem: There are no W_a-measurements U such that $U(W_o \otimes W_a)U^{-1}$ is a mixture of eigenstates of $1 \otimes A$ for all initial states W_o. (He assumes that the object observable O has at least two distinct eigenvalues.) His theorem is a consequence of the theorem of Sec. II. Suppose u_1 and u_2 are two eigenvectors of O associated with different eigenvalues, and hence O-distinguishable, and suppose that $U(P_{u_1} \otimes W_a)U^{-1}$ and $U(P_{u_2} \otimes W_a)U^{-1}$ are $1 \otimes A$-distinguishable and are both expressible as mixtures of eigenstates of $1 \otimes A$. Then the hypotheses of the theorem of Sec. II are satisfied. Therefore $U(P_u \otimes W_a)U^{-1}$ is not a mixture of eigenstates of $1 \otimes A$, if u is the superposition $g_1u_1 + g_2u_2$ with nonzero coefficients g_1 and g_2. On the other hand, if vectors u_1 and u_2 with the assumed properties do not exist, then evidently Fine's theorem would also hold.

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[1]M. H. Fehrs and A. Shimony, preceding paper, Phys. Rev. D 9, 2317 (1974).

[2]Professor A. Fine emphasized this point in private correspondence.

[3]A. Fine, Phys. Rev. D 2, 2783 (1970).