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Requirements for scattering solutions and the causal Green's function of the Bethe-Salpeter equation

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We investigate the conditions necessary for the existence of scattering solutions to the Bethe-Salpeter equation. We work with the causal Green's function in both timelike and spacelike regions of $x - x'$. Our results indicate that high-momentum or small-relative-time limits must be taken. In the small-relative-time limit, suitable restrictions on the interaction $I(x)$ must also be considered.

I. INTRODUCTION

In recent years, increasing interest has been placed on the use of the Bethe-Salpeter (BS) equation[1] as a vehicle for the study of the scattering of elementary particles.[2,3] The usual approach[4,5] demands the reproducibility of the Feynman-Dyson perturbation series in the ladder approximation with the resultant demand that causal boundary conditions must be used in the construction of the Green's function. It has been shown by Schwartz and Zemach[2] that the causal Green's function for large spacelike separations reduces to the form

$$G_c(x, x') = \frac{ie^{iq|\vec{x}-\vec{x}'|}}{8\pi(\omega_1+\omega_2)|\vec{x}-\vec{x}'|} \tag{1.1}$$

necessary for the description of a scattering process. The timelike regions, however, do not reduce to such a form. We investigate in this paper what precise conditions are necessary in order for the causal Green's function to lead to scattering solutions when both timelike and spacelike regions are included.

The simplest approach would be to demand that the integral representing the particular solution to the BS integral equation,

$$\int G_c(x, x')I(x')\psi(x')d^4x' \ , \tag{1.2}$$

where $I(x')$ represents the interaction and $\psi(x')$ represents the BS amplitude, gives a vanishing contribution when $x - x'$ is timelike. We find that this can be accomplished at the cost of requiring either the momentum to approach infinity or the relative time to approach small values. Using the large-momentum limit, which has played such a prominent role in recent particle theories (parton models,[6] current-algebra sum rules,[7] etc.), or the small-relative-time limit, the scattering solutions retained the nonrelativistic two-particle scattering form,

$$\psi(\vec{x}) = \phi(\vec{x}) + f(\theta)\frac{e^{iqr}}{r} \ . \tag{1.3}$$

The infinite-momentum limit has also been a useful tool in studying the symmetries of the BS equation. Recently, Kim and Zaoui[8] have shown that an O(3)-invariant integral equation resulted from taking the infinite-momentum limit to the BS equation.

We shall follow the procedure used by Huang and De Facio[9,10] in presenting the causal Green's function. However, we wish to point out some errors in their derivation and present the corrected results. These are presented in Sec. II along with our notation and a topology of the various space-time domains necessary to perform our analysis. Section III is devoted to the study of the conditions necessary to maintain scattering boundary conditions, and is separated into high momentum and small-relative-time cases. Section IV contains the summary.

We choose units such that $\hbar = c = 1$ and the inner product between four-vectors p and x is $p \cdot x = \vec{p}\cdot\vec{x} - p_0 x_0$.

II. FORM OF THE CAUSAL GREEN'S FUNCTION

We let x_1 and x_2 denote the space-time four-vectors which locate particles 1 and 2. The relative coordinate and momentum

$$\begin{aligned} x &= x_1 - x_2 , \\ p &= \mu_2 p_1 - \mu_1 p_2 \end{aligned} \tag{2.1}$$

are next introduced, where $\mu_1 + \mu_2 = 1$ is the only constraint placed on μ_1 and μ_2. We let ω_1 and ω_2 denote the center-of-momentum energies of particles 1 and 2, and let q denote the magnitude of the three-momentum for each particle in the c.m. frame. A convenient energy variable is

$$\nu = \mu_2\omega_1 - \mu_1\omega_2 , \tag{2.2}$$

which allows the Green's function equation to be written in the form

$$[\vec{p}\cdot\vec{p} - (p_0-\nu+\omega_1)^2 + m_1^2] \times [\vec{p}\cdot\vec{p} - (p_0-\nu+\omega_2)^2 + m_2^2] G(x,x') = \delta^4(x-x') . \tag{2.3}$$

The Fourier transform of $G(x,x')$ leads us to an off-the-mass-shell Ω plane in which we can select the appropriate contours to define a causal Green's function. If we denote the spatial part of $x-x'$ as R and the time part as T,

$$\vec{R} = \vec{x} - \vec{x}' , \quad T = t - t' , \tag{2.4}$$

the causal Green's function can be expressed as

$$^{+}G_c(x,x') = -\frac{ie^{-i\nu T}}{4\pi^2 R}\int_0^\infty dk\, k\sin(kR)\left\{\frac{e^{i(\omega_1-\omega_{1k})T}}{\omega_{1k}[(\omega_{1k}-\omega_1-\omega_2)^2-\omega_{2k}^2]} + \frac{e^{-i(\omega_2+\omega_{2k})T}}{\omega_{2k}[(\omega_{2k}+\omega_1+\omega_2)^2-\omega_{1k}^2]}\right\} \tag{2.5}$$

for $T>0$ and

$$^{-}G_c(x,x') = \frac{ie^{-i\nu T}}{4\pi^2 R}\int_0^\infty dk\, k\sin(kR)\left\{\frac{e^{i(\omega_1+\omega_{1k})T}}{\omega_{1k}[\omega_{2k}^2-(\omega_{1k}+\omega_1+\omega_2)^2]} + \frac{e^{-i(\omega_2-\omega_{2k})T}}{\omega_{2k}[\omega_{1k}^2-(\omega_{2k}-\omega_1-\omega_2)^2]}\right\} \tag{2.6}$$

for $T<0$. The superscripts $\pm$ refer to the sign of T, and ω_{ik} is given by

$$\omega_{ik} = (k^2+m_i^2)^{1/2} , \tag{2.7}$$

where $i=1,2$. The evaluation of the above integrals can be performed by going to the complex k plane with $q\to q+i\epsilon$.

In the work by Huang and De Facio,[9,10] the cut structures of the integrands due to ω_{ik} were taken correctly; however, the sign of ω_{ik} was not taken correctly everywhere in the complex k plane. With the correct sign, the causal Green's functions have the form (for $m_1=m_2=m$ and $\omega_1=\omega_2=\omega$)

$$^{\pm}G_c^\sigma(x-x') = \frac{ie^{-i\nu T}}{16\pi\omega}\frac{e^{iqR}}{R} + \frac{ie^{i(\omega-\nu)T}}{16\pi^2\omega R} f_1(R,T) - \frac{ie^{-i(\omega+\nu)T}}{16\pi^2\omega R} f_2(R,T) \tag{2.8}$$

for spacelike intervals of $x-x'$ and

$$^{\pm}G^\tau(x-x') = \pm\frac{ie^{i(\omega-\nu)T}}{16\pi^2\omega R} f_3(R,\pm T) \mp \frac{ie^{-i(\omega+\nu)T}}{16\pi^2\omega R} f_4(R,\pm T) \tag{2.9}$$

for timelike intervals of $x-x'$. The superscripts σ and τ refer to the spacelike and timelike intervals of $x-x'$, and

$$f_1(R,T) = A(R,T) - \omega\overline{A}(R,T) , \tag{2.10}$$

$$f_2(R,T) = A(R,T) + \omega\overline{A}(R,T) , \tag{2.11}$$

$$f_3(R,\pm T) = i[B(R,\mp T) - C(R,\mp T)] \pm \omega[\overline{B}(R,\mp T) + \overline{C}(R,\mp T)] , \tag{2.12}$$

$$f_4(R,\pm T) = i[B(R,\mp T) - C(R,\mp T)] \mp \omega[\overline{B}(R,\mp T) + \overline{C}(R,\mp T)] , \tag{2.13}$$

with

$$A(R,T) = -i\int_0^\infty dz\, \frac{z e^{-(m^2+z^2)^{1/2}R}\sinh(zT)}{z^2+\omega^2} , \tag{2.14}$$

$$\overline{A}(R,T) = \int_0^\infty dz\, \frac{e^{-(m^2+z^2)^{1/2}R}\cosh(zT)}{z^2+\omega^2} , \tag{2.15}$$

$$B(R,\pm T) = \int_0^m dz\, \frac{z\sinh[(m^2-z^2)^{1/2}R]e^{\pm izT}}{z^2-\omega^2} , \tag{2.16}$$

$$\overline{B}(R,\pm T) = i\int_0^m dz\, \frac{\sinh[(m^2-z^2)^{1/2}R]e^{\pm izT}}{z^2-\omega^2} , \tag{2.17}$$

$$C(R,\pm T) = \int_0^\infty dz\, \frac{z\sinh[(m^2+z^2)^{1/2}R]e^{\pm zT}}{z^2+\omega^2} , \tag{2.18}$$

and

$$\overline{C}(R,\pm T) = \int_0^\infty dz\, \frac{\sinh[(m^2+z^2)^{1/2}R]e^{\pm zT}}{z^2+\omega^2} . \tag{2.19}$$

One can readily observe that $A(R,T)$ and $\overline{A}(R,T)$ are of order e^{-mR}, and therefore

$$^{\pm}G_c^\sigma(x-x') = \frac{ie^{-i\nu T}}{16\pi\omega}\frac{e^{iqR}}{R} + O(e^{-mR}) , \tag{2.20}$$

in agreement with the results of Schwartz and Zemach.[2] However, the timelike causal Green's functions $^{\pm}G_c^\tau$ do not have the above form.

If the interaction $I(x_1,x_2)$ has the symmetry

$$I(x_1,x_2) = I(x) , \tag{2.21}$$

then the BS amplitude using the causal Green's function takes the form

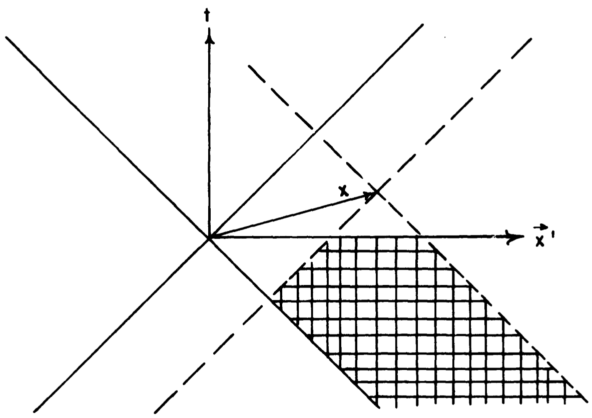

FIG. 1. The domain $\mathcal{S}^-_{\tau+}$ for the x' integration, given that x is spacelike with time component positive. The solid lines represent the light cone for the four-vector x, while the dashed lines represent the inverted light cone for the four-vector $x-x'$.

TABLE I. Possible domains for the x' integration with σ, τ, (+), and (−) referring to the spacelike, timelike, and positive- and negative-time-component properties, respectively, of the relevant four-vectors.

x	x'	$x-x'$	D_α
$\sigma+$	$\sigma+$	$\sigma\pm, \tau\pm$	$\mathcal{S}^+_{\sigma+}, \mathcal{S}^+_{\sigma-}, \mathcal{S}^+_{\tau+}, \mathcal{S}^+_{\tau-}$
$\sigma+$	$\sigma-$	$\sigma+, \tau+$	$\mathcal{S}^-_{\sigma+}, \mathcal{S}^-_{\tau+}$
$\sigma+$	$\tau+$	$\sigma\pm, \tau-$	$\mathcal{T}^+_{\sigma+}, \mathcal{T}^+_{\sigma-}, \mathcal{T}^+_{\tau-}$
$\sigma+$	$\tau-$	$\sigma+, \tau+$	$\mathcal{T}^-_{\sigma+}, \mathcal{T}^-_{\tau+}$
$\sigma-$	$\sigma+$	$\sigma-, \tau-$	$\mathcal{S}^+_{\sigma-}, \mathcal{S}^+_{\tau-}$
$\sigma-$	$\sigma-$	$\sigma\pm, \tau\pm$	$\mathcal{S}^-_{\sigma+}, \mathcal{S}^-_{\sigma-}, \mathcal{S}^-_{\tau+}, \mathcal{S}^-_{\tau-}$
$\sigma-$	$\tau+$	$\sigma-, \tau-$	$\mathcal{T}^+_{\sigma-}, \mathcal{T}^+_{\tau-}$
$\sigma-$	$\tau-$	$\sigma\pm, \tau+$	$\mathcal{T}^-_{\sigma+}, \mathcal{T}^-_{\sigma-}, \mathcal{T}^-_{\tau+}$
$\tau+$	$\sigma+$	$\sigma\pm, \tau+$	$\mathcal{S}^+_{\sigma+}, \mathcal{S}^+_{\sigma-}, \mathcal{S}^+_{\tau+}$
$\tau+$	$\sigma-$	$\sigma+, \tau+$	$\mathcal{S}^-_{\sigma+}, \mathcal{S}^-_{\tau+}$
$\tau+$	$\tau+$	$\sigma\pm, \tau\pm$	$\mathcal{T}^+_{\sigma+}, \mathcal{T}^+_{\sigma-}, \mathcal{T}^+_{\tau+}, \mathcal{T}^+_{\tau-}$
$\tau+$	$\tau-$	$\tau+$	$\mathcal{T}^-_{\tau+}$
$\tau-$	$\sigma+$	$\sigma-, \tau-$	$\mathcal{S}^+_{\sigma-}, \mathcal{S}^+_{\tau-}$
$\tau-$	$\sigma-$	$\sigma\pm, \tau-$	$\mathcal{S}^-_{\sigma+}, \mathcal{S}^-_{\sigma-}, \mathcal{S}^-_{\tau-}$
$\tau-$	$\tau+$	$\tau-$	$\mathcal{T}^+_{\tau-}$
$\tau-$	$\tau-$	$\sigma\pm, \tau\pm$	$\mathcal{T}^-_{\sigma+}, \mathcal{T}^-_{\sigma-}, \mathcal{T}^-_{\tau+}, \mathcal{T}^-_{\tau-}$

$$\psi(x)=\phi(x)+\sum_\alpha \int_{D_\alpha} d^4x' G_c^\alpha(x-x')I(x')\psi(x'), \quad (2.22)$$

where $\phi(x)$ is the homogeneous solution, D_α represents the αth domain with α referring to the various spacelike, timelike, and the sign of the time-component properties of the four-vector $x-x'$. For integration purposes, it is useful to break the domains of the four-vectors x, x', and $x-x'$ into smaller regions such that one can specify a common Green's function. Since the Green's functions are distinct for $x-x'$ being spacelike or timelike and T being positive or negative, we shall dissect the entire region of x' into smaller domains, with the α index previously mentioned referring to the above possible regions of $x-x'$. We shall write $\mathcal{S}^i_{\beta h}$ and $\mathcal{T}^i_{\beta h}$ to denote spacelike (σ) and timelike (τ) regions of x', respectively. The i superscript refers to the sign of t', (+) for $t'>0$ and (−) for $t'<0$. The β subscript refers to the spacelike (σ) or timelike (τ) properties of the four-vector $x-x'$ and the h subscript refers to the sign of T, (+) for $T>0$ and (−) for $T<0$. For a given four-vector x, the possible regions can be read off from a diagram similar to Fig. 1, which is the example for the case $x\in\sigma_+$ (spacelike with $t>0$). The shaded area in Fig. 1 corresponds to $\mathcal{S}^-_{\tau+}$, the solid 45° lines are the light cone for the four-vector x', and the dashed lines inclined at 45° are the inverted light cone for the four-vector $x-x'$. Table I gives a complete topology for the x' regions in terms of the properties of x, x', and $x-x'$.

III. SCATTERING BOUNDARY CONDITIONS AND THE INFINITE-MOMENTUM AND SMALL-RELATIVE-TIME LIMITS

Using the relationships listed in Sec. II, we can express Eq. (2.22) in terms of the relevant Green's functions. The simplest method is to first specify the domains for the four-vector x and then use Table I to find the integration region D_α for the four-vector x'. If we write the BS amplitude ψ with subscripts βh to denote the space-time regions for x (not $x-x'$ now), the amplitudes $\psi_{\beta h}$ become

$$\begin{aligned}\psi_{\beta h}(x)=\phi(x)+\frac{i}{16\pi^2\omega}\Bigg\{&\int_{D^1_{\beta h}} d^4x'\, e^{-i\nu T}\psi(x')\frac{I(x')}{R}\left[\pi e^{iqR}+e^{i\omega T}f_1(R,T)-e^{-i\omega T}f_2(R,T)\right]\\ &+\int_{D^2_{\beta h}} d^4x'\, e^{-i\nu T}\psi(x')\frac{I(x')}{R}\left[e^{i\omega T}f_3(R,T)-e^{-i\omega T}f_4(R,T)\right]\\ &-\int_{D^3_{\beta h}} d^4x'\, e^{-i\nu T}\psi(x')\frac{I(x')}{R}\left[e^{i\omega T}f_3(R,-T)-e^{-i\omega T}f_4(R,-T)\right]\Bigg\} \quad (3.1)\end{aligned}$$

where

$$\begin{aligned}
D^1_{\sigma+} &= \mathcal{S}^+_{\sigma+} \oplus \mathcal{S}^+_{\sigma-} \oplus \mathcal{S}^-_{\sigma+} \oplus \mathcal{T}^+_{\sigma+} \oplus \mathcal{T}^+_{\sigma-} \oplus \mathcal{T}^-_{\sigma+}\,,\\
D^2_{\sigma+} &= \mathcal{S}^+_{\tau+} \oplus \mathcal{S}^-_{\tau+} \oplus \mathcal{T}^-_{\tau+}\,,\\
D^3_{\sigma+} &= \mathcal{S}^+_{\tau-} \oplus \mathcal{T}^+_{\tau-}\,,\\
D^1_{\sigma-} &= \mathcal{S}^+_{\sigma-} \oplus \mathcal{S}^-_{\sigma+} \oplus \mathcal{S}^-_{\sigma-} \oplus \mathcal{T}^+_{\sigma-} \oplus \mathcal{T}^-_{\sigma+} \oplus \mathcal{T}^-_{\sigma-}\,,\\
D^2_{\sigma-} &= \mathcal{S}^-_{\tau+} \oplus \mathcal{T}^-_{\tau+}\,,\\
D^3_{\sigma-} &= \mathcal{S}^+_{\tau-} \oplus \mathcal{S}^-_{\tau-} \oplus \mathcal{T}^+_{\tau-}\,,\\
D^1_{\tau+} &= \mathcal{S}^+_{\sigma+} \oplus \mathcal{S}^+_{\sigma-} \oplus \mathcal{S}^-_{\sigma+} \oplus \mathcal{T}^+_{\sigma+} \oplus \mathcal{T}^+_{\sigma-}\,,\\
D^2_{\tau+} &= \mathcal{S}^+_{\tau+} \oplus \mathcal{S}^-_{\tau+} \oplus \mathcal{T}^+_{\tau+} \oplus \mathcal{T}^-_{\tau+}\,,\\
D^3_{\tau+} &= \mathcal{T}^+_{\tau-}\,,\\
D^1_{\tau-} &= \mathcal{S}^+_{\sigma-} \oplus \mathcal{S}^-_{\sigma+} \oplus \mathcal{S}^-_{\sigma-} \oplus \mathcal{T}^-_{\sigma-} \oplus \mathcal{T}^-_{\sigma+}\,,\\
D^2_{\tau-} &= \mathcal{T}^-_{\tau+}\,,\\
D^3_{\tau-} &= \mathcal{S}^+_{\tau-} \oplus \mathcal{S}^-_{\tau-} \oplus \mathcal{T}^+_{\tau-} \oplus \mathcal{T}^-_{\tau-}\,.
\end{aligned} \tag{3.2}$$

As mentioned previously, the terms involving f_1 and f_2 do lead to scattering solutions, while the terms involving f_3 and f_4 lead to nonscattering solutions. Let us illustrate this point by considering the total effect that $B(R, \pm T)$ will have on the BS amplitude $\psi_{Bh}(x)$. The relevant contribution has the form

$$\int_{D^*} d^4x'\, e^{-i(\nu\pm\omega)T}\psi(x')\frac{I(x')}{R}\int_0^m dz\,\frac{z\sinh(\eta R)e^{\pm izT}}{z^2-\omega^2}\,, \tag{3.3}$$

where D^* is the relevant domain chosen from Eq. (3.2) and $\eta=(m^2-z^2)^{1/2}$. The $e^{-\eta R}$ part of $\sinh(\eta R)$ cause no difficulty; therefore, let us consider only the $e^{\eta R}$ part. The integral of interest becomes

$$\int_{D^*} d^4x'\, e^{-i(\nu\pm\omega)T}\psi(x')\frac{I(x')}{R}\int_0^m dz\,\frac{z\,e^{\eta R}e^{\pm izT}}{z^2-\omega^2}\,. \tag{3.4}$$

We wish to separate the $\vec{x}$ and $\vec{x}'$ parts of $1/R$ and $e^{\eta R}$. This can be accomplished by the expansion

$$\frac{e^{\eta R}}{R} = 4\pi\eta\sum_{l=0}^{\infty} j_l(-i\eta r_<)h_l^{(1)}(-i\eta r_>) \times\sum_{m_l=-l}^{l} Y^*_{l,m_l}(\theta',\phi')Y_{l,m_l}(\theta,\phi), \tag{3.5}$$

where $r=|\vec{x}|$, $r'=|\vec{x}'|$, and where $r_<$ and $r_>$ are the smaller and larger of r and r'.

Equation (3.4) can now be rewritten as

$$4\pi\int_{D^*} d^4x'\, e^{-i(\nu\pm\omega)T}\psi(x')I(x')\int_0^m dz\left[\frac{z\eta e^{\pm izT}}{z^2-\omega^2}\sum_{l=0}^{\infty} j_l(-i\eta r_<)h_l^{(1)}(-i\eta r_>)\sum_{m_l=-l}^{l} Y^*_{l,m_l}(\theta',\phi')Y_{l,m_l}(\theta,\phi)\right]. \tag{3.6}$$

We can expand D^* into domains such that $r>r'$ and $r<r'$:

$$D^* = D^*_> \oplus D^*_<\,. \tag{3.7}$$

Equation (3.6) can be expanded into two parts, giving the integral

$$\begin{aligned}
&4\pi\int_{D^*_>} d^4x'\, e^{-i(\nu\pm\omega)T}\psi(x')I(x')\int_0^m dz\left[\frac{z\eta e^{\pm izT}}{z^2-\omega^2}\sum_{l=0}^{\infty} j_l(-i\eta r')h_l^{(1)}(-i\eta r)\sum_{m_l=-l}^{l} Y^*_{l,m_l}(\theta',\phi')Y_{l,m_l}(\theta,\phi)\right]\\
&\quad+4\pi\int_{D^*_<} d^4x'\, e^{-i(\nu\pm\omega)T}\psi(x')I(x')\int_0^m dz\left[\frac{z\eta e^{\pm izT}}{z^2-\omega^2}\sum_{l=0}^{\infty} j_l(-i\eta r)h_l^{(1)}(-i\eta r')\sum_{m_l=-l}^{l} Y^*_{l,m_l}(\theta,\phi)Y_{l,m_l}(\theta',\phi')\right].
\end{aligned} \tag{3.8}$$

It would be helpful at this point to consider the behavior of the functions $h_l^{(1)}(-i\eta r')$ and $j_l(-i\eta r')$. The $l=0$ case is sufficient to inform us of the relevant properties for our scattering boundary conditions,

$$\begin{aligned}
h_0^{(1)}(-i\eta r') &= \frac{e^{\eta r'}}{\eta r'}\,,\\
j_0(-i\eta r') &= \frac{\sinh(\eta r')}{\eta r'}\,.
\end{aligned} \tag{3.9}$$

Since the z integration ranges from 0 to m, the worst possible behavior for large r' is $e^{mr'}/r'$. If suitable restrictions are placed on $I(x')$, the x' integrations are well behaved. However, we are left with an amplitude whose large-r behavior seemingly increases exponentially, violating the scattering boundary conditions. We shall introduce two different methods to handle this difficulty, the large-momentum limit and the small-relative-time limit.

A. Large-momentum limit

In the large-momentum limit, we may neglect terms contributing to the causal Green's function which are of order $1/\omega^2$. From Eqs. (2.8)–(2.19) we can see that all the f_i's can be neglected and the BS amplitude can be written as

$$\psi_{Bh}(x) = \phi(x) + \frac{i}{16\pi\omega}\int_{D^1_{Bh}} d^4x'\, e^{-i\nu T}\frac{e^{iqR}}{R} I(x')\psi(x')\,. \tag{3.10}$$

The troublesome timelike regions of $x - x'$ are conveniently eliminated from the integration over x'. For $r \gg r'$, the above reduces to the familiar form

$$\psi_{Bh}(x) = \phi(x) + \frac{e^{i(qr - \nu t)}}{r} f(\vec{q}' - \vec{q}), \tag{3.11}$$

where

$$f(\vec{q}' - \vec{q}) = \frac{i}{16\pi\omega}\int d^4x'\, e^{i\nu t'} e^{i\vec{q}'\cdot\vec{x}'} I(x')\psi(x')\,. \tag{3.12}$$

B. Small-relative-time limit

It is desirable to set up a procedure which would not require the large-momentum limit since there are many applications of the BS equation to nonasymptotic energy regions.[11] We return to $B(R, \pm T)$ and its contribution to the BS amplitude in the form of Eq. (3.3). Collecting all of the t' dependence in Eq. (3.3), we have

$$\int_{D^*} dt'\, e^{i(\nu \pm \omega)t'} e^{\pm i z t'} I(x')\psi(x'), \tag{3.13}$$

where z runs over the real numbers from 0 to m. Let us assume that the BS amplitude can be written in the form

$$\psi(x') = e^{-i\nu t'}\psi_s(\vec{x}'). \tag{3.14}$$

Substituting Eq. (3.14) into (3.13) and dropping the irrelevant $\psi_s(\vec{x})$ factor, the integral (3.13) becomes

$$\int_{t' \in D^*} dt'\, e^{\pm i(\omega \pm z)t'} I(x')\,. \tag{3.15}$$

It would be useful at this stage to write down explicitly the D^* domains. The integral in (3.15) has the two forms,

$$\int_{t' \in D^2_{Bh}} dt'\, e^{i(z \pm \omega)t'} I(x') \tag{3.16}$$

and

$$\int_{t' \in D^3_{Bh}} dt'\, e^{-i(z \pm \omega)t'} I(x')\,. \tag{3.17}$$

It would be highly desirable if we could set conditions on $I(x')$ such that the above two integrals (3.16) and (3.17) would vanish. Let us assume that $I(x')$ has the form

$$I(x') = I_0(\vec{x}')e^{-a|t'|}, \tag{3.18}$$

where a is a positive real constant. If we remember that the $r \gg r'$ condition really implies that $I(x')$ is nonzero only for small (compared with r) values of r', the regions D^2_{Bh} and D^3_{Bh} will overlap with the nonzero regions of $I(x')$ only for large values $|t'|$, providing that the relative time t is zero or close to zero. This point is illustrated in Fig. 2 for the case of zero relative time, where the horizontally shaded region is the nonvanishing domain of $I(x')$ and the two cross-hatched areas are the overlap regions. The boundaries of the overlap regions formed from the dashed lines in Fig. 2 give half of the limits for the integrals in (3.16) and (3.17), with plus or minus infinity giving the other half. Since the slope of the dashed line (the inverted Minkowski cone for $x - x'$) is 1, the magnitude of the finite limits is between $r - b$ and $r + b$, where b is the range of the *short-ranged* interaction $I(x')$. If the relative time t was not zero, the magnitude of the finite limits would just be shifted by an amount t and would now be between $|r - b \pm t|$ and $|r + b \pm t|$.

Specifically, the integrals (3.16) and (3.17) become

$$\int_{-\infty}^{-(r-b-t)} dt'\, e^{-a|t'|} e^{i(z \pm \omega)t'} \tag{3.19}$$

and

$$\int_{r-b+t}^{\infty} dt'\, e^{-a|t'|} e^{-i(z \pm \omega)t'}, \tag{3.20}$$

where we have omitted the irrelevant $I_0(\vec{x}')$ factor and have represented the finite limits by just

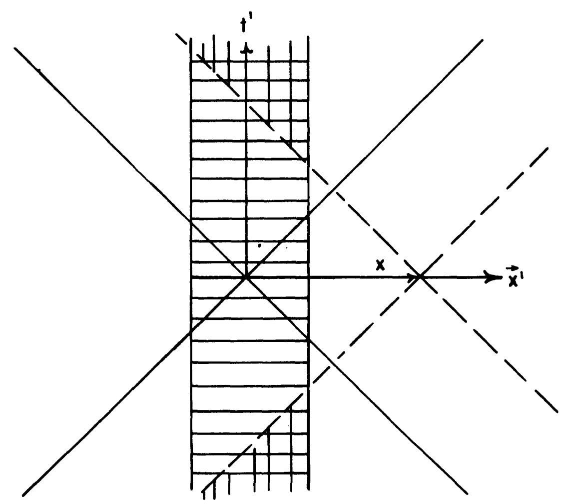

FIG. 2. Minkowski diagram showing that for $t = 0$ the overlap region with $I(x')$ (cross-hatched area) occurs only for $|t'|$ large. The solid lines represent the light cone for x', while the dashed lines represent the inverted light cone for $x - x'$.

$\pm(r - b \pm t)$ for simplicity. After we perform the indicated integrations in Eqs. (3.19) and (3.20) for the case $r - b \mp t > 0$, the term is of the form

$$Q(k, \omega, r, t)e^{\pm at}e^{-ar}, \tag{3.21}$$

where

$$Q(k, \omega, r, t) = \frac{\exp[ab - i(z \pm \omega)(r - b \mp t)]}{a + i(z \pm \omega)}. \tag{3.22}$$

Referring back to Eqs. (3.8) and (3.9), we see that the e^{-ar} factor has exactly the form required to cancel the e^{mr} factors appearing in Eq. (3.8) [from $h_l^{(1)}(-i\eta r)$ and $j_l(-i\eta r)$]. For $a > m$, the $B(R, -T)$ contributions to the BS amplitude vanish for small values of $|t|$ in the asymptotic region $r \gg b$. It is obvious that the same statement can be made for $\overline{B}(R, \pm T)$.

The $C(R, \pm T)$ and $\overline{C}(R, \pm T)$ terms can also be treated by this type of analysis, but instead of Eqs. (3.21) and (3.22), we have

$$\frac{\exp[(a+z)b + iz(r - b \mp t)]}{(a+z) - i\omega} e^{-(a+z)r} e^{\pm(a+z)t}. \tag{3.23}$$

The form of Eq. (3.23) will cancel the exponentially increasing behavior caused by the factor $\sinh[(z^2 + m^2)^{1/2}R]$ appearing in Eqs. (2.18) and (2.19). The sinh factor gives rise to a contribution of the form $\exp[(z^2 + m^2)^{1/2}r]$. However, for $a > m$ we always have $(a + z) > (z^2 + m^2)^{1/2}$. Therefore, for small values of relative time $|t|$, the $C(R, \pm T)$ and $\overline{C}(R, \pm T)$ contributions will also vanish in the asymptotic region. A large value for z does not invalidate our argument since we are in the region where $|T| > R$ and from Eq. (2.18) the net exponential factor will always damp. We now have eliminated in Eq. (3.1) the integrals over the domains $D^2_{\beta h}$ and $D^3_{\beta h}$. As was previously noted, $f_1(R, T)$ and $f_2(R, T)$ can be neglected in the asymptotic region and we are left with the same results of Sec. III A, Eqs. (3.10)–(3.12).

IV. SUMMARY

We find that the high-momentum limit will allow the BS equation to have a plane-wave solution plus outgoing scattered-wave solutions. The scattering boundary conditions can also be obtained if the relative time $|t|$ is small and if the integrals of the form (3.15) decrease faster than e^{-mr}. These conditions on the interaction can be met if $I(x')$ has the form

$$I(\vec{x}', t') = I_0(\vec{x}')e^{-a|t'|},$$

with $a > m$ and $I_0(\vec{x}')$ having a short range $b \ll r$.

From a slightly different perspective, we can understand why extra conditions are intrinsically necessary in the study of the BS equation. Consider the timelike causal Green's functions ${}^{\pm}G_c^T(R, T)$ given in Eqs. (2.8) and (2.9). These functions contain factors of the form $e^{\pm i\omega t}$ which will reappear in the inhomogeneous solution to the BS equation. However, the terms of the scattering solution must all contain the same exponential time factor (conservation of energy). The homogeneous solution $\phi(x)$ and part of the ${}^{\pm}G_c^0(R, T)$ contribution to the inhomogeneous solution both have exactly the factor $e^{-i\nu t}$; therefore, conditions must be placed on the remaining terms to either cancel the $e^{\pm i\omega t}$ factors or eliminate entirely the remaining terms. The conditions listed above accomplish exactly this. For the part of ${}^{\pm}G_c^0(R, T)$ which contains $e^{\pm i\omega t}$ factors, there is no problem since these terms are of order e^{-mr}. However, they also vanish in both the high-momentum and small-relative-time limits.

Finally we note that exactly the same analysis can be carried out for the nonequal-mass case ($m_1 \neq m_2$ and $\omega_1 \neq \omega_2$). There are more terms for the nonequal-mass problem; however, they are all of the same form as presented here.

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