ural in algebraic terms. They are the steps from commutative * algebras to * algebras to algebras.

[7]A mark is an equivalence class of throws. A throw of a line L is an ordered quadruple of points $(P_0 P_1 P_\infty P)$ on L. See O. Veblen and J. W. Young [*Projective Geometry I* (Ginn, Boston, Mass., 1910), p. 157] for the operations of the ring of marks. A. N. Whitehead [*Axioms of Projective Geometry* (Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge, 1906)] rather anticipates q logic; for him, "Geometry is the science of cross-classification." And he meant projective geometry especially.

[8]Cf. V. S. Varadarajan, *Geometry of Quantum Theory* (Van Nostrand Rheinhold, New York, 1968). I have adapted the standard term channel from J. N. Blatt and V. F. Weisskopf [*Theoretical Nuclear Physics* (Wiley, New York, 1952)], dropping purity but keeping idempotence. Elsewhere channels and cochannels are called states and tests, effectors and receptors, ...

[9]A. H. Taub and J. W. Givens [*Geometry of Complex Domains* (Princeton Univ. Press, Princeton, 1955)] are a good source for projective concepts. Every projective concept is also an rq logical one.

[10]Termed antipolarity in Ref. 9.

[11]J. Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, Mass., 1970).

[12]D. Finkelstein, G. Frye, and L. Susskind, following paper, Phys. Rev. D 9, 2231 (1974).

[13]T. D. Lee, Phys. Rev. Lett. 26, 801 (1971).

PHYSICAL REVIEW D VOLUME 9, NUMBER 8 15 APRIL 1974

Space-time code. V

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(Received 5 December 1972)

The concept of a quantum dynamics is recapitulated. The Dirac equation is obtained from a pure quantum dynamics as the limit of classical time. The theory is defective in projective gauge invariance and semantic consistency, but illustrates the relation between dynamical and experimental elements of q dynamics, and is finite, Lorentz-invariant, and local.

I. INTRODUCTION

In this work we recapitulate the present status of pure quantum (q) mechanics[1] (Sec. II) and show how the Dirac equation may be obtained as the mixed cq theory resulting from a q mechanics in the limit of classical time (Sec. III). The procedure is marred by a certain arbitrariness discussed in Sec. IV but provides a guide toward a fuller q dynamics with interactions.

The formulation of mechanics that emerges from these mathematical models is stable under the transition from classical mechanics to quantum mechanics and provides a plausible successor for quantum mechanics. It implies the following conception of the world:

(1) Both the classical space-time continuum and quantized fields are semimacroscopic statistical contructs, part of the surface structure of the world manifested in processes that are long compared to an elementary time τ.

(2) The deep structure contains neither space-time nor fields. The microscopic world is a discrete complex of discrete binary entities, elementary quantum processes. Such a world is not plenum but plexus, obeying Mach's principle in the strongest possible form: There is no space between matter, no spatial relations without interaction.

(3) The dynamical law is not a differential equation but one stator $|D)$ constructed by finite algebraic operations and yielding the Feynman amplitude in the appropriate limit $\tau \to 0$. The amplitude for any process E is the inner product of $|D)$ with a costator $)E|$.

(4) Particles are recognized by discrete chromosomelike patterns of elementary process. For example a most simple $|\dot{D})$ involving only a line complex (processes in simple series) gives rise to the Minkowski space-time and the proper-time Dirac equation for the electron as $\tau \to 0$, while a double strand is similarly related to Maxwell's equation and the photon.

II. q DYNAMICS

The basic entity is the q process. We start from a primitive q process or monad χ. (Here χ is only the name of a quantum, not an algebraic quantity of some sort.) Like any quantum, χ is associated with a linear space $L(\chi)$, subspaces of which ideally represent χ channels, ways of precontrolling a χ. χ stators, vectors of $L(\chi)$, are written as $|\rightarrow\rangle$ or $\langle\leftarrow|$. The interior arrows are used when it is necessary to tell stators $|\rightarrow\rangle$ in $L(\chi)$ from complex conjugate stators $|\leftarrow\rangle$ in the complex conjugate space $L^C(\chi)$. Costators, vectors of the dual space $L^T(\chi)\equiv L(\chi^T)$, are written $\rangle\rightarrow|$ or $|\leftarrow\langle$. Contraction of a stator $|\rangle$ and a costator $\rangle|$ is written $|\rangle|$. Subspaces of the dual space ideally represent χ cochannels, ways of postcontrolling a χ. At present we foresee operational meaning only for highly composite assemblages of χ's, processes we can actually, and not merely ideally, control.

The spaces $L(\chi)$ and $L^T(\chi)$ represent the class logic of the monad χ. We keep the theory Lorentz-invariant by keeping the χ logic Lorentz-invariant. Only the subspaces of $L(\chi)$, the subspaces of $L^T(\chi)$, the inclusion relations within $L(\chi)$, the inclusion relations within $L^T(\chi)$, and the orthogonality relations $|\rightarrow\rangle\rightarrow|=0$ between $L(\chi)$ and $L^T(\chi)$ are given meaning. If the stators $|A\rangle$ of a channel A are orthogonal to the costators $\rangle B|$ of a cochannel B, $|A\rangle B|=0$, a process from channel A will never pass cochannel B. For Lorentz invariance $L(\chi)$ is taken to be two-dimensional. The invariance group of the logic is then the antiprojective group on two complex homogeneous variables, AP(2, C), isomorphic to the Lorentz group. We write $\hat{2}$ for this χ.

An assembly or product of n elementary processes or $\hat{2}$'s is called an n-ad: monad, dyad,. . . .

The general process π is a q assembly of $\hat{2}$'s with a further element of structure, a causal or chronological order C. As a q assembly, the process π is constructed from the monad $\hat{2}$ by algebraic procedures of quantification or second quantization familiar from many-body theory. Thus π, too, is represented in a linear space, $L(\pi)$, whose stators are tensors of arbitrary rank over $L(\hat{2})$. The chronological order C among the monads of π is represented by a partial order C of the indices of the tensors in $L(\pi)$, and we suppose $L(\pi)$ provided with this structure. This partial order can be graphically represented by a network or 1-complex on whose 1-cells are placed the indices of the π stators. Such a network of chronologically ordered processes we call a plexus; the ordered tensors which are its stators we call plexors. Plexors play the part in q mechanics that Feynman path amplitudes do in cq.

The physical theory is given by the plexor space $L(\pi)$ (kinematics), a projection $)D|D)$ in $L(\pi)$ (dynamics), and rules for associating with various experiments or environments E projections $)E|E)$ in $L^T(\pi)$ (semantics). The physical results of the theory are all expressible in terms of amplitudes $|D)E|$ whose vanishing means the experiment $)E|$ invariably gives a null result according to the dynamics $|D)$.

With an appropriate choice for the linear space of stators $|\rightarrow\rangle$, $L(\pi)$, $)D|D)$, and $)E|E)$ this formalism reproduces the Hamiltonian dynamics of ordinary (cq) quantum mechanics. With another choice, it reproduces the Feynman-amplitude theory for relativistic particle processes. These theories use a classical background space-time. We do not.

III. THE DIRAC EQUATION

A. The Simplest Dynamics

Let $\rangle\rightarrow\rangle$ designate the unit operator on $L(\chi)$. The simplest $|D)$, whatever the monad χ may be, is a linearly ordered product of unit operators or tensors,

$$|D_0)=\oplus\ \rangle\rightarrow\rangle\rangle\rightarrow\rangle\cdots\rangle\rightarrow\rangle\ ,$$

where the chronological order is that of the arrows, the ellipsis indicates that the number of factors σ is arbitrary, and the $\oplus$ is a direct sum over $\sigma=0, 1, 2, \ldots$. This $|D)$ can be thought of as describing a process consisting of successive processes χ^T and χ, with a perfect correlation between each χ^T and its chronologically following χ. Each unit tensor $\rangle\rightarrow\rangle$ is a singlet stator for a $\chi^T\chi$ pair. The complex of this plexor is a single line of σ segments, representing the linear chronological order of these processes. Each term in this $|D)$ can be regarded as a special product, the sequential product $\odot$, of its factors $\rangle\rightarrow\rangle$:

$$|D_0)=\bigoplus_{\sigma=0}^{\infty}\ \bigodot_{n=1}^{\sigma}\rangle\rightarrow\rangle\ ^{\boldsymbol{\cdot}}.$$

The sequential product $|A)\odot|B)$ of two plexors $|A)$, $|B)$ is formed when $|A)$ and $|B)$ are linear by (1) connecting the final segment of $|A)$ to the initial of $|B)$ to get the product complex and (2) forming the direct product of the tensors $|A)$ and $|B)$ to get the product tensor.

If $|A)$ and $|B)$ are not linear but branching, there may be more than one final segment of $|A)$, more than one initial segment of $|B)$. If a universal definition of $\odot$ is desired, it seems natural to sum over all possible connections of final segments of $|A)$ and initial segments of $|B)$. When we consider a branching $|D)$, dynamics of interaction and production processes, we shall make $|D)$ skew-symmetric in its initial channels and also in its final to ensure the spin-statistics connection, following Feynman.

B. Outer and Inner Products and Sums: A Lemma

Let a and b belong to an algebra A. We distinguish between two familiar kinds of operations upon a and b as *inner* and *outer*. The inner + and $\times$ are the ordinary operations in the algebra A. To carry out an outer operation we create a new algebra A', a replica of A, take the replica b' in A' of b in A, and form a direct sum or product of a and b'. The sequential and parallel products $\odot$ and $\oplus$ are outer products.

$|D_0)$ converts outer products into inner.

Indeed let $)A|$ be the sequential product of $\sigma - 1$ factors $\rangle 1\rangle, \rangle 2\rangle, \ldots, \rangle\sigma-1\rangle$ and $|D_0)$ be the sequential product of σ $\rangle\to\rangle$'s:

$$
\begin{aligned}
)A| &= \quad \rangle 1\rangle \;\; \rangle 2\rangle \cdots \rangle\; \sigma-1\rangle\,,\\
|D_0) \;&= \rangle\to\rangle\;\; \rangle\to\rangle \quad \cdots \qquad \rangle\to\rangle\;.
\end{aligned}
$$

We have aligned $|D_0)$ with $)A|$ vertically to bring together those dual channels or indices that are to be mated when we form their contraction, which we shall write tr $)A|D_0)$. This contraction is a plexor of the form $\rangle a\rangle$ since only two channels remain unmated, the costator $\rangle\,|$ at the left end and the stator $|\rangle$ at the right.

Lemma.

$$\operatorname{tr})A|D_0) = \rangle a\rangle = \rangle\, 1\rangle\, 2\rangle \cdots \rangle\sigma-1\rangle \tag{1}$$

is the inner product of the factors $\rangle 1\rangle, \ldots, \rangle\sigma-1\rangle$ that appear in the outer product $)A|$.

Since an outer product of quantities infinitesimally close to 1 leads to an outer sum of infinitesimals, $|D_0)$ also converts outer sums into inner.

Indeed, with $|D_0)$ and $)A|$ as above, let us define algebraic operations $+, \times$ upon $)A|$'s permitting us to form an exponential exp $)A|$. For + we take the + of the linear space of $)A|$'s. For $\times$ we take

$$
\begin{aligned}
&(\rangle 1\rangle \odot \cdots \odot \rangle\sigma\rangle) \times (\rangle 1'\rangle \odot \cdots \odot \rangle\sigma'\rangle)\\
&\qquad\qquad \equiv (\rangle\, 1\rangle\, 1'\rangle) \odot \cdots \odot (\rangle\sigma\rangle\sigma'\rangle)\,. \qquad (2)
\end{aligned}
$$

Here $\rangle\, 1\rangle\, 1'\rangle$ is an inner product of two quantities (operators) $\rangle 1\rangle$ and $\rangle 1'\rangle$. The product $\times$ brings together quantities associated with the same time and lets quantities at different times commute. It is a kind of outer product.

Let then $)A|$ and $)a|$ be the outer and the inner sum of the same sequence of operators. (Some factors of the unit operator $\rangle\to\rangle$ are left implicit.) Using the outer $\times$ and inner product we define two exponentials $\exp A$ and $\exp a$. The lemma for this is

$$\operatorname{tr}|D_0)\exp A| = \exp a\;. \tag{3}$$

This is an immediate consequence of (1).

C. Coordinates

To form experimental stators $|E)$ we use polyadic plexors $\sum\rangle p\rangle$ which are sums of many dyad plexors $\rangle p\rangle$. Let

$$
\begin{aligned}
)\textstyle\sum\rangle p\rangle| \equiv\;& \rangle p\rangle\;\rangle\to\rangle \cdots \rangle\to\rangle\\
&+\rangle\to\rangle\;\rangle p\rangle \;\cdots \rangle\to\rangle + \cdots\\
&+\rangle\to\rangle\;\rangle\to\rangle \cdots \rangle p\,\rangle\,,\\
|D_0) \qquad =\;& \rangle\to\rangle\;\rangle\to\rangle \quad \cdots \quad \rangle\to\rangle\;,
\end{aligned}
$$

where the vertical alignment of channels of D_0 and $\sum\rangle p\rangle$ in this expression tells how the channels mate in contractions like tr $)\sum\rangle p\rangle|D_0)$. It is important that the $)E|$ factors $\rangle p\rangle$ fall between and connect different $|D_0)$ factors $\rangle\to\rangle$.

Suppose that space-time coordinates of a process π are defined by

$$x^\mu = \tau \textstyle\sum\rangle\gamma^\mu\rangle\;, \tag{4}$$

using the above $\sum$ operation on four Dirac γ matrices. Then an experimental stator describing a process of destruction with spinor stator $\rangle\alpha|$ and a process of creation with spinor stator $|\beta\rangle$ separated by a space-time interval y^μ is

$$)E^\alpha{}_\beta(y^\mu)| = \rangle\alpha|\delta_s(x^\mu - y^\mu)|\beta\rangle\;.$$

Since the four-coordinates x^μ do not commute, there is no well-defined δ function that can be used in this definition; the functional calculus fails. We define the smeared delta function δ_s arbitrarily by a Fourier integral

$$\delta_s(x^\mu - y^\mu) = \int_K \exp ik_\mu(x^\mu - y^\mu)\,d^4k/(2\pi)^2$$

over a suitable region K of k_μ space. In this, the exponentials are to be computed in an algebra using the outer product $\times$ of Eq. (2).

We do not extend the k integral over all of k space. The operators γ^μ have integer spectrum. In the c limit where the commutators of the γ^μ are neglected, the k translations $k\to k+2\pi/\tau$ leave the exponentials unchanged. To make an irredundant orthonormal set of exponentials a suitable subset K of k space, a kind of unit cell, is supposed to be sufficient.

There are many problems connected with this definition. They will be taken up in Sec. IV.

The coordinates x^μ were one dyad shorter than the dynamical stator $|D_0)$. Their exponentials have the same plexus, are also shorter than $|D_0)$. The spin stators $\rangle\alpha|$ and $|\beta\rangle$ bring $)E|$ up to the same length as $|D_0)$. The mating $|D_0)E|$ is then a well-defined complex number. It is useful to work with Fourier-transformed $)E^{\alpha}{}_{\beta}(k)|$ and $)E(k)|$ defined by

$$)E(k)| = \exp_s ik_\mu \chi^\mu ,$$

$$)E^{\alpha}{}_{\beta}(k)| = 1^{\alpha})E(k)|1_{\beta} ,$$

where 1^α and 1_β are unit spinors.[2]

Then the amplitude $a = |D_0)E(k)|$ may be computed as

$$a = |D_0)\exp_s ik_\mu y^\mu$$
$$= |D_0)\exp_c ik_\mu x^\mu .$$

Since x^μ is an outer sum (1) is applicable and yields

$$a = \exp ik_\mu \gamma^\mu n\tau ,$$

where n, the number of γ^μ's in the outer sum defining x^μ, is an operator in its own right. Here an outer sum has become an inner in accord with (3) and all n terms in the inner sum are the same.

This amplitude a obeys the Dirac equation in the proper-time form[3]

$$\left(k_\mu \gamma^\mu + i\frac{d}{ds}\right) a = 0 , \tag{5}$$

where $s = n\tau$. This propagator can be used to determine the space-time geometry, either as a causal space or a Minkowski pseudometric space. In this way we arrive at a statistical geometry from the q dynamics $|D_0)$.

This dynamics comes from the electron model of Ref. 1 as follows: The four-component Dirac spinor χ^α is the stator of an entity χ related to the processes given in Ref. 1 by

$$\chi = I \oplus H, \quad \chi^\alpha = \varphi^A \oplus \varphi_{\dot{B}} ,$$

and the unit operator $\rangle \rightarrow \rangle$ of the present work is the sum of unit operators of I and H:

$$\rangle \rightarrow \rangle_\chi = \begin{matrix} \rangle \rightarrow \rangle_I \\ \oplus \\ \langle \rightarrow \langle_H \end{matrix} ,$$

where vertically aligned angular brackets abutt the same point of the plexus. The terms in this sum agree in the direction of electric current, not the current defined by the brackets. Then the present

$$|D_0) = \begin{matrix} \rangle \rightarrow \rangle \; \rangle \rightarrow \rangle \cdots \rangle \rightarrow \rangle_I \\ \oplus \\ \langle \rightarrow \langle \; \langle \rightarrow \langle \cdots \langle \rightarrow \langle_H \end{matrix}$$

is the sum of the e dynamics of diagram e of Fig. 4 (Ref. 1) and its Hermitian conjugate, a superposition of a negative electron traveling one way and a positive electron traveling the other way.

IV. MAXWELL'S EQUATION

It is well known that Maxwell's equations for the electromagnetic field $F_{\mu\nu}$ can be written, with a tensor-class spinor $F = \gamma^{\mu\nu}F_{\mu\nu}$ and the usual differential operator $\partial = \gamma^\mu \partial_\mu$, as $\partial F = 0$.[4] The techniques of Sec. III, applied to the double-stranded plexor, γ, of Fig. 4 (Ref. 1), yield a proper-time propagator satisfying as $\tau \rightarrow 0$

$$\partial \cdot F + i\partial_s F = 0 ,$$

the dot product of two Dirac operators being half their anticommutator when one is of vector class and one tensor, as here. The linear space of solutions of this proper-time equation contains a subspace obeying the transformation law and differential equations of the free-space electromagnetic field, namely, the subspace of those F's of tensor class satisfying the missing Maxwell equation $\partial \wedge F = 0$ and $\partial_s F = 0$.

It would be good to extract the photon process from the full dynamics before taking the limit $\tau \rightarrow 0$. However, the problem of dissecting a family of processes into its stationary parts, crucial for the theory of elementary-particle processes, has not yet been well posed for $\tau > 0$, where there is no Hamiltonian to diagonalize.

V. THE PROBLEMS

This development of the Dirac equation raises the following questions, among others.

The new element added in this paper is the formula (4) for the space-time coordinates of a process. It is easy to motivate this: (4) is the obvious special relativistic modification of the formula

$$x^\mu = \sum \sigma^\mu \tau$$

of the SU(2, C) theory we began from. It would be easy to axiomatize (4) group theoretically. But these are arguments from ignorance. Coordinates have an operational meaning. Two objects have the same coordinates when they interact. An expression for coordinates is meaningful in q dynamics as a limitation upon the interactions of the theory, for both coordinates and interactions lead to concepts of locality and these two concepts of locality must be consistent with each other. (4) must emerge only as an approximation from a more exact theory with interactions.

The expression (4) for the coordinates is a sum (over dyads) of stators. This sum is not a concept of projective geometry. Only the ray of a stator is supposed to have meaning. One ray may be represented by any of its vectors. The choice of a vector to represent a ray we will call the projective gauge. Our starting point is a theory with projective gauge invariance. What has destroyed this invariance? Some physical entity has been left out of our picture.

The operator γ^μ is a geometric object under SL(2, C) but not under GL(2, C), which splits γ^μ into a sum of two geometric objects of different weight:

$$\gamma^\mu = \tfrac{1}{2}(1 + i\gamma_5)\gamma^\mu + \tfrac{1}{2}(1 - i\gamma_5)\gamma^\mu .$$

This means another symmetry of the theory is broken by the environment $|E($, another physical entity has been left out.

Part of what we left out is gravity, space-time curvature. In adding vectors γ^μ from each monad pair we establish a connection between directions at two nearby space-time events, the job ordinarily filled by the affine connection of the gravitational field.

The operational bases of these concepts of quantum theory and general relativity are not present in the dynamics $|D_0)$ we have employed. Both space-time curvature and transition probability, both (Riemannian) pseudometric and (Hilbert) metric are inaccessible in a world without interaction. Curvature can be detected only in loops, and there is no loop in our $|D)$ plexus. Transition probability cannot be measured without channels and counters, and these require interactions for their construction. Thus the present theory fails the test of semantic consistency put forward by Weizsäcker: It fails to describe the means by which we can know the entities of the theory. The true $|D)$ must have more complicated topology than the linear $|D_0)$ we have used here. Loops and interactions are both forms of nonlinear topology in plexus dynamics. The missing concepts must be given operational meaning in terms of such topological structure.

In such works as Snyder,[5] Yang,[6] and Weizsäcker,[7] the c space-time continuum is the c limit of a q system, at least insofar as abstract geometry is concerned. One problem has been how to go from geometry to dynamics when the local action methods of cq physics are gone. It has not been clear what should be meant by the passing of time when time is a q coordinate.

The present work inverts the problem. We give up the practice, common to c and cq physics, of building a dynamics over a given space-time, and build space-time over a given dynamics instead. This tests a different philosophy of time and matter, taking as primary not object systems and space-time points, but dynamical processes. The relation

$$\Delta x = \tau \sum \gamma$$

gives the increment in space-time coordinates for a general process in terms of q coordinates of elementary processes. We find that for dynamical calculations the differential and integral calculus may be replaced by the plexor calculus. The c space-time arises as the c limit of a discrete q system for dynamics as well as for geometry. The Feynman amplitude of cq physics and the action of c physics are successive limiting cases of our dynamical stator. We show how to go from such a discrete mechanics to the continuum laws of Dirac and Maxwell as $\tau \to 0$.

The actual computation of cross sections raises problems we have solved here. It requires expressions not so much for geometrical quantities like Δx but for the conjugate dynamical ones like energy-momentum. We have given suitable expressions. They are not unique or exact. According to q dynamics the quantities themselves are ambiguous, approximate until the limit of c time is taken.

Thus we are now ready for trial calculations of cross sections connected with either phenomenological or fundamental q dynamical theories.

ACKNOWLEDGMENTS

Attempts similar to $\Delta x = \tau\sum\gamma$ were made by R. P. Feynman and C. F. v. Weizsäcker long ago. A skepticism about the evidence for c space-time at the nuclear level has been strongly expressed by T. D. Lee. One of us is indebted to these gentlemen for stimulating discussions of their unpublished work and to the Max Planck Institute (Erlebensbedingungen), Starnberg, for hospitality and encouragement.

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[1]Here we summarize the main conclusions of the preceding four papers of this sequence. D. Finkelstein [preceding paper, Phys. Rev. D 9, 2219 (1974)] gives some references to the literature.

[2]Such quantum space-time plane waves have been more thoroughly studied by M. Hamilton and G. Sandri (private communication).

[3]This proper-time Dirac equation is that of V. Fock [Phys. Z. Sowjetunion 12, 404 (1937)] and Y. Nambu [Prog. Theor. Phys. 5, 82 (1950)], also employed by J. Schwinger [Phys. Rev. 82, 664 (1951)]. Some most suggestive studies of proper time in quantum mechan-

ics are M. Bunge and A. J. Kâlnay, Prog. Theor. Phys. 42, 1445 (1969) and C. Piron (unpublished).
[4]This algebra is that of M. Riesz [*Clifford Numbers and Spinors,* Lecture series No. 38 (Univ. of Maryland Press, College Park, Maryland, 1958)] and D. Hestenes [*Space-Time Algebra* (Gordon and Breach, New York, 1966)] (except for his i).
[5]H. Snyder, Phys. Rev. 71, 38 (1947).
[6]C. N. Yang, Phys. Rev. 72, 874 (1948).
[7]C. F. v. Weizsäcker, Naturwissenschaften 20, 545 (1955); Z. Naturforsch. 13A, 245 (1958); etc.