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Index of refraction for scalar, electromagnetic, and gravitational waves in weak gravitational fields

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We consider the solutions of the scattering of scalar, electromagnetic, and gravitational waves by the gravitational field of a single particle, for the case of small wave amplitudes and weak gravitational fields. Scatterings are considered for both incident plane waves and incident spherical waves. For plane waves incident on a thin sheet of matter composed of free particles, the superimposed wave solutions give rise to a phase change arising from the coordinate dependence of the speed of light on the gravitational potential, focusing of the incident wave by the sheet, and, in some cases, a phase change due to dispersion of the wave by the matter. For gravitational waves, the index of refraction n is given by $n-1=2\pi G\rho/\omega^2$, assuming $n-1$ is small, and for electromagnetic waves $n=1$ to the same order. The index of refraction for scalar waves depends on the form of the scalar-wave equation used. The generation of back-scattered waves is also treated. Calculations are repeated for spherical waves incident on a thin spherical shell of matter. The propagation of δ-function wave packets is then treated in order to show that the solutions are consistent with causality, even though, in some cases, the group velocity exceeds the velocity of light.

I. INTRODUCTION

In flat space-time, solutions of wave equations describing fields mediated by massless particles are propagated sharply along null rays. The most familiar example of this is the case of electromagnetic waves.[1] When electromagnetic waves interact with charged particles, scattered waves are produced which interfere with the incident waves. If the density of scatterers is sufficiently uniform, the resultant wave propagation through the matter is described by an index of refraction, which in general is frequency-dependent. One effect of such dispersion is that electromagnetic signals are, in effect, no longer propagated sharply on the light cone, but rather are smeared out inside the light cone.

In curved space-time even the source-free solutions of massless wave equations are, in general, no longer sharply propagated along the light cone.[2] In addition to the direct, sharply propagated signal, one finds a tail,[3] representing the smearing out of the signal inside the light cone due to the nonvanishing curvature tensor. This is the case even if the matter, which generates the curvature through Einstein's field equations, does not interact directly with the wave. We will assume, where possible, that the matter which generates the curvature is inert so far as the waves are concerned.

For regions of strong gravitational fields, there is no superposition principle for the gravitational field, and wave propagation in each such geometry must be considered separately. An example of such a treatment is found in the solution of wave equations in the Schwarzschild geometry,[4,5] which exhibit such effects as smearing out of the signal inside the light cone and back scattering of the waves off the background curvature. In regions of weak gravitational fields the Riemann tensor due to a mass distribution can be approximated by the sum of Riemann tensors due to each particle in the mass distribution. Although not valid for strong-field regions, such an approximation should be useful in determining what are the dominant effects of gravitation on signal propagation in most regions of space-time.

In this paper we will consider the solution for scalar, electromagnetic, and gravitational waves in the presence of a single particle. We will then consider the wave solutions in suitable configurations of matter by superposition in order to determine the gravitational field contribution to such bulk matter properties as the index of refraction for each kind of wave. For scalar waves such solutions are of only academic interest. For electromagnetic waves one may find what corrections, if any, are needed for the standard electromagnetic dispersion formulas. For gravitational waves, such calculations may have relevance for the propagation of pulses of gravitational waves possibly emitted near the center of our galaxy,[6] which may have passed through dense regions of matter, or for gravitational waves which travel to us from cosmological distances.

In the recent literature there have appeared a number of discussions which relate to the interaction of gravitational waves with bulk matter, e.g., gases or fluids. Szekeres[7] has found the index of refraction of gravitational waves propagating through matter which is composed of "atoms," in which the incident wave induces quadrupole moments in each atom. Polnarev[8] and Chesters[9] have discussed the interaction and dispersion of gravitational waves in a hot gas, with collisions and without collisions. Madore[10] has considered gravitational wave propagation through a region devoid of matter ($R_{\mu\nu}=0$), influenced only by the curvature tensor; similar calculations are made also for electromagnetic wave propagation. In this paper we consider waves interacting with a cold gas of free particles, initially all at rest. The expression we derive for $n-1$, where n is the index of refraction of gravitational waves, is much larger than that generated by induced quadrupole moments (by the square of the ratio of the wave length to the size of the atom) and larger than that generated by a hot gas (by the ratio of mass-energy density to pressure).

In Sec. II we define the metric and derive the wave equations for scalar, electromagnetic, and gravitational waves to first order in the gravitational potential ϕ. In Sec. III we solve the problem of the scattering of an incident plane wave by a single mass for each of the three kinds of waves. We then consider an infinite thin sheet of such scatters and derive the index of refraction for scalar, electromagnetic, and gravitational waves, including the back-scattered waves for the three cases. In Sec. IV we reexamine the same calculations from the point of view of spherical-wave packets, first solving the single-particle scattering problem as well as the effect of a thin shell of matter and a spherical volume of matter. In Sec. V we describe the evolution of δ-function plane-wave packets and δ-function spherical-wave packets. Section VI gives a discussion of the results and conclusions.

II. WAVE EQUATIONS

We assume a metric $g_{\mu\nu}$ given by[11]

$$g_{00}=(1+2\phi),\quad g_{0i}=0,\quad g_{ij}=-\delta_{ij}(1-2\phi),\quad (2.1)$$

where the potential $\phi=-GM/r$ for a single point mass, and where contributions of order ϕ^2 are consistently ignored. The potential for a mass distribution is then obtained by adding the potentials due to each particle in the mass distribution. The energy densities in the scalar, electromagnetic, and gravitational waves that will be considered are assumed to generate, through the Einstein field equations, a negligible contribution to the metric. Thus the wave equations will also be linear.

A. Scalar waves

We consider scalar waves emitted by a distant source propagating through a region in which particles interact with the wave only through gravitational fields, i.e., through the geometry of space-time. Thus the scalar waves satisfy a source-free scalar wave equation throughout the region of interest. The generalization of the source-free, flat-space-time, scalar wave equation to a curved space-time is not unique. One prescription[12] is to replace ordinary derivatives by covariant derivatives. However, one could also add terms proportional to the Riemann tensor or its products and derivatives, which vanish in flat space-time. If one chooses not to introduce any dimensional constants, then the form of the curved-space-time scalar wave equation is

$$\psi_{;\lambda}{}^{;\lambda}+aR\psi=0\,, \qquad (2.2)$$

where a is a dimensionless constant and R is the curvature scalar. The choice $a=0$ corresponds to the standard prescription and the choice $a=\frac{1}{6}$ yields a conformally invariant wave equation.[13]

Using the metric (2.1) in (2.2) and keeping terms up to first order in ϕ yields an approximate scalar wave equation in which, since ϕ is static, we can assume a time dependence of ψ of the form

$$\psi(\vec{r},t)=\psi(\vec{r})e^{-i\omega t}\,. \qquad (2.3)$$

Then, using the fact that in terms of order ϕ, ψ satisfies the flat-space-time equation, the scalar wave equation can be cast in the form

$$(\nabla^2+\omega^2)\psi=(4\omega^2\phi+2a\phi_{,kk})\psi\,. \qquad (2.4)$$

B. Electromagnetic waves

We consider now the case of electromagnetic waves emitted by a distant source passing through a region of space which, for simplicity, contains only electromagnetically inert matter. The source-free wave equation for the electromagnetic potentials is generalized to curved space-time in a fairly unique way, if one keeps gauge invariance and if one does not add dimensional constants. Thus we take the covariant wave equation

$$A_{\mu;\lambda}{}^{;\lambda}-A_{\lambda;\mu}{}^{;\lambda}=0 \qquad (2.5)$$

as our curved-space-time wave equation. Substitution of the metric (2.1) into (2.5) yields electromagnetic wave equations for A_0 and A_i, to first order in ϕ, which have been discussed previously.[14] We choose the gauge condition

$$A_{0,0}-A_{k,k}=4\phi A_{k,k}+2\phi_{,k}A_k \qquad (2.6)$$

to simplify our expressions. We then assume a time dependence of A_μ as given in (2.3) and make use of the fact that, in terms already of order ϕ, we can assume that the unperturbed wave is spacelike and transverse, consistent with the gauge condition (2.6). This results in the source-free wave equations

$$(\nabla^2+\omega^2)\tilde{A}_0=4i\omega\phi_{,k}\tilde{A}_k\,, \qquad (2.7)$$

$$(\nabla^2+\omega^2)\tilde{A}_i=4\omega^2\phi\tilde{A}_i-2\phi_{,ki}\tilde{A}_k+\phi_{,kk}\tilde{A}_i\,, \qquad (2.8)$$

where $\tilde{A}_0=(1-\phi)A_0$ and $\tilde{A}_i=(1+\phi)A_i$. The perturbed wave will, in general, no longer be spacelike and transverse, although one is still free to make a new gauge choice. We will not consider the explicit solution of (2.7) since the potential A_0 can be determined from the gauge condition (2.6), which, for an incident spacelike transverse wave, becomes

$$-i\omega A_0=A_{k,k}+2\phi_{,k}A_k\,. \qquad (2.9)$$

C. Gravitational waves

We consider gravitational waves emitted by a distant source propagating through a region of space-time described by the metric (2.1). In this case we cannot, *a priori,* assume a source-free equation, as the particles generating the potential ϕ necessarily interact with the incident wave. The wave equation is, however, unique if one assumes that the gravitational interaction is described fully by Einstein's field equations. We consider the wave to be described by a perturbation $h_{\mu\nu}$ about some background metric $g^{(0)}_{\mu\nu}$, i.e.,

$$g_{\mu\nu}=g^{(0)}_{\mu\nu}+h_{\mu\nu} \qquad (2.10)$$

and we expand the field equations to first order in $h_{\mu\nu}$, introducing the perturbation in the stress-energy tensor $\delta T_{\mu\nu}$. This expansion has been discussed previously.[15] Letting

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}g^{(0)}_{\mu\nu}g^{(0)\alpha\beta}h_{\alpha\beta} \qquad (2.11)$$

and defining f_μ by

$$f_\mu=\bar{h}_{\mu\nu}{}^{;\nu}\,, \qquad (2.12)$$

we arrive at the wave equation for $\bar{h}_{\mu\nu}$ [15]:

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha}-f_{\mu;\nu}-f_{\nu;\mu}+g_{\mu\nu}f_\alpha{}^{;\alpha}-2\bar{h}_{\alpha\beta}R^\alpha{}_{\mu\nu}{}^\beta+\bar{h}_{\mu\alpha}R^\alpha{}_\nu +\bar{h}_{\nu\alpha}R^\alpha{}_\mu-h_{\mu\nu}R+g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta}=-16\pi G\delta T_{\mu\nu}\,. \qquad (2.13)$$

We next consider the substitution of the metric (2.1) into (2.13), keeping terms up to first order in ϕ. To simplify the resulting equations we make the choice of gauge

$$f_\mu=2\phi_{,k}\bar{h}_{k\mu} \qquad (2.14)$$

and make use of the fact that in terms of order ϕ, the incident gravitational wave, in accord with (2.14), can be chosen to be spacelike, traceless, and transverse. Further, we assume a time dependence as in (2.3). This results in the wave equation for the spatial components of $\bar{h}_{\mu\nu}$,

$$(\nabla^2+\omega^2)\tilde{h}_{ij}=4\omega^2\phi\tilde{h}_{ij}+16\pi G\delta T_{ij}\,, \qquad (2.15)$$

where $\tilde{h}_{ij}=\bar{h}_{ij}(1+2\phi)$. One can similarly write wave equations for $\bar{h}_{0i}$ and $\bar{h}_{00}$. However, as in the electromagnetic case, it is simpler to derive these components from the spatial components using the gauge condition (2.14). Specifically, for the time dependence chosen and for an incident wave which is spacelike, traceless, and transverse, this yields, to order ϕ,

$$-i\omega\bar{h}_{00}=\bar{h}_{0k,k}\,, \qquad (2.16)$$

$$-i\omega\bar{h}_{0k}=\bar{h}_{kl,l}+2\phi_{,l}\bar{h}_{kl}\,. \qquad (2.17)$$

In order to solve (2.15) it is necessary to specify the perturbation in the stress-energy tensor as a result of the incident gravitational wave. We assume that the stress-energy tensor $T_{\mu\nu}$ is that of a gas of free particles, the particles being point particles initially at rest. The equation of motion for each particle is the geodesic equation, which gives rise to a coordinate acceleration which vanishes in the gauge, or coordinate system, in which the incident wave is spacelike and traceless. Thus the coordinate velocity remains zero. Of course, there will be contributions to the acceleration of any one particles due to the Newtonian potential of all of the other particles. For a fixed time t this will give rise to velocity components of order ϕ and thus contributions to δT_{ij} of order $\phi^2 T_{00}$, which we are neglecting.

We see that since particles at rest remain at rest[16] in response to the incident wave, there are no contributions to δT_{ij} from particle velocities. Likewise, there are no contributions to δT_{ij} from perturbations in the metric itself since the original $T^{ij}=0$ and the wave has only spacelike components. Thus the spatial components $\tilde{h}_{ij}$ of the wave, analogous to the scalar wave ϕ and electromagnetic wave $\tilde{A}_i$, obey a source-free wave equation:

$$(\nabla^2+\omega^2)\tilde{h}_{ij}=4\omega^2\phi\tilde{h}_{ij}\,. \qquad (2.18)$$

If we let

$$\tilde{h}_{ij}=\epsilon_{ij}\tilde{\psi}\,, \qquad (2.19)$$

we see that the wave equation (2.18) is solved by a constant polarization tensor ϵ_{ij} and an amplitude $\tilde{\psi}$ which obeys the scalar wave equation

$$(\nabla^2+\omega^2)\tilde{\psi}=4\omega^2\phi\tilde{\psi}\,. \qquad (2.20)$$

This is the same as the wave equation for the scalar potential (2.4) with the particular value of $a=0$.

III. INDEX OF REFRACTION

A. Scattering by a single particle

We wish to solve (2.4), (2.8), and (2.20) for a gas of particles in order to derive the index of refraction. The right-hand sides of these equations are linearly dependent on the Newtonian potential ϕ, which can be decomposed into the sum of potentials due to each point particle. Moreover, on the right-hand sides of these equations, the wave amplitudes can be taken to be those of the incident waves. Thus if one can determine the scattered wave due to each particle in the gas, one can then superimpose the scattered waves from all of the particles to obtain the scattered wave from the gas. By comparing the phase of the scattered plus incident wave with that of the incident wave, we can then deduce the index of refraction of the gas.

We first focus on the simplest of the three potential equations, (2.20), with $\phi=-Gm/r$, and assume that our incident wave is a plane wave of wave number k in the z direction:

$$\tilde{\psi}_I \sim e^{ikz} . \tag{3.1}$$

Although (2.20) is an approximate equation in the sense that ϕ^2 terms are ignored, we can in fact solve (2.20) exactly. We first notice that if we identify

$$\omega^2 \rightarrow k^2 ; \quad 4Gm\omega^2 \rightarrow \frac{-2\mu Z_1 Z_2 e^2}{\hbar^2} ,$$

then Eq. (2.20) is just Schrödinger's equation for scattering of two charged particles (charge $Z_1 e$ and $Z_2 e$) with center-of-mass energy $E=\hbar^2 k^2/2\mu$, where μ is the reduced mass. For asymptotic plane waves (ignoring logarithmic phase factors) of the form (3.1), the solution for $\tilde{\psi}$ is best derived in parabolic coordinates. In terms of parameters of Eq. (2.20), this solution is[17]

$$\tilde{\psi} = Ce^{ikz} F(2iGm\omega, 1, ik(r-z)) , \tag{3.2}$$

where $F(a, b, \zeta)$ is the confluent hypergeometric function and C is an arbitrary constant.

If we keep terms in the power-series expansion of F up to those linear in Gm, we can derive a useful approximate expression for $\tilde{\psi}$,

$$\tilde{\psi} = Ce^{ikz} \left\{ 1 - 2Gmi\omega \left[\gamma - i\frac{\pi}{2} + \ln k(r-z) + \int_{k(r-z)}^{\infty} \frac{du}{u} e^{iu} \right] \right\}, \tag{3.3}$$

where γ is the Euler-Mascheroni constant. The first two terms in the square brackets of (3.3) can be eliminated by a renormalization of the incident wave to unity. In the exact solution (3.2) the logarithmic term in $k(r-z)$, for large $k(r-z)$, arises from a phase factor[17] which, in (3.3) appears expanded to first order in Gm. The last term in (3.3) represents the scattered wave.

The solution to (2.20) can be used to generate, to first order in Gm, solutions to (2.4) and (2.8). First consider the solution to the wave equation (2.20) with incident wave

$$\psi_I \sim e^{ik(z-z')} = e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} .$$

The solution of (2.20) appropriate to this wave is just

$$\tilde{\psi}' = \tilde{\psi} e^{-ikz'} . \tag{3.4}$$

Define the symmetrized gradient $\tilde{\nabla}_k$ to be $\tilde{\nabla}_k = (\partial/\partial x^k) + (\partial/\partial x^{k'})$ and operate on (2.20) with $\tilde{\nabla}_k$. Then $\tilde{\nabla}_k \tilde{\psi}'$ satisfies

$$(\nabla^2 + \omega^2)\tilde{\nabla}_k \tilde{\psi}' = 4\omega^2 \phi_{,k} \tilde{\psi}' \tag{3.5}$$

since, to first order in ϕ, $\tilde{\nabla}_k \tilde{\psi}'=0$ for the incident wave. Therefore, the solution to (2.4) can be given in terms of $\tilde{\psi}'$ defined in (3.4) as

$$\psi = \tilde{\psi}' + \frac{a}{2\omega^2} \tilde{\nabla}_k \tilde{\nabla}_k \tilde{\psi}' . \tag{3.6}$$

The electromagnetic equation (2.8) can be similarly solved. Letting ϵ_i be the polarization vector of the incident wave (which is constant), the solution of (2.8), to first order in ϕ, is given by

$$\tilde{A}_i = \epsilon_i \tilde{\psi}' - \frac{1}{2\omega^2} \epsilon_k \tilde{\nabla}_k \tilde{\nabla}_i \tilde{\psi}' + \frac{1}{4\omega^2} \epsilon_i \tilde{\nabla}_k \tilde{\nabla}_k \tilde{\psi}' . \tag{3.7}$$

The solutions for A_i and $\bar{h}_{ij}$ are incomplete as they stand. One must also compute A_0 from (2.9) and the components $\bar{h}_{00}$ and $\bar{h}_{0k}$ from (2.16) and (2.17). However, in our application of these solutions, due to the symmetry of the physical system, there are no net contributions to A_0 or to $\bar{h}_{00}$ and $\bar{h}_{0k}$.

B. Phase shift and index of refraction

Our derivation of the index of refraction will parallel well-known derivations[18] for the case of electromagnetic waves. We first assume that there is an infinite sheet of scatterers at $z=0$ within a thickness Δz. Let N be the number density of scatterers, which for simplicity, we assume are all of the same mass m. Then the mass density $\rho=Nm$. We initially assume that the density ρ falls off exponentially with the distance x from an axis of symmetry, i.e.,

$$\rho(x) = \rho_0 e^{-x/x_0} . \tag{3.8}$$

We will let $x_0 \to \infty$ whenever possible. The number density can always be chosen so large that the granularity in the matter distribution can be ignored.

Consider a wave e^{ikz} incident on the plane $(z<0)$ and the total wave (incident + scattered) after leaving the plane $(z>0)$ in the form $e^{i(kz+\delta\phi)}$, where $\delta\phi$ is the phase change as a result of the presence of the plane of scatterers. The phase velocity in the plane is the speed of light divided by the index of refraction n, which gives the phase change $\delta\phi=(n-1)\omega\Delta z/c$. In the limit that Δz is small, $\delta\phi$ is also small, so that the relation between the final wave form and the index of refraction is

$$\psi_{\text{total}} \cong e^{ikz}[1+i\omega(n-1)\Delta z/c] . \tag{3.9}$$

C. Logarithmic phase factors and focusing

We can anticipate two effects that should be found in any wave solution. One concerns the fact that the coordinate velocity of light deviates from c by terms of order ϕ. This should give rise to terms in the phase of the original wave which depend on ϕ. The second effect results from the fact that null geodesics are deflected in the presence of matter, and this should give rise to amplitude changes in the wave.

The coordinate speed of null rays is found from the metric (2.1). For a null ray between two points $\vec{r}\,'$ and $\vec{r}$, which differ only in their z components, the elapsed time for propagation between $\vec{r}\,'$ and $\vec{r}$ differs from the coordinate separation by a line integral involving the potential. For $\phi=-Gm/r$, this integral is proportional to

$$\int_{z'}^{z} \phi\, dz = Gm \ln\left(\frac{r-z}{r'-z'}\right) = Gm[\ln k(r-z) - \ln k(r'-z')] , \tag{3.10}$$

where the factors of k have been added to make the arguments of the logarithms dimensionless. The incident wave is therefore expected to have the form

$$\psi_I = e^{ik(z-2Gm\ln k(r-z))} e^{-i\omega t} \psi_I' , \tag{3.11}$$

where ψ_I' is a phase factor that depends only on $\vec{r}\,'$ and t'. If (3.11) is expanded to first order in Gm, we reproduce the logarithmic term in (3.3), giving the physical significance of that term.

We are now in a position to see how the logarithmic term in (3.3) should be handled in the sum over all particles in the plane. Since this term arises from the fact that the coordinate velocity of light is not c, we should first compute the phase expected by evaluating the integral $\int \phi\, dz$ for the total potential due to all the masses. For an infinite plane of such masses, this gives an infinite result for a constant finite mass density ρ.

There are two alternate ways of treating this difficulty. First, we could cut off the mass density for large distances using (3.8). This would give us an expression for ω/k as the coordinate phase velocity, which would not be c, even in the high-frequency limit. If we then wanted to find the physical phase velocity, defined to be c in the high-frequency limit, we would, to order ϕ, subtract from the coordinate phase velocity the difference between the high-frequency limit of the coordinate phase velocity and c. After this subtraction there is no divergence in the potential contribution as the mass density in (3.8) becomes uniform, i.e., as $x_0 \to \infty$. The net effect of this procedure is that logarithmic phase factors may be ignored if one is computing the physical phase velocity and the physically significant index of refraction.

The second approach, which leads to the same result, is to make sure that when the wave with no matter present is compared with the wave with the sheet of matter present, the two points z' and z are the same optical distance apart in the two cases, rather than at the same coordinate separation. By the same optical distance we mean the same number of wave lengths between z and z' generated by a high-frequency source with the time dependence $e^{-i\omega t}$.

The second geometrical effect relates to the fact that null geodesics are deflected by a mass m. This deflection angle is $\approx 4Gm/bc^2$, where b is the minimum distance between the mass and geodesic. To calculate the deflection expected in the case of a plane of masses, let $x=0$ be the center of the axis of symmetry of the density $\rho(x)$, and let x be the distance from the center to the point in the plane at which we wish to compute the net deflection. Only mass within the circular disk of radius x about the center contributes to the deflection. Thus we may let x_0 in (3.8) be as large as we wish, and we can then consider the density within the disk to be uniform. The deflection of the null ray at x by an angle $\theta(x)$ means that at a height z above the plane there is a displacement $\Delta x = -\theta(x)z$. Evaluating $\Delta x/x$ by integrating the x component of the deflection angle over the disk gives the expected amount of focusing. This implies an increase in the wave amplitude of an amount

$$\psi_f = 4\pi G\rho z \Delta z e^{ikz} . \tag{3.12}$$

D. Indices of refraction

We consider first the superposition of scattered solutions of (2.20), which relates to the index of refraction of gravitational waves. Using the wave solution (3.3) with the incident amplitude set equal to 1 and the logarithmic term discarded as discussed in Sec. III C, we find the total wave, incident plus scattered, for a uniform density thin plane, to be

$$\bar{\psi}=e^{ikz}\left(1+4\pi G\rho z\Delta z+\frac{2\pi G\rho i\Delta z}{\omega}\right), \tag{3.13}$$

where we have set $\omega=k$ in terms of order ϕ. Comparison of the first scattered contribution with the focusing calculation (3.12) shows that we have reproduced via the wave solution what was expected from geometrical optics. The last term in (3.13) is what we wish to identify with the phase change resulting from an index of refraction n. Comparing (3.13) with (3.9) gives the expression for n,

$$n=1+\frac{2\pi G\rho}{\omega^2}\ \text{(gravitational waves)}, \tag{3.14}$$

for gravitational waves passing through matter of density ρ. Note that this is the same expression as is obtained for electromagnetic waves[1] passing through a gas of free-charged particles, with the replacement of the charge density by the mass density, the charge per unit mass by 1, and the mks constant $(1/4\pi\epsilon_0)$ by $-G$, i.e., this is the same substitution that takes the Coulomb force law into the Newtonian one.

We next consider the integration over the plane of the scattered scalar waves given by (3.6). The logarithmic phase shift is contained in the first term on the right-hand side of (3.6), so the full solution (3.3) times $e^{-ikz'}$ must be used in the second term. We find, using the definition (3.4) and letting $z'=0$, $k=\omega$, that

$$\bar{\nabla}_k\bar{\nabla}_k\bar{\psi}'=-\frac{4Gm\omega^2}{r}e^{ikr}. \tag{3.15}$$

The total wave ψ is then found to be

$$\psi=e^{ikz}\left[1+4\pi G\rho z\Delta z+\frac{2\pi G\rho i\Delta z}{\omega}(1-2a)\right]. \tag{3.16}$$

As before we identify the focusing term in (3.16), and comparing (3.16) with (3.9), we find the expression for n:

$$n=1+\frac{2\pi G\rho}{\omega^2}(1-2a)\ \text{(scalar waves)}. \tag{3.17}$$

For electromagnetic waves we first compute the middle term on the right-hand side of (3.7). We use the fact that ϵ_k is transverse to z, and note that in the integration in the plane over all angles (at fixed r) the average value of $\vec{\epsilon}\cdot\vec{r}$ vanishes and the average value of $\vec{\epsilon}\cdot\vec{r}x_l$ is $\frac{1}{2}(r^2-z^2)\epsilon_l$. Therefore we only obtain a scattered polarization in the direction of the incident polarization. This feature allows a factorization of the wave amplitude $\tilde{A}_i$ into a constant polarization ϵ_i and a scalar function $\bar{\psi}$. The middle term on the right-hand side of (3.7) contributes to $\bar{\psi}$, on the average over angles, an amount for each mass m of

$$\Delta\bar{\psi}=\frac{Gmi}{2\omega}\left[(e^{ikr}-e^{ikz})\frac{z}{r^3}-ike^{ikr}\left(\frac{1}{r}+\frac{z}{r^2}\right)\right]. \tag{3.18}$$

Integrating the e^{ikz} term in (3.18) over the plane gives the contribution

$$\bar{\psi}_{\text{direct}}=\frac{-\pi G\rho i\Delta z}{\omega}\frac{z}{|z|}e^{ikz}, \tag{3.19}$$

which is important for $z<0$. Renormalization of the incident wave to unity doubles the final contribution in (3.19) for $z>0$. The integration of the e^{ikr} terms over the plane yields a contribution to $\bar{\psi}$, which is exactly the negative of twice (3.19) for $z>0$. Thus the middle term in (3.7) contributes, on the average over the plane, nothing to the scattered wave.

This leaves us with the first and third terms on the right-hand side of (3.7). However, comparing with the scalar solution (3.6), we see that $\bar{\psi}$ should be the same as the scalar solution ψ, given by (3.16), with the particular value $a=\frac{1}{2}$. But for $a=\frac{1}{2}$ the last term in (3.16) vanishes, and we are left only with the focusing term. Thus we conclude that, for electromagnetic waves,

$$n=1\ \text{(electromagnetic waves)} \tag{3.20}$$

up to order ϕ.

It should be noted that a computation of A_0, either from (2.7) or (2.9) gives zero for the plane of masses, indicating that the transmitted wave remains spacelike. In a similar manner, it can also be shown from (2.16) and (2.17) that for the plane of masses $\bar{h}_{00}$ and $\bar{h}_{0k}$ are also zero for the case of gravitational waves. Thus in either case, there is no need to perform a further gauge transformation on the final wave before computing the phase shift and index of refraction.

In addition to shifting the phase of the forward propagated incident wave, the scattering from the plane of masses produces a back-scattered wave with z dependence e^{-ikz}. For gravitational waves we find a back-scattered wave analogous to (3.13):

$$\bar{\psi}_B=\frac{2\pi G\rho i\Delta z}{\omega}e^{-ikz}. \tag{3.21}$$

For scalar waves ψ_B, analogous to (3.16), is found to be $(1-2a)$ times (3.21), and for electromagnetic waves $\bar{\psi}_B$, as in the calculation of the index of refraction (3.20), is zero.

IV. SPHERICAL WAVES

A. Green's function

One can avoid the use of an infinite plane of masses by assuming that the incident wave has spherical wave fronts and is interacting with a spherical shell of masses of thickness $\Delta r'$ at a distance r' from the source of the waves. As in Sec. III D, the total wave (scattered + incident) is compared with the incident wave to find the index of refraction.

The solution, to first order in Gm, of the differential equation

$$(\nabla^2+\omega^2)G(\vec{r},\vec{r}',\omega)=-\frac{Gm}{r}\frac{e^{i\omega R}}{R}, \tag{4.1}$$

with $\vec{R}=\vec{r}-\vec{r}'$, has been given previously.[15] Explicitly,

$$G(\vec{r},\vec{r}',\omega)=\frac{Gmi}{2\omega}\left[\frac{e^{i\omega R}}{R}\ln\left(\frac{rR+\vec{r}\cdot\vec{R}}{r'R+\vec{r}'\cdot\vec{R}}\right)-\int_0^\infty\frac{du\,e^{i\omega(u+r'+\rho)}}{(u+r')\rho}\right], \tag{4.2}$$

where

$$\rho(\vec{r},\vec{r}',u)=\left[r^2-\left(\frac{\vec{r}\cdot\vec{r}'}{r'}\right)^2+\left(\frac{\vec{r}\cdot\vec{r}'}{r'}+u\right)^2\right]^{1/2}. \tag{4.3}$$

It can be shown that in the limit $r'\gg r$, $4\omega^2G$ reproduces the solution we have found for incident plane waves (3.3). In addition, one finds the z' dependence of the wave that was expected from our discussion of logarithmic phase factors, as in Eq. (3.10), verifying that the z' coordinate is in fact associated with the source of the wave.

B. Spherical shells

We restrict ourselves now to a spherically symmetric source of scalar waves at the origin and consider the solution of (2.4) with $a=0$ for a spherical shell of mass scatterers of number density N at a distance r' from the origin with thickness $\Delta r'$. The wave is observed at a point R from the origin, with $R>r'$. The incident wave is then $e^{i\omega R}/R$. The contribution of the scattering off the masses in the shell is then seen, comparing (2.4) with (4.1), to be $4\omega^2GN$ times the volume integral of the Green's function (4.2) over the shell, where $\vec{r}$ designates the radius vector of the observer's point from each scatterer and $\vec{r}'$ designates the radius vector of the origin from the scatterer. For simplicity $\vec{R}=\vec{r}-\vec{r}'$, which is fixed for each element of the shell, is chosen to be the z direction. The scattered contribution can be written as $\psi_s^{(1)}+\psi_s^{(2)}$. The first scattered contribution, $\psi_s^{(1)}$, for $R>r'$, is

$$\psi_s^{(1)}=8\pi G\rho_0 i\omega r'^2\Delta r'\left[1+\ln(R/r')\right](e^{i\omega R}/R), \tag{4.4}$$

where ρ_0 is the constant mass density. It has been noted in Sec. III C that one expects corrections due to the fact that the coordinate speed of light is not 1. If one defines a distance measure l such that the speed of light $dl/dt=1$, then the phase of the wave would be expected to be $\omega(t-l)$. But

$$l=\int_0^R(1-2\phi)dr\,.$$

Using the potential ϕ for the shell considered gives

$$l=R+8\pi G\rho_0 r'^2\Delta r'\left[1+\ln(R/r')\right].$$

Thus we see that (4.4) is an expansion of a term in the phase proportional to ϕ to first order in ϕ. If $r'\to\infty$, multiplying (4.4) by R to keep the incident wave amplitude finite, we find that $\psi_s^{(1)}$ diverges as was found in the corresponding case of the sum of the logarithmic phase terms over the plane of scatterers.

The second scattered contribution, $\psi_s^{(2)}$, for $R>r'$, is

$$\psi_s^{(2)}=4\pi G\rho_0 r'^2\Delta r'\frac{e^{i\omega R}}{R}\times\left[\frac{1}{r'}-\frac{1}{R}+\int_R^\infty\frac{dv}{v^2}e^{2i\omega(v-R)}-\int_{r'}^\infty\frac{dv}{v^2}e^{2i\omega v}\right]. \tag{4.5}$$

Note that if $R=r'+z$, so that z is then the distance from the observer to the nearest point on the shell, and if $z\ll r'$, then $1/r'-1/R\approx z/r'^2$. This gives a correction to the wave amplitude which is just that expected from the arguments based on focusing (3.12) and also verified in the plane-wave solution (3.13). Clearly, by symmetry, focusing in the normal sense cannot be the cause of enhancement in the spherical case. However, we can understand the origin of that term for the spherical wave in the modification of the geometry due to the potential of the spherical shell. Specifically, we expect the wave amplitude to be proportional to $A^{-1/2}$, where A is the proper surface area at coordinate R. Since $A=4\pi R^2(1-2\phi)$, using the potential of the shell and normalizing the wave amplitude to be $e^{i\omega R}/R$ at r' reproduces the first correction term of (4.5).

We can simplify the integrals in (4.5) greatly by assuming that $\omega r'$ is large and by integrating by parts. In the limit that r' and R are large, keeping $R-r'=z$ fixed, we reproduce the plane-wave solution found in (3.13). In addition we find a contribution from the back-scattering of the waves off the opposite side of the spherical shell. Ignoring the back-scattered contribution therefore gives the same index of refraction as was found in (3.14).

For scalar waves with $a\neq 0$ we find an additional scattered wave (for $R>r'$):

$$\psi_s^{(3)}=-\frac{4\pi i G\rho_0 a\Delta r'}{\omega}\,\frac{e^{i\omega R}}{R}\,(1-e^{2i\omega r'})\,.$$

Thus we find that the index of refraction, for $a\neq 0$, is again given by (3.17).

C. Spherical volumes

Thus far we have considered only cases in which the source and observer were spatially separated from the matter through which the wave propagates. If the source or observer is immersed in the scattering region, then the combination of forward-scattered and back-scattered waves produces a change in phase of the final wave which is not simply that given by the previously derived indices of refraction. Specifically, we consider a source of spherically symmetric scalar waves at the center of a sphere of uniform density and radius $\mathcal{R}$, with the observer again located at a distance R from the center. Ignoring the coordinate dependence of the speed of light, we obtain the scattered wave from the volume of radius $\mathcal{R}$,

$$\psi_s=-\frac{4\pi G\rho_0}{3R}\left[\mathcal{R}^3\int_{\mathcal{R}}^{\infty}\frac{dv}{v^2}\,g(v)+\int_0^{\mathcal{R}}dv\,g(v)\left(v-\frac{3ai}{\omega}\right)\right],\tag{4.6}$$

where

$$g(v)=e^{i\omega(R+2v)}-e^{i\omega(|R-v|+v)}\,.$$

Suppose first that the observer is outside the mass distribution, i.e., $R>\mathcal{R}$. Then, in the large-$\omega\mathcal{R}$ limit (4.6) reduces to

$$\psi_s=\frac{4\pi G\rho_0}{3}\,\frac{e^{i\omega R}}{R}\left[\tfrac{3}{2}\mathcal{R}^2-\frac{\mathcal{R}^3}{R}+\frac{i\mathcal{R}}{2\omega}\left(\frac{\mathcal{R}^2}{R^2}-6a\right)+O\left(\frac{1}{\omega^2}\right)\right].\tag{4.7}$$

The first two terms represent the amplitude change in the wave due to the dependence of the surface area on the potential of the mass distribution. The third term is the one that arises from the phase change due to an index of refraction. From (4.7) one deduces an effective index of refraction for scalar waves

$$``n"=1+\frac{2}{3}\,\frac{\pi G\rho_0}{\omega^2}\left(\frac{\mathcal{R}^2}{R^2}-6a\right),\ \mathcal{R}<R\tag{4.8}$$

where the effects of back-scattered waves are included.

Next suppose that both the source and observer are inside the mass distribution, i.e., $\mathcal{R}>R$. Then, in the large-ωR limit (4.6) reduces to

$$\psi_s=\frac{2\pi G\rho_0}{3}\,\frac{e^{i\omega R}}{R}\left[R^2+\frac{iR}{\omega}(1-6a)+O\left(\frac{1}{\omega^2}\right)\right],\tag{4.9}$$

which implies an effective index of refraction

$$``n"=1+\frac{2}{3}\,\frac{\pi G\rho_0}{\omega^2}(1-6a),\quad \mathcal{R}>R\,.\tag{4.10}$$

Note that if $a=\frac{1}{6}$, then there is no phase change due to the effective index of refraction. This is understandable since for $a=\frac{1}{6}$ the scalar wave equation is conformally invariant. A spherically symmetric uniform distribution of free particles is described in the interior by one of the Friedmann cosmologies, here taken to be at a stage of maximum expansion. But the Friedmann models are conformally flat,[19] and therefore one expects solutions of conformally invariant wave equations to propagate as in flat space-time, i.e., sharply along null geodesics. Thus, one would not expect a phase change, or a deviation from one of the effective index of refraction, for the case in which both the source and observer are immersed in a uniform mass distribution, in agreement with our results. However, if either the source or observer is outside the sperically symmetric mass distribution then one generally expects some dispersion, as the exterior metric is then the Schwarzschild metric, which is not conformally flat.

For electromagnetic and gravitational waves the general analysis is complicated by the spatial dependence of the polarization and the lack of spherical symmetry of the waves. However, in the case that both the source and observer are immersed in the uniform density mass distribution, we can make some simplifying statements. Consider the electromagnetic wave equation (2.8). On an average over a volume of scatterers, with no preferred direction, the average value of $\phi_{,ki}$ is $\frac{1}{3}\delta_{ki}\phi_{,ll}$. Alternatively, the potential of a spherically symmetric uniform mass density is $2\pi G\rho r^2/3$, giving $\phi_{,ki}=4\pi G\rho\delta_{ki}/3=\delta_{ki}\phi_{,ll}/3$. Thus within this volume $\tilde{A}_i$ satisfies the equation

$$(\nabla^2+\omega^2)\tilde{A}_i=4\omega^2\phi\tilde{A}_i+\tfrac{1}{3}\phi_{,kk}\tilde{A}_i\,.\tag{4.11}$$

We can define a spacelike, transverse wave by taking

$$A_i=(\epsilon_i+\nabla_i\vec{\epsilon}\cdot\vec{\nabla}/\omega^2)\bar{\psi},\tag{4.12}$$

where, to lowest order in ϕ, $\bar{\psi}=e^{i\omega R}/R$, and where $\vec{\epsilon}$ is a constant vector. Comparing with (2.4), we see that $\bar{\psi}$, to order ϕ, satisfies the scalar wave equation with the choice $a=\frac{1}{6}$. Thus we have the same propagation properties as the conformally invariant scalar wave equation, i.e., no deviation of the effective index of refraction from one. This is to be expected, however, since the electromagnetic wave equations are conformally invariant,[20] and therefore the solutions should propagate sharply as in flat space-time.[21]

For gravitational waves we can construct potentials from a scalar function $\tilde{\psi}$ by projection as in (4.12). It is clear from the wave equation (2.18) that the scalar function $\tilde{\psi}$ must again satisfy the scalar wave equation (2.4) with the parameter $a=0$. Thus the expected effective index of refraction for gravitational waves in the case of both observer and source being immersed in the matter distribution is (4.10) with $a=0$.

V. WAVE PACKETS

Up to this point we have considered only monochromatic waves interacting with matter. We next ask about the influence of the matter on the propagation of wave packets. Ordinarily the speed with which a wave packet propagates is the group velocity $d\omega/dk$. For the indices of refraction we have computed here, this leads to an immediate difficulty. From (3.14), (3.17), and (3.20) we can form the dispersion relations $\omega(k)$ and compute the group velocity $d\omega/dk$. For gravitational waves, and for scalar waves with $a<\frac{1}{2}$, the group velocity exceeds the speed of light. Fortunately, we can described the propagation of a δ-function wave packet in closed form. This will be sufficient to show that causality is not violated; this also gives a way of describing the propagation of wave packets of a finite time duration, by a superposition of δ functions.

Consider first our single-particle scattering solution for plane waves. From the discussion in Sec. III we find (by taking Fourier transforms) that a plane-wave packet $\delta(t'-t+z)$, satisfying (2.20), incident on a single scatterer, gives rise to a transmitted wave packet

$$\tilde{\psi}=\delta(t'-t+z)-2Gm\left[\delta'(t'-t+z)\ln(r-z)+\int_{r-z}^{\infty}\delta'(t'-t+z+u)\frac{du}{u}\right], \tag{5.1}$$

where the dependence of the wave packet on the source variables (primed coordinates) has been suppressed and where δ' indicates differentiation with respect to the argument. The first two terms indicate direct propagation of the wave packet along the z direction with the velocity of light; the last term is zero until there is time for the wave packet to scatter off the mass m and then propagate radially outward to the observer with the speed of light. The first two terms can be combined, using the arguments in Sec. III about the coordinate dependence of the speed of light. This gives a modified retarded time in the δ function

$$\delta(t'-t+z-2Gm\ln(r-z)),$$

which reproduces the first two terms of (5.1) when "expanded" to first order in Gm. Thus we have demonstrated that our single-particle solution is consistent with causality, i.e., no signal is received faster than the speed of light.

We next consider the same wave packet incident on the plane of scatterers, and consider only the scattered contributions, ignoring the potential dependence of the directly propagated δ function as well as the focusing term. From (3.13), or from the superposition of (5.1), we find

$$\tilde{\psi}_s=2\pi G\rho\Delta zH(t-t'-z), \tag{5.2}$$

where H is the step function defined to be 1 if the argument is positive and 0 if the argument is negative. Thus the initial δ-function wave packet incident on the plane produces a δ-function transmitted packet plus a jump in the value of $\tilde{\psi}$ (from 0) of amount $2\pi G\rho\Delta z$. Note that since

$$\frac{\partial}{\partial t}\tilde{\psi}_s=2\pi G\rho\Delta z\delta(t'-t+z),$$

energy propagation still takes place along the null line $t=z+t'$.

It is necessary that the plane of scatters be infinite in extent in order that (5.2) hold. If the plane of masses terminates at a distance R from the observer, then (5.2) becomes

$$\tilde{\psi}_s=2\pi G\rho\Delta z[H(t-t'-z)-H(t-t'-R)]-2\pi G\rho\Delta z\left[\frac{(R-z)^2+2z(R-z)}{(t-t'-z)^2}H(t-t'-R)\right], \tag{5.3}$$

indicating that the step change in $\tilde{\psi}$ is constant only so long as there are masses in the plane to scatter the signal to the observer at the speed of light. If the mass density decreases gradually, as in (3.8), then the value of $\tilde{\psi}_s$ decays to zero after the initial jump.

If we consider the other wave solutions in Sec. III, we find similar behavior. For the case of the scalar wave equation with $a\neq 0$, related to (3.16), we find a scattered contribution which is $(1-2a)$ times (5.2) for an infinite plane or $(1-2a)$ times the first term of (5.3) plus the second term in (5.3) for a plane which terminates a distance R from the observer. Note that the character of the solutions is similar whether $a<\frac{1}{2}$ or $a>\frac{1}{2}$, indicating group velocities $>c$ or $<c$, respectively. Thus for the dispersion relations considered here, the wave packet is so distorted that the group velocity is meaningless, even if it is less than the speed of light. So long as axial symmetry in the plane is preserved, we find no net scattered elec-

tromagnetic wave, as in Sec. III.

We next consider the case of spherically symmetric wave packets, which are solutions of the scalar wave equation with $a=0$. As in the plane-wave case we start with an incident wave which is a δ function, i.e., an initial wave $\tilde{\psi}=\delta(t'-t+R)/R$. From (4.2) we find the final wave packet due to scattering off a single mass m:

$$\tilde{\psi}_s=\delta(t'-t+R)/R+2Gm\left[\frac{\delta'(t'-t+R)}{R}\ln\left(\frac{rR+\vec{r}\cdot\vec{R}}{r'R+\vec{r}'\cdot\vec{R}}\right)-\int_0^\infty\frac{\delta'(t'-t+r'+u+\rho)du}{(u+r')\rho}\right]. \quad (5.4)$$

As in the plane-wave solution (5.1), the first term in the bracket arises from the coordinate dependence of the speed of light and can be combined with the first term to give a direct propagation contribution. The last term gives no contribution until there is time for the signal to scatter off the mass m and the scattered signal to propagate to the observer with the speed of light.

For the case of a spherical δ function incident on a spherical shell, ignoring the terms arising from the coordinate dependence of the propagation speed and from amplitude modification due to the geometric effects of the potential, we find from (4.5) a scattered wave, analogous to (5.2), for $R>r'$:

$$\tilde{\psi}_s=\frac{8\pi G\rho_0 r'^2\Delta r'}{R}\left[\frac{H(t-t'-R)}{(t-t'+R)^2}-\frac{H(t-t'-R-2r')}{(t-t'-R)^2}\right]. \quad (5.5)$$

The first term represents a jump in the value of $\tilde{\psi}$ by an amount $2\pi G\rho_0\Delta r'(r'^2/R^3)$ when the signal has had time to scatter off the forward part of the shell, which then decays with the typical decay time of R. The second term represents a jump in the value of $\tilde{\psi}$ by an amount $-2\pi G\rho_0\Delta r'/R$ when the signal has had time to back-scatter off the opposite side of the shell, which then decays with typical decay time r'. Note that if $R\approx r'$, i.e., the observer is close to the shell, then the two jumps are approximately equal in magnitude, but opposite in sign. Moreover, the first jump that occurs agrees with the jump in $\tilde{\psi}$ found in (5.2), except that in the latter case there is no decay of the new value of $\tilde{\psi}$ since R, the distance to the source, is infinite for incident plane waves.

A similar situation arises when we consider the scalar wave equation with $a\neq 0$. This gives rise to an additional contribution to the scattered wave from (4.5):

$$\psi_s'=\frac{4\pi G\rho_0 a\Delta r'}{R}[H(t-t'-R)-H(t-t'-R-2r')]. \quad (5.6)$$

When there is time for the back-scattered signal to arrive, the value of ψ_s' jumps back to 0.

Spherical volumes are similarly treated. Starting from (4.6) we find the scattered part of an incident δ-function wave packet (ignoring amplitude modifications of the incident packet) for $R>\mathfrak{R}$:

$$\psi_s=\frac{8\pi G\rho_0}{3R}\left\{H(t-t'-R)\left[\frac{\mathfrak{R}^3}{(t-t'+R)^2}-\tfrac{1}{8}(1-6a)(t-t'-R)-\tfrac{3}{2}a\mathfrak{R}\right]\right.$$
$$\left.-H(t-t'-R-2\mathfrak{R})\left[\frac{\mathfrak{R}^3}{(t-t'-R)^2}-\tfrac{1}{8}(1-6a)(t-t'-R)-\tfrac{3}{2}a\mathfrak{R}\right]\right\}. \quad (5.7)$$

The first bracket represents contributions which begin when the signal has had time to propagate directly to the observer and the second bracket represents contributions which begin when the signal has back-scattered off the far side of the spherical mass distribution. Note that the last two terms in each bracket do not contribute when $t>t'+R+2\mathfrak{R}$, indicating that the decay of the signal is then given by the first term in each bracket.

If the observer and source are both immersed in the mass distribution, we find that the scattered δ-function wave packet (ignoring amplitude modifications of the incident packet) becomes, for $R<\mathfrak{R}$,

$$\psi_s=\frac{8\pi G\rho_0}{3R}\left\{\tfrac{1}{4}R(1-6a)H(t-t'-R)+H(t-t'+R-2\mathfrak{R})\left[\frac{\mathfrak{R}^3}{(t-t'+R)^2}-\tfrac{1}{8}(1-6a)(t-t'+R)-\tfrac{3}{2}a\mathfrak{R}\right]\right.$$
$$\left.-H(t-t'-R-2\mathfrak{R})\left[\frac{\mathfrak{R}^3}{(t-t'-R)^2}-\tfrac{1}{8}(1-6a)(t-t'-R)-\tfrac{3}{2}a\mathfrak{R}\right]\right\}. \quad (5.8)$$

The first step function represents contributions which begin when the signal has had time to directly propagate to the observer; the second step function gives contributions which start when the back-scattered signal off the forward surface of the spherical distribution reaches the observer; the third step function starts when the back-scattered signal off the far side of the spherical distribution reaches the observer. As in the case for $R > \mathcal{R}$, when enough time has elapsed so that the observer sees all contributions, only the first term of each of the last two brackets contributes, and those describe the decay of the scattered contribution in time.

One should note in (5.8) that if $a=\frac{1}{6}$, as in the case of conformally invariant waves or as in the case of electromagnetic waves considered as in Sec. IV C, there are no scattered contributions until the signal has had time to propagate to the edge of the mass distribution and then propagate back to the observer. For gravitational waves, treated as in Sec. IV C, the appropriate value of a is 0, giving a jump in the value of $\tilde{\psi}$ of amount $2\pi G\rho_0/3$ which remains constant until a signal scattered off the edge of the mass distribution reaches the observer. Note that this discontinuity in $\tilde{\psi}$ is independent of R, the distance of the observer from the source. Thus, so long as the observer remains inside the uniform mass distribution, the scattered signal grows in relation to the incident one (which falls off as $1/R$) as the distance from the source to observer increases.

VI. DISCUSSION OF THE RESULTS

We have found in this paper formulas for the index of refraction of scalar, electromagnetic, and gravitational waves arising from the influence of the gravitational fields of the matter on wave propagation. These suffice to derive the various refractive and dispersive effects that are normally found in optics discussions. Throughout our analyses we assumed weak gravitational fields, so that the results are applicable only to situations in which $G\rho/\omega^2$ is small, covering most situations one is likely to encounter. From an observational point of view the results are not too encouraging. For gravitational waves, the dispersive effects would be small corrections to effects which are, if anything, barely observable at present. For electromagnetic waves, in which one might have hoped to see small corrections, the analysis gives no dispersion to the order calculated. Scalar waves, unfortunately, do not exist except, perhaps, in the Brans-Dicke theory of gravity and then their interaction and corrections would be on a scale comparable to those of gravitational waves.

In addition to computing the indices of refraction, we have shown how the wave solutions predict a phase change, if one is using a coordinate system in which the speed of light is dependent on the gravitational potential, and how the wave amplitude is enhanced in passing through the matter distribution. We have taken the point of view that the phase velocity in the limit of large frequencies defines the speed of light, since high-frequency waves propagate principally along null geodesics. Thus the indices of refraction are the physical ones which give rise to dispersive effects. The amplitude enhancement, due to focusing, is not considered part of index of refraction in our analysis, but must be considered in any analysis of wave propagation through matter.

If the incident waves are not monochromatic, but rather are finite duration wave packets, then one cannot rely on analogies with corresponding propagation characteristics of electromagnetic waves in a gas of charged particles. To aid in such considerations, we have solved for the effect of propagation of δ-function wave packets in certain configurations. Because of the assumption of weak gravitational fields the results quoted are easily extended to other configurations. In addition, the knowledge of propagation character of δ-function wave packets is sufficient to describe the propagation character of any wave packet, since any function can be expressed as a superposition of δ functions.

The results we have derived for gravitational waves may be compared with previous dispersive effects calculated using different mechanisms.[7-10] For matter in states that are normally found in astrophysical systems, the expressions we have found give the dominant, albeit small, effects.

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