Hard-photon theorems for radiative pion-proton scattering*

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(Received 26 December 1972; revised manuscript received 13 December 1973)

Low's prescription for imposing gauge invariance is used to construct a reasonably general form of the amplitude for $\pi^{\pm}p \to \pi^{\pm}p\gamma$, excluding magnetic moment radiation from the internal strongly interacting structure. The explicit k dependence of the amplitude is exhibited and the breakdown of Low's result (that the first two terms in the expansion in k are model-independent) is examined in the hard-photon regime. We then consider possible hard-photon theorems and a program to determine unknown form factors in the off-shell scattering amplitude, in the proton magnetic moment, and in the radiation of the resonant πp system, e.g., magnetic moment radiation from the $\Delta(1236)$.

I. INTRODUCTION

The explicit functional dependence on photon energy k of the amplitude[1,2] and cross section[3,4] for *nonresonant* radiative scattering is restricted by current conservation in the soft-photon limit. Specifically, Low[1] has shown that when the radiative amplitude is written in the form

$$\mathfrak{M}_\mu \epsilon^\mu = (a/k) + b + ck + \cdots, \quad (1)$$

where each term (power of k) is separately explicitly gauge-invariant, the first two terms are exactly determined by the elastic amplitude and the static electromagnetic properties of the external particles. (ϵ^μ is the radiated photon's polarization). Burnett and Kroll[3] have obtained a similar result for the gauge-invariant radiative cross section of unpolarized particles in terms of the unpolarized elastic cross section. In the limit $k \to 0$, only the first two terms of $\mathfrak{M}_\mu \epsilon^\mu$ contribute, and the amplitude and cross section are model-independent. These results are frequently referred to as soft-photon theorems. For high-energy photons (hard photons), off-shell effects in the two-body scattering amplitude and at the electromagnetic vertices of the external particles which in Low's treatment contribute to the $ck + \cdots$ terms are expected to become important. (As in Ref. 1, "elastic" denotes no energy loss to the electromagnetic field in the scattering process while the off-shell amplitude is the corresponding two-body scattering amplitude which conserves energy and momentum but not mass.)

Resonances in the elastic amplitude present additional problems. First, the electromagnetic properties of the resonances[5-10] are expected to contribute to at least order $ck + \cdots$ in $\mathfrak{M}_\mu \epsilon^\mu$. Specifically, it has been proposed[7-9] that studies of *resonant* $\pi^+p \to \pi^+p\gamma$ *scattering* around the first πp resonance could lead to a determination of $\mu(\Delta^{++})$, the magnetic dipole moment of the $\Delta^{++}(1236)$. [Situations where the electromagnetic properties of resonances contribute to order (a/k) or b in Low's treatment have been discussed elsewhere[5,6] but here we will assume they contribute to order $ck + \cdots$.] Second, a rapid energy variation of the elastic amplitude is expected to limit the range of photon energies for which soft-photon theorems may be expected to yield useful predictions. In Low's treatment the coefficients a and b of Eq. (1) contain the elastic amplitude $T_0(\bar{s}, t)$ and its derivative, $dT_0(\bar{s}, t)/ds$, respectively, as multiplicative factors. Here s is the c.m. scattering energy squared, $\bar{s}$ is an average s corresponding to emission of a photon of energy k before and after a two-body scattering process, and t is the square of the momentum transfer to the nonradiating colliding particles (taken to be without charge or magnetic moment in Low's treatment). Now $\bar{s}$ is a function of k, so $T_0(\bar{s}, t)$ is in turn an implicit function of k. Third, Low's result that the first two terms of $\mathfrak{M}_\mu \epsilon^\mu$ are model-independent is frequently interpreted as implying the use of on-shell kinematics (all particles in the two-body scattering process on their mass shell) in determining $T_0(\bar{s}, t)$. In fact, the t dependence of $T_0(s, t)$ only occurs through the c.m. scattering angle θ. But $\cos\theta$ is a function of the virtual mass of the off-shell particle,[11] which is in turn a function of k. (This point will be further discussed in Secs. II and VI.) Thus a and b of Eq. (1) may also be complicated implicit functions of k for $k \to 0$ due to the choice of kinematics and in order to demonstrate gauge invariance in each order of k of the amplitude. This implicit dependence will be minimal for nonresonant s-wave radiative scattering, but for resonant $\pi p \to \pi p\gamma$ scattering the rapid energy variation of the elastic amplitude and the assumption of on-shell kinematics will induce corrections to the $(a/k) + b$ terms in Low's treatment which are of order $ck + \cdots$.

Fischer and Minkowski[9] have argued that ck corrections in Low's treatment due to a rapid energy variation of the elastic amplitude may be of the same size as those due to internal-structure radiation of the πp system [in particular $\Delta^{++}(1236)$ magnetic-moment radiation, which is also expected to contribute to order ck]. They have shown how to treat this problem for resonant $\pi p \to \pi p \gamma$ scattering by replacing the derivatives of Low's treatment with finite-difference ratios. However, after considering possible off-shell effects, they start with the elastic amplitude and on-shell kinematics; hence, their result is only expected to agree with data in the soft-photon limit.

The purpose of this paper is to consider contributions to the $ck+\cdots$ terms of $\mathfrak{M}_\mu \epsilon^\mu$ except those from the internal structure of the πp system. Therefore, we generalize the results of Refs. 1 and 9 by applying Low's procedure to the off-shell πp amplitude to find the form imposed by current conservation on the unknowns in the charge part of $\mathfrak{M}_\mu \epsilon^\mu$. The proton's magnetic-moment terms are then added at the end.

Our result is rewritten to exhibit its model dependence. The model-independent part is of the form $(\alpha/k)+\beta$ as obtained in Ref. 9. The coefficients α and β contain the elastic amplitude and the finite-difference ratio analog of its derivative with respect to s, respectively. But the elastic amplitude is evaluated at the exact c.m. scattering energy. Thus, while the form $(\alpha/k)+\beta$ satisfies over-all gauge invariance, $(\alpha/k)+\beta$ are not individually gauge-invariant. When this form is compared with Low's result, it is found to contain Low's result plus all higher order (in k) corrections to it. Thus, this form yields the model-independent part of $\mathfrak{M}_\mu \epsilon^\mu$ for all k. The model-dependent part is of the form γk, where γ is an implicit function of k.

We call the result of these considerations a hard-photon theorem, in the spirit that, given a model-independent theoretical calculation or an experimental evaluation of the unknowns in the off-shell amplitude and electromagnetic vertices of the external particles, a difference between the predictions of the hard-photon theorem and data for $k \gg 0$ would be evidence for internal-structure radiation. In Sec. V the functional dependence of the unknowns on the external variables is extracted so that later a program for experimentally determining the unknowns from studies of $\pi p \to \pi p \gamma$ scattering may be considered. In Sec. VI we consider the theoretical status of the unknowns in the off-shell amplitude and conclude that for a range of $k \gg 0$ a kinematically corrected hard-photon theorem exists which is nearly model-independent as far as the necessary s-dependent corrections to the off-shell amplitude are concerned. A program to test these considerations is also suggested, based on the expected properties of the internal-structure radiation.

II. KINEMATICS, NOTATION, AND LOW'S PROCEDURE

In this section the steps of Low's procedure are reviewed in a general way. The specific notation for $\pi p \to \pi p \gamma$ scattering is introduced, and most of the notation which will be used in later sections is collected together. As noted in Ref. 1, the extension to other radiative-scattering problems is straightforward.

Four-vectors for radiative πp scattering are defined as

$$\Pi(q_i)+P(p_i)\to \Pi(q_f)+P(p_f)+\gamma(k),$$

where $q_i=(E_{q_i},\vec{q}_i)$, etc. The four-vectors for the corresponding off-shell two-body scattering process are defined as

$$\Pi(\Delta_i)+P(M_i)\to \Pi(\Delta_f)+P(M_f).$$

The virtual masses are

$$\Delta_i^{\,2}=(q_i-k)^2=\mu^2-2q_i\cdot k,$$
$$\Delta_f^{\,2}=(q_f+k)^2=\mu^2+2q_f\cdot k,$$
$$M_i^{\,2}=(p_i-k)^2=m^2-2p_i\cdot k,$$
$$M_f^{\,2}=(p_f+k)^2=m^2+2p_f\cdot k,$$

with on-shell values of Δ^2 and M^2 of μ^2 and m^2 (see Fig. 1). The off-shell scattering amplitude is to be evaluated at

$$s_i=Q_i^{\,2}=(p_i+q_i)^2=m^2+\mu^2+2p_i\cdot q_i,$$
$$s_f=Q_f^{\,2}=(p_f+q_f)^2=m^2+\mu^2+2p_f\cdot q_f,$$
$$t_p=(p_i-p_f)^2=2m^2-2p_i\cdot p_f,$$

and

$$t_\pi=(q_i-q_f)^2=2\mu^2-2q_i\cdot q_f,$$

where $\sqrt{s_i}$ ($\sqrt{s_f}$) is the total c.m. energy when a final (initial) particle is off shell, and t_p (t_π) is the momentum-transfer squared when the pion (proton) is off shell. Now $\bar{s}=\frac{1}{2}(s_i+s_f)$, so that $s_i-\bar{s}=\bar{s}-s_f=2Q_f\cdot k=2Q_i\cdot k$. A slashed variable, e.g., $\not{Q}$, is defined by $\not{Q}=\gamma\cdot Q=\gamma_0 Q_0-\vec{\gamma}\cdot\vec{Q}$, etc.

When the initial (final) pion is off shell,

$$\cos\theta = (2E_{p_i}E_{p_f} + t_p - 2m^2)/2|\vec{p}_i|\,|\vec{p}_f| , \qquad (2a)$$

but p_i (p_f) clearly [Fig. 1(f)] depends on the initial (final) virtual-pion four-momentum, Δ_i (Δ_f) which in turn depends on k. When the proton is off shell a similar expression is obtained from t_π. In the case of elastic scattering $|\vec{p}_i| = |\vec{p}_f| = |\vec{p}_{\rm c.m.}|$ and

$$\cos\theta_{\rm el} = 1 + t_p/2\,|\vec{p}_{\rm c.m.}|^2 . \qquad (2b)$$

In general, the πp amplitude Υ is a function of s_i, s_f, t_π, t_p and the virtual masses Δ_i, Δ_f and M_i, M_f. On one hand, when the variables are specified we suppress all but the virtual mass actually off shell and t is understood according to the preceding paragraph when no confusion can result. The symbol $T(s, t, \ldots)$ is used for this case. Thus, the elastic amplitude and its constituents are $T_0(s) = -A(s) + \not{Q}B(s)$. On the other hand, *symbolically* the off-shell amplitude is $\Upsilon(\Delta, M)$, $\Upsilon(\Delta)$, and $\Upsilon(M)$ when any of the particles, pions only, and protons only, respectively, can be off shell; s and t are to be understood according to the above discussion. In general, Δ and M represent Δ_i, Δ_f and M_i, M_f, respectively; otherwise the virtual masses are explicitly specified.

Low's result rests on the fact that contributions to $\mathfrak{M}_\mu \epsilon^\mu$ may be divided into two types of terms: first, terms [Fig. 1(a)–1(d)] in which the photon is radiated from the charge or magnetic moment of an external particle (we use E_μ for these terms), and second, terms in which the photon is radiated from the charge or magnetic moment of the internal structure of the colliding system. We use I_μ for these terms. We use the following notation in this and later sections to describe specific pieces of the total radiative amplitude $\mathfrak{M}_\mu \epsilon^\mu$. The contributions from the charge radiation of the pion, proton, and total magnetic moment radiation are M_μ^π, M_μ^q, and M_μ^λ, respectively. According to the preceding discussion $M_\mu^\alpha = E_\mu^\alpha + I_\mu^\alpha$, where $\alpha = \pi$, q, or λ. So $\mathfrak{M}_\mu = M_\mu^\pi + M_\mu^q + M_\mu^\lambda = M_\mu + I_\mu^\lambda$, where $M_\mu = M_\mu^\pi + M_\mu^p + E_\mu^\lambda$ is the contribution obtained from Low's prescription. We also use the symbol E_μ^p for the total proton contribution, i.e., $E_\mu^p = M_\mu^q + E_\mu^\lambda$. Here I_μ^λ contains, e.g., the as-yet-unmeasured magnetic-moment contribution from the $\mu[\Delta(1236)]$ [Fig. 1(e)] as well as other gauge-invariant unknowns.

In these terms Low's prescription involves the following steps:

(1) The sum of terms contributing to E_μ is written down. The most general form of $\Upsilon(s, t, \Delta, M)$ is used to describe the following (or preceding) two-body scattering process. But s depends on whether photon emission occurs before or after scattering. So Υ is taken to be a function of $\bar{s}$,

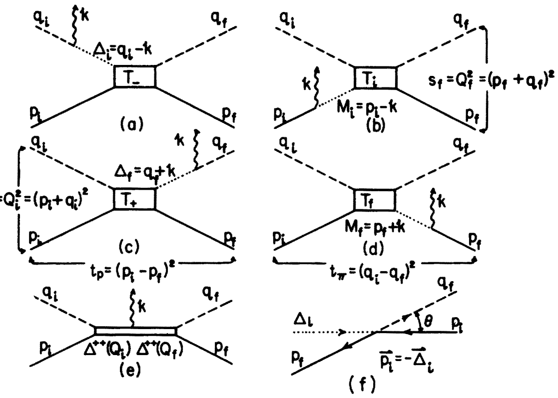

FIG. 1. (a)–(d) indicate the kinematic variables introduced to evaluate the bremsstrahlung from the colliding particles E_μ. The initial (final) four-momenta of the pion and proton are q_i (q_f) and p_i (p_f), respectively. The πp off-shell amplitudes $T_\pm$ and $T_{f(i)}$ are defined in the text, and are evaluated at s_f (s_i), when one of the initial (final) particles radiates, and at t_p (t_π) when a pion (proton) radiates. (e) is the Feynman diagram corresponding to photon radiation from the $\Delta^{++}(1236)$ formed as an intermediate state in the reaction $\pi^+ p \to \pi^+ p \gamma$. The magnetic-moment radiation from such diagrams contributes to I_μ^λ. (f) indicates the variables of the off-shell two-body scattering process when the initial pion is off shell. Because $\vec{p}_i = -\vec{\Delta}_i$, $E_{p_i} = (\Delta_i^2 + m^2)^{1/2}$ in the c.m. system of the colliding particles. These are the quantities used in Eq. (2a) of the text for this case.

the t to the pair of nonradiating particles, and the virtual masses of the radiating particles.

(2) The quantities multiplying Υ are rewritten to explicitly exhibit their $1/k$ dependence in each term of E_μ. However, Υ itself depends implicitly on k.

(3) Gauge invariance is imposed to determine the form of I_μ^π and I_μ^q; i.e.,

$$k^\mu \mathfrak{M}_\mu = 0 = k^\mu M_\mu^\pi + k^\mu M_\mu^q ,$$

or $k^\mu M_\mu^\pi = 0$, $k^\mu M_\mu^q = 0$ separately. This follows since M_μ^λ is separately gauge-invariant. The Dirac equation is used when nucleons are involved in this step, and the derivatives of $k^\mu E_\mu^\pi$, $k^\mu E_\mu^q$ are evaluated at $k = 0$.

(4) I_μ^π, I_μ^q or $I_\mu^\pi \epsilon^\mu, I_\mu^q \epsilon^\mu$ are recognized from step (3).

(5) E_μ^π, E_μ^q is expanded in an explicit power series in k with derivatives evaluated at $k = 0$ and added to I_μ^π, I_μ^q to yield the explicit k dependence of $(M_\mu^\pi + M_\mu^q)\epsilon^\mu$.

The separately gauge-invariant E_μ^λ terms may either be carried through the above steps along with the charge terms or added at the end after being developed in an explicit power series in k. Dirac's equation may be used when nucleons are involved in this step. Derivatives of $\Upsilon(\Delta, M)$ with respect to Δ, M (evaluated at $k=0$) generated in step (3) are exactly canceled from the b term in step (5), and M_μ is then determined solely by the static values of the electromagnetic vertices, $T_0(s)$ and $dT_0(s)/ds$, as $k\to 0$. This is Low's result.

In Secs. III and IV the derivatives in Low's procedure are replaced by finite-difference ratios defined as follows: Let F be a scalar function of the independent variables and let

$$D_1F(s,x)=\frac{F(s,x)-F(\bar{s},x)}{s-\bar{s}}\,,$$
$$D_2F(s,x)=\frac{F(s,x)-F(s,x_0)}{x^2-x_0^2}\,, \tag{3}$$

where x is Δ or M, and x_0 is μ or m. Then, e.g., when the pion is off shell,

$$\begin{aligned}F(s_i,\Delta_f^2)-F(s_f,\Delta_i^2)&\equiv[F(s_i,\Delta_f^2)-F(\bar{s},\Delta_f^2)]+[F(\bar{s},\Delta_f^2)-F(\bar{s},\mu^2)]\\&\quad-\{[F(s_f,\Delta_i^2)-F(\bar{s},\Delta_i^2)]+[F(\bar{s},\Delta_i^2)-F(\bar{s},\mu^2)]\}\\&=(s_i-\bar{s})D_1F(s_i,\Delta_f^2)+(\bar{s}-s_f)D_1F(s_f,\Delta_i^2)+(\Delta_f^2-\mu^2)D_2F(\bar{s},\Delta_f^2)\\&\quad-(\Delta_i^2-\mu^2)D_2F(\bar{s},\Delta_i^2)\\&=k\cdot Q_i[D_1F(s_i,\Delta_f^2)+D_1F(s_f,\Delta_i^2)]+2q_f\cdot kD_2F(\bar{s},\Delta_f^2)+2q_i\cdot kD_2F(\bar{s},\Delta_i^2)\,,\end{aligned} \tag{4}$$

where the last part of Eq. (4) results when $s_i-\bar{s}$, etc. are expressed in terms of the four-vector defined early in this section. A similar expression with $\Delta\to M$, $\mu\to m$, $q_f\to p_f$, and $q_i\to p_i$ holds when a nucleon is off shell.

Finally, since the finite-difference ratios are in turn scalars, this procedure can be extended to include the equivalent of second derivatives, i.e., D_1^2F or D_1D_2F.

III. PION BREMSSTRAHLUNG

When one of the pions is off shell $\Upsilon(\Delta)$ contains no new invariants, and when the final pion is off shell

$$\begin{aligned}T_+(s_i,t_p,\Delta_f^2)&=-A(s_i,t_p,\Delta_f^2)\\&\quad+\tfrac{1}{2}(\not{q}_{\rm in}+\not{q}_{\rm out})B(s_i,t_p,\Delta_f^2)\\&=-A(s_i,\Delta_f^2)+(\not{Q}+\tfrac{1}{2}\not{k})B(s_i,\Delta_f^2)\\&=T_0(s_i,\Delta_f^2)+\tfrac{1}{2}\not{k}B(s_i,\Delta_f^2)\,,\end{aligned} \tag{5a}$$

where $T_0=-A+\not{Q}B$. Then

$$T_-(s_f,\Delta_i^2)=T_0(s_f,\Delta_i^2)-\tfrac{1}{2}\not{k}B(s_f,\Delta_i^2)\,, \tag{5b}$$

for the initial pion off shell. This follows since, e.g., when the final pion is off shell, $\not{q}_{\rm out}=\not{\Delta}_f=\not{k}+\not{q}_f$, $\not{Q}=\frac{1}{2}(\not{q}_i+\not{q}_f)$, and so $\not{q}_{\rm in}+\not{q}_{\rm out}=2\not{Q}+\not{k}$. Equations (5) exhibit the explicit k dependence of $\Upsilon(\Delta)$. The invariant scalar amplitudes $A(s,t,\Delta^2)$ and $B(s,t,\Delta^2)$ are the extrapolation off shell of the elastic amplitudes and hence are implicit functions of k.

The product of the vertex function and the pion propagator at the π-γ-π vertex ($k^2=0$, $\epsilon\cdot k=0$) has been shown (page 975 of Ref. 1) to introduce no unknown form factors as a result of the Ward[12]-Takahashi[13] identity. (An equivalent proof is given on page 526 of Ref. 9.) This point will be further discussed in the Appendix. We suppress the pion's charge e_π until the final result. The Dirac equation is not used, so the nucleon spinors are ignored here. We now follow the steps of Low's procedure.

Step (1):

$$E_\mu^\pi\epsilon^\mu=\frac{(2q_f+k)\cdot\epsilon}{\Delta_f^2-\mu^2}T_+(s_i,\Delta_f^2)+T_-(s_f,\Delta_i^2)\frac{(2q_i-k)\cdot\epsilon}{\Delta_i^2-\mu^2}\,. \tag{6}$$

Step (2):

$$E_\mu^\pi\epsilon^\mu=\frac{q_f\cdot\epsilon}{q_f\cdot k}T_+(s_i,\Delta_f^2)-T_-(s_f,\Delta_i^2)\frac{q_i\cdot\epsilon}{q_i\cdot k}\,. \tag{7}$$

Step (3):

$$\begin{aligned}I_\mu^\pi k^\mu&=-k^\mu E_\mu^\pi\\&=-[T_+(s_i,\Delta_f^2)-T_-(s_f,\Delta_i^2)]\\&=-[T_0(s_i,\Delta_f^2)-T_0(s_f,\Delta_i^2)]-\tfrac{1}{2}\not{k}[B(s_i,\Delta_f^2)+B(s_f,\Delta_i^2)]\\&=-k\cdot Q_i[D_1T_0(s_i,\Delta_f^2)+D_1T_0(s_f,\Delta_i^2)]-2q_f\cdot kD_2T_0(\bar{s},\Delta_f^2)-2q_i\cdot kD_2T_0(\bar{s},\Delta_i^2)-\tfrac{1}{2}\not{k}[B(s_i,\Delta_f^2)+B(s_f,\Delta_i^2)]\,.\end{aligned}$$

Step (4):

$$I^{\pi}_{\mu}\epsilon^{\mu}=-\epsilon\cdot Q_i\,[D_1T_0(s_i,\Delta_f{}^2)+D_1T_0(s_f,\Delta_i{}^2)]-2q_f\cdot\epsilon D_2T_0(\bar{s},\Delta_f{}^2)-2q_i\cdot\epsilon D_2T_0(\bar{s},\Delta_i{}^2)$$
$$-\tfrac{1}{2}\not\epsilon[B(s_i,\Delta_f{}^2)+B(s_f,\Delta_i{}^2)]\,. \tag{8}$$

Step (5): Expanding Eq. (7) with Eqs. (3) and (4) and adding Eq. (8) yields

$$M^{\pi}_{\mu}\epsilon^{\mu}=e_\pi\left\{\frac{q_f\cdot\epsilon}{q_f\cdot k}T_0(s_i)-\frac{q_i\cdot\epsilon}{q_i\cdot k}T_0(s_f)\right\}+e_\pi\{\tfrac{1}{2}[\not\pi_f B(s_i)+\not\pi_i B(s_f)]-\epsilon\cdot Q_i\,[D_1T_0(s_f,\Delta_i{}^2)+D_1T_0(s_i,\Delta_f{}^2)]$$
$$+2q_f\cdot\epsilon[D_2T_0(s_i,\Delta_f{}^2)-D_2T_0(\bar{s},\Delta_f{}^2)]$$
$$+2q_i\cdot\epsilon[D_2T_0(s_f,\Delta_i{}^2)-D_2T_0(\bar{s},\Delta_i{}^2)]\}$$
$$+e_\pi\{q_f\cdot k\not\pi_f D_2B(s_i,\Delta_f{}^2)-q_i\cdot k\not\pi_i D_2B(s_f,\Delta_i{}^2)\}\,, \tag{9}$$

with

$$\not\pi_f=\frac{q_f\cdot\epsilon\not k-q_f\cdot k\not\epsilon}{q_f\cdot k},\quad \not\pi_i=\frac{q_i\cdot\epsilon\not k-q_i\cdot k\not\epsilon}{q_i\cdot k}\,.$$

Due to the simplicity of $\Upsilon(\Delta)$, no approximation or expansion was necessary other than that required in step (3). In its most compact form, $M^{\pi}_{\mu}\epsilon^{\mu}$ is the sum of Eqs. (7) and (8). Then, $M^{\pi}_{\mu}\epsilon^{\mu}$ is explicitly of the form $(\alpha/k)+\beta$, where α and β contain $\Upsilon(\Delta)$, but are not explicitly gauge-invariant. Given a knowledge of $\Upsilon(\Delta)$ and using off-shell two-body kinematics, this expression gives the exact form of $M^{\pi}_{\mu}\epsilon^{\mu}$ for all k. When $\Upsilon(\Delta)$ in Eq. (7) is expanded by Eqs. (3) and (4), then a term of order γk is naturally generated, as may be seen in the last term in curly braces of Eq. (9).

IV. PROTON BREMSSTRAHLUNG

When one of the protons is off shell $\Upsilon(M)$ becomes more complicated and takes the form

$$T_f(s_i,t_\pi,M_f)=\Lambda_+(\not M_f)T_0(s_i,t_\pi,M_f)+\Lambda_-(M_f)T_0'(s_i,t_\pi,M_f)\,, \tag{10a}$$

with $\Lambda_\pm(M_f)=(M_f\pm\not M_f)/2M_f$ when the final proton is off shell, $\not M_f=\not p_f+\not k$, and $M_f=(\not M_f{}^2)^{1/2}$. When the initial proton is off shell,

$$T_i(s_f t_\pi,M_i)=T_0(s_i,t_\pi,M_i)\Lambda_+(M_i)+T_0'(s_i,t_\pi,M_i)\Lambda_-(M_i)\,, \tag{10b}$$

with

$$\not M_i=\not p_i-\not k,\quad M_i=+(\not M_i{}^2)^{1/2}$$

and

$$\Lambda_\pm(M_i)=(M_i\pm\not M_i)/2M_i\,.$$

$T_0(s,t,M)=-A(s,t,M)+\not QB(s,t,M)$ is the off-shell extrapolation of $T_0(s,t)$, while $T_0'(s,t,M)=-A'(s,t,M)+\not QB'(s,t,M)$ and $T_0'(M)=T_0(-M)$.[14] The projection operators $\Lambda_\pm(\not M)$ satisfy the relations $\Lambda_\pm{}^2(M)=\Lambda_\pm(M)$, $\Lambda_\pm(M)\Lambda_\mp(M)=0$. The free nucleon propagator can be written as $1/(\not M-m)=\Lambda_+(M)/(M-m)-\Lambda_-(M)/(M+m)$, to reduce algebraic steps. The explicit k dependence of T_f,T_i is contained in the projection operators while, as before, $A(s,t,M)$ and $B(s,t,M)$ are implicit functions of k.

The N-γ-N vertex with one proton off shell also takes a more complicated form. The charge coupling is still exactly known[12,13] (see Appendix), but $E^{\lambda}_{\mu}\epsilon^{\mu}$ now contains unknown form factors, $F_2^{\pm}(M)$. The total amplitude for external proton bremsstrahlung is then

$$E^{p}_{\mu}\epsilon^{\mu}=\bar{u}(p_f)\left[\left(\not\epsilon+\not\epsilon\not k\frac{F_2(M_f)}{2m}\right)\frac{1}{\not M_f-m}T_f(s_i,t_\pi,M_f)+T_i(s_f,t_\pi,M_i)\frac{1}{\not M_i-m}\left(\not\epsilon+\frac{F_2(M_i)}{2m}\not\epsilon\not k\right)\right]u(p_i)\,,$$

where $F_2(m)=\mu_p-1$ is the static value of the proton's anomalous magnetic moment. We consider $M^{q}_{\mu}\epsilon^{\mu}$ and $E^{\lambda}_{\mu}\epsilon^{\mu}$ separately.

The first step of Low's procedure is to write down the amplitude for the charge radiation from the external proton lines. We obtain, using $1/(\not M-m)=\Lambda_+(M)/(M-m)-\Lambda_-(M)/(M+m)$ and the properties of the projection operators,

$$E^{q}_{\mu}\epsilon^{\mu}=\bar{u}(p_f)\left\{\not\epsilon\left[\frac{\Lambda_+(M_f)}{M_f-m}T_0(s_i,M_f)-\frac{\Lambda_-(M_f)}{M_f+m}T_0'(s_i,M_f)\right]+\left[T_0(s_f,M_i)\frac{\Lambda_+(M_i)}{M_i-m}-T_0'(s_f,M_i)\frac{\Lambda_-(M_i)}{M_i+m}\right]\not\epsilon\right\}u(p_i)\,. \tag{11}$$

Step (2) is implemented by inserting in $E^q_\mu \epsilon^\mu$ the expansions

$$\frac{\Lambda_\pm(M_f)}{M_f \mp m} = \frac{1}{2M_f} \pm \frac{(M_f \pm m)}{2M_f}\frac{(\not{M}_f + m)}{2p_f \cdot k}$$

$$\simeq \frac{1}{2M_f} + \left[\frac{\frac{1}{2}\pm\frac{1}{2}}{2p_f\cdot k} - \frac{1}{4m^2} + \frac{3}{8}\frac{(p_f\cdot k)}{m^4} + \cdots\right] \times (\not{M}_f + m) \qquad (12a)$$

and

$$\frac{\Lambda_\pm(M_i)}{M_i \mp m} = \frac{1}{2M_i} \mp \frac{(M_i \pm m)}{2M_i}\frac{(\not{M}_i + m)}{2p_i \cdot k}$$

$$\simeq \frac{1}{2M_i} - \left[\frac{\frac{1}{2}\pm\frac{1}{2}}{2p_i\cdot k} + \frac{1}{4m^2} + \frac{3}{8}\frac{(p_i\cdot k)}{m^4} + \cdots\right] \times (\not{M}_i + m). \qquad (12b)$$

With the use of the Dirac equation to evaluate $\bar{u}(p_f)\not{k}(\not{M}_f + m) = \bar{u}(p_f)2p_f\cdot k$ and $(\not{M}_i + m)\not{k}u(p_i) = 2p_i\cdot k u(p_i)$, step (3) of Low's procedure becomes

$$k^\mu I^q_\mu = -k^\mu E^q_\mu$$

$$= -\bar{u}(p_f)\bigg\{[T_0(s_i,M_f) - T_0(s_f,M_i)] + \frac{1}{2}\left(\frac{\not{k}}{M_f} - \frac{p_f\cdot k}{m^2} + \frac{3}{2}\frac{(p_f\cdot k)^2}{m^4} + \cdots\right)[T_0(s_i,M_f) - T_0'(s_i,M_f)]$$

$$+ [T_0(s_f,M_i) - T_0'(s_f,M_i)]\frac{1}{2}\left(\frac{\not{k}}{M_i} - \frac{p_i\cdot k}{m^2} - \frac{3}{2}\frac{(p_i\cdot k)^2}{m^4} + \ldots\right)\bigg\}u(p_i).$$

Expanding the first term in square brackets by use of Eqs. (3) and (4), step (4) becomes

$$I^q_\mu\epsilon^\mu = -\bar{u}(p_f)\bigg\{\epsilon\cdot Q_i[D_1T_0(s_i,M_f) + D_1T_0(s_f,M_i)] + 2p_f\cdot\epsilon D_2T_0(\bar{s},M_f) + 2p_i\cdot\epsilon D_2T_0(\bar{s},M_i)$$

$$+ \frac{1}{2}\left[\frac{\not{\epsilon}}{M_f} - \frac{p_f\cdot\epsilon}{m^2} + \frac{3}{2}\frac{(p_f\cdot\epsilon)}{m^2}\frac{(p_f\cdot k)}{m^2} + \cdots\right][T_0(s_i,M_f) - T_0'(s_i,M_f)]$$

$$+ [T_0(s_f,M_i) - T_0'(s_f,M_i)]\frac{1}{2}\left[\frac{\not{\epsilon}}{M_i} - \frac{p_i\cdot\epsilon}{m^2} - \frac{3}{2}\left(\frac{p_i\cdot\epsilon}{m^2}\right)\left(\frac{p_i\cdot k}{m^2}\right) + \cdots\right]\bigg\}u(p_i).$$

Step (5) follows immediately after using the Dirac equation to evaluate

$$\bar{u}(p_f)\not{\epsilon}(\not{M}_f + m) = \bar{u}(p_f)[2p_f\cdot\epsilon + \not{\epsilon}\not{k}]$$

and

$$(\not{M}_i + m)\not{\epsilon}u(p_i) = (2p_i\cdot\epsilon + \not{\epsilon}\not{k})u(p_i),$$

and when the expansion [Eq. (12)] is inserted in Eq. (11) and multiplied by the corresponding $2p\cdot k$, then all but the first four terms in $I^q_\mu\epsilon^\mu$ cancel. This leaves

$$M^q_\mu\epsilon^\mu = \bar{u}(p_f)\bigg\{\left[\frac{p_f\cdot\epsilon}{p_f\cdot k}T_0(s_i,M_f) - T_0(s_f,M_i)\frac{p_i\cdot\epsilon}{p_i\cdot k}\right] + \epsilon\cdot Q_i[D_1T_0(s_i,M_f) + D_1T_0(s_f,M_i)] - 2p_i\cdot\epsilon D_2T_0(\bar{s},M_i)$$

$$- 2p_f\cdot\epsilon D_2T_0(\bar{s},M_f) + \frac{\not{\epsilon}\not{k}}{2p_f\cdot k}T_0(s_i,M_f) + \not{\epsilon}\not{k}\left(-\frac{1}{4m^2} + \frac{3}{8}\frac{p_f\cdot k}{m^4} + \cdots\right)[T_0(s_i,M_f) - T_0'(s_i,M_f)]$$

$$- [T_0(s_f,M_i) - T_0'(s_f,M_i)]\left(\frac{1}{4m^2} + \frac{3}{8}\frac{p_i\cdot k}{m^4} + \cdots\right)\not{\epsilon}\not{k} - T_0(s_f,M_i)\frac{\not{\epsilon}\not{k}}{2p_i\cdot k}\bigg\}u(p_i).$$

Now expanding the first term in square brackets by means of Eqs. (3) and (4) and comparing the series in the last two terms to the expansion [Eq. (12)], we obtain

$$M^q_\mu\epsilon^\mu = \bar{u}(p_f)\bigg(\left\{\frac{p_f\cdot\epsilon}{p_f\cdot k}T_0(s_i) - T_0(s_f)\frac{p_i\cdot\epsilon}{p_i\cdot k}\right\}$$

$$+ \bigg\{-\epsilon\cdot Q_i[D_1T_0(s_i,M_f) + D_1T_0(s_f,M_i)] + 2p_i\cdot\epsilon[D_2T_0(s_f,M_i) - D_2T_0(\bar{s},M_i)]$$

$$+ 2p_f\cdot\epsilon[D_2T_0(s_i,M_f) - D_2T_0(\bar{s},M_f)] + \frac{\not{\epsilon}\not{k}}{2p_f\cdot k}T_0(s_i,M_i) - T_0(s_f,M_i)\frac{\not{\epsilon}\not{k}}{2p_i\cdot k}\bigg\}$$

$$-\left\{\frac{\not\epsilon\not k}{2p_i\cdot k}\left(\frac{M_f-m}{2m_p}\right)[T_0(s_i,M_f)-T_0'(s_i,M_f)]-[T_0(s_f,M_i)-T_0'(s_f,M_i)]\left(\frac{M_i-m}{2M_i}\right)\frac{\not\epsilon\not k}{2p_i\cdot k}\right\}\right)u(p_i)\,. \tag{13}$$

Each of the curly braces represents a term in the expression $(\alpha/k)+\beta+\gamma k$.

When, say, the final proton is off shell, the magnetic moment term $E_\mu^\lambda\epsilon^\mu$ becomes

$$\bar{u}(p_f)\left\{\frac{\not\epsilon\not k}{2m}[\Lambda_+(M_f)F_2^+(M_f)+\Lambda_-(M_f)F_2^-(M_f)]\times\frac{1}{M_f-m}T_f(s_i,M_f)\right\}u(p_i)\,, \tag{14}$$

and a similar expression results when the initial proton is off shell. The unknown form factors $F_2^\pm(M)$ have been discussed by several authors.[15-18] Clearly $F_2^+(m)=\mu_p-1$, the anomalous magnetic moment, while $F_2^-(m)$ is uncertain,[18] but $F_2^-(M)=F_2^+(-M)$.[15] Using the properties of the projection operators, we write, for $E_\mu^\lambda\epsilon^\mu$,

$$\bar{u}(p_f)\left\{\frac{\not\epsilon\not k}{2m}\left[\frac{\Lambda_+(M_f)}{M_f-m}F_2^+(M_f)T_0(s_i,M_f)-\frac{\Lambda_-(M_f)}{M_f+m}F_2^-(M_f)T_0'(s_i,M_f)\right]+\left[F_2^+(M_i)T_0(s_f,M_i)\frac{\Lambda_+(M_i)}{M_i-m}-F_2^-(M_i)T_0'(s_f,M_i)\frac{\Lambda_-(M_i)}{M_i-m}\right]\frac{\not\epsilon\not k}{2m}\right\}u(p_i)\,, \tag{15}$$

where

$$\bar{u}(p_f)\not\epsilon\not k\frac{\Lambda_\pm(M_f)}{M_f\pm(-m)}=\bar{u}(p_f)\frac{[(M_f\pm m)\not\epsilon\not k\pm 2m\phi_f]}{[M_f\pm(-m)]M_f}\simeq\bar{u}(p_f)\left[(1\pm1)\frac{m(\not\epsilon\not k+\phi_f)}{4p_f\cdot k}-\frac{\phi_f}{2m}\right],$$

$$\frac{\Lambda_\pm(M_i)\not\epsilon\not k}{M_i\pm(-m)}u(p_i)=\frac{[(M_i\pm m)\not\epsilon\not k\mp 2m\phi_i]}{[M_i\pm(-m)]2M_i}u(p_i)\simeq\left[(1\pm1)\frac{m(\phi_i-\not\epsilon\not k)}{4p_i\cdot k}+\frac{\phi_i}{2m}\right]u(p_i)\,,$$

and $m\phi_f=p_f\cdot k\not\epsilon-p_f\cdot\epsilon\not k$ and $m\phi_i=p_i\cdot k\not\epsilon-p_i\cdot\epsilon\not k$. Using the indicated approximations to the operations in Eq. (15) and using Eqs. (3) and (4) to project out the b term of Eq. (1), then Eq. (15) may be rewritten as

$$E_\mu^\lambda\epsilon^\mu\simeq\bar{u}(p_f)\left\{\frac{(\not\epsilon\not k+\phi_f)}{2p_f\cdot k}(\mu_p-1)T_0(s_i)-T_0(s_f)(\mu_p-1)\frac{(\not\epsilon\not k-\phi_i)}{2p_i\cdot k}\right\}u(p_i)$$
$$+\bar{u}(p_f)\left\{(\not\epsilon\not k+\phi_f)D_2F_2^+(M_f)T_0(s_i,M_f)-\frac{\phi_f}{4m^2}[F_2^+(M_f)T_0(s_i,M_f)-F_2^-(M_f)T_0'(s_i,M_f)]+D_2F_2^+(M_i)T_0(s_f,M_i)(\not\epsilon\not k-\phi_i)+[F_2^+(M_i)T_0(s_f,M_i)-F_2^-(M_i)T_0'(s_f,M_i)]\frac{\phi_i}{4m^2}\right\}u(p_i)\,. \tag{16}$$

$\Upsilon(M)$ is more complicated, and it was necessary to expand explicitly in k at step (2) the factors multiplying $T_0(s,M)$ and $T_0'(s,M)$ in $M_\mu^q\epsilon^\mu$. The expansion was carried out to terms of order k^2, and the terms were carried through to step (4), where the corresponding counterterms were recognized. This procedure indicated the form of the counterterm for all orders of k. At step (5) the higher-order counterterms were canceled, but a power series multiplied by the gauge-invariant quantity $\not\epsilon\not k$ appeared. The leading terms in the series are the Dirac magnetic moment terms, i.e., $(\not\epsilon\not k/2p\cdot k)T_0(s)$. The Dirac magnetic-moment terms were extracted and the remainder recognized as generators of power series with leading terms of order γk. If the expansion [Eq. (12)] had been terminated after terms leading to the Dirac magnetic-moment terms, as is usually done, then $M_\mu^q\epsilon^\mu$ would be explicitly of the form $(\alpha/k)+\beta$, where α and β contain $\Upsilon(M)$ but are not explicitly gauge-invariant. When $\Upsilon(M)$ is expanded by Eqs. (3) and (4), terms of order γk are naturally generated from the Dirac magnetic-moment term itself. So there are two sources of γk and higher terms in $M_\mu^q\epsilon^\mu$. One source, which is basically due to the spin of the proton, arises when gauge invariance is imposed on all orders of k. The other source arises in the same way as for the spinless pion. $E_\mu^\lambda\epsilon^\mu$ may either be added to $M_\mu^q\epsilon^\mu$ in the form of Eq. (15) or after manipulation to extract the terms of order k^0 in the form of Eq. (16) to yield $M_\mu^p\epsilon^\mu$. Now, given a knowledge of $\Upsilon(M)$ and $F_2^{\pm}(M)$ and using off-shell two-body kinematics, Eqs. (13) plus (15) give the exact form of $M_\mu^p \epsilon^\mu$, the total proton contribution in Low's treatment, for all k.

V. DISCUSSION

We now consider the result of this treatment. First, we compare this result to those of others.[1,2,9] Second, we look at it with the goal of discovering the form imposed by current conservation on the unknowns in $M_\mu \epsilon^\mu$.

The total πp radiative amplitude, except for internal magnetic-moment radiation $M_\mu \epsilon^\mu$, is the sum of Eqs. (9), (13), and (16). In order to compare our results with others, we collect together all the terms from these equations which are not explicitly of order ck or higher, i.e.,

$$M_\mu \epsilon^\mu \simeq (\alpha/k) + \beta = e_\pi\{A\} + \{B\} - \epsilon\cdot Q_i\{C\} + 2\{D\}, \tag{17}$$

where

$$A = \left|\frac{q_f\cdot\epsilon}{q_f\cdot k} T_0(s_i, t_p) - \frac{q_i\cdot\epsilon}{q_i\cdot k} T_0(s_f, t_p)\right| + \tfrac{1}{2}[\not{\pi}_f B(s_i, t_p) + \not{\pi}_i B(s_f, t_p)],$$

$$B = \left|\frac{2p_f\cdot\epsilon + \mu_p \not{\epsilon}\not{k}}{2p_f\cdot k} T_0(s_i, t_\pi) - T_0(s_f, t_\pi)\frac{2p_i\cdot\epsilon + \mu_p\not{\epsilon}\not{k}}{2p_i\cdot k}\right| + \tfrac{1}{2}(\mu_p - 1)\left|\frac{\not{\phi}_f T_0(s_i, t_\pi)}{p_f\cdot k} + \frac{T_0(s_f, t_\pi)\not{\phi}_i}{p_i\cdot k}\right|,$$

$$C = e_\pi[D_1 T_0(s_i, \Delta_f^2) + D_1 T_0(s_f, \Delta_i^2)] + [D_1 T_0(s_i, M_f) + D_1 T_0(s_f, M_i)],$$

$$\begin{aligned} D = {} & e_\pi q_f\cdot\epsilon[D_2 T_0(s_i, t_p, \Delta_f^2) - D_2 T_0(\bar{s}, t_p, \Delta_f^2)] \\ & + e_\pi q_i\cdot\epsilon[D_2 T_0(s_f, t_p, \Delta_i^2) - D_2 T_0(\bar{s}, t_p, \Delta_i^2)] \\ & + p_f\cdot\epsilon[D_2 T_0(s_i, t_\pi, M_f) - D_2 T_0(\bar{s}, t_\pi, M_f)] \\ & + p_i\cdot\epsilon[D_2 T_0(s_f, t_\pi, M_i) - D_2 T_0(\bar{s}, t_\pi, M_i)]. \end{aligned}$$

When $T_0(s)$ is substituted into Eq. (17) for $\Upsilon(\Delta, M)$, it reduces to the same expression (in an expanded form) as that obtained by Fischer and Minkowski [Ref. 9, Eq. (40)] except for the D term. This term contains the differences of the analogs of derivatives of $\Upsilon(\Delta, M)$ with respect to Δ and M evaluated at different s. Since Fischer and Minkowski use the recipe of Adler and Dothan,[2] which drops the derivatives with respect to the virtual masses at the second step of the recipe, they naturally do not obtain it; in fact, finite-difference ratios equivalent to the $D_2 T_0$'s used in this paper are not defined in Ref. 9. Clearly, $D_2 T_0$ vanishes if (1) $\Upsilon(\Delta, M)$ is independent of Δ and M, or (2) the extrapolation off shell is independent of s. The latter follows if the only Δ and M dependence of $\Upsilon(\Delta, M)$ occurs in form factors like $F_2(M)$. However, the Δ and M dependence of $\Upsilon(\Delta, M)$ is expected to occur both in kinematic corrections to $T_0(s)$ and in unknown form factors. Regardless of this complication, the difference of these terms cancels in the soft-photon limit, and it is this cancellation that leads to Low's result. Using Eqs. (3) and (4), this term can be rewritten as

$$\begin{aligned} 2k\cdot Q_i\{ & e_\pi[q_f\cdot\epsilon D_1 D_2 T_0(s_i, t_p, \Delta_f^2) \\ & - q_i\cdot\epsilon D_1 D_2 T_0(s_f, t_p, \Delta_i^2)] \\ & + [p_f\cdot\epsilon D_1 D_2 T_0(s_i, t_\pi, M_f) \\ & - p_i\cdot\epsilon D_1 D_2 T_0(s_f, t_\pi, M_i)]\}, \end{aligned} \tag{18}$$

where $D_1 D_2 T_0$ is the analog of the second derivatives with respect to s and Δ or M. In this form it is clear that for nonzero $D_1 D_2 T_0$'s the term vanishes only at $k = 0$.

Now consider the C term of Eq. (17): It is the sum of finite-difference ratios with respect to s. If there is a resonance in the elastic amplitude, i.e., $T_0 \propto 1/(s - \tilde{M}^2)$, where $\tilde{M} = M_r - \frac{1}{2}i\Gamma$, then it is this term which leads to a double-resonance-pole contribution to the b term of Eq. (1), i.e., $b \propto 1/(s - \tilde{M}^2)^2$. Such a term is expected if charge radiation from the resonance contributes to $\mathfrak{M}_\mu \epsilon^\mu$ [see Ref. 9, Eq. (42)]. Fischer and Minkowski have pointed out that it is important not to double-count possible charge radiation from the resonating πp system when models are introduced to calculate the contribution of $I_\mu^\lambda \epsilon^\mu$ from resonances. We note that this term cannot contribute to $\mathfrak{M}_\mu \epsilon^\mu$ because it is multiplied by the Lorentz-invariant product of four-vectors, $\epsilon\cdot Q_i$, and in the radiation gauge $\epsilon = (0, \vec{\epsilon})$, $\vec{\epsilon}\cdot\vec{k} = 0$, $k^2 = 0$, while $Q_i = q_i + p_i = (\sqrt{s_i}, 0)$; so $\epsilon\cdot Q_i = 0$. We also note that $\epsilon\cdot Q_f = 0$ in Eq. (42) of Ref. 9. This result is also justified as follows. Consider the situation in which the π and p interact to form a $\Delta(M)$ at rest in the Q_i system. Then spontaneously $\Delta(M) \to \gamma + \Delta(M - k)$. Electric dipole ($E_1$) radiation is forbidden, since $\Delta(M)$ and $\Delta(M - k)$ are presumed to have the same parity. Of course, magnetic dipole (M_1) and electric quadrupole (E_2) radiation are permitted. Thus, to lowest order, charge (E_1) radiation from a *single* πp resonant state cannot contribute to the radiative amplitude. E_1 radiation will arise from parity-charging transitions but without a double-resonance-pole form.

The A and B terms of Eq. (17) must reduce to Low's result. Low considered two cases: (a) a charged spin-zero boson scattering from a neutral spin-zero boson [the A term in Eq. (17) corresponds to a charged spin-zero boson scattering from a neutral spin-$\frac{1}{2}$ fermion; the difference between the two situations accounts for the second part of this term]; and (b) a neutral spin-zero boson scattering from a spin-$\frac{1}{2}$ fermion of charge e and anomalous magnetic moment $\mu_p - 1$. This accounts for the B term of Eq. (17). To obtain Low's result for case (a), expand $T_0(s, t_p)$ in the $(1/k)$ terms about $\bar{s}$ using Eqs. (3) and (4). Then

$$\frac{q_f\cdot\epsilon}{q_f\cdot k}T_0(s_i)-\frac{q_i\cdot\epsilon}{q_i\cdot k}T_0(s_f)\equiv\left[\frac{q_f\cdot\epsilon}{q_f\cdot k}T_0(\bar{s})-\frac{q_i\cdot\epsilon}{q_i\cdot k}T_0(\bar{s})\right]$$
$$+\left[\left(q_f\cdot\epsilon+\frac{q_f\cdot\epsilon}{q_f\cdot k}p_f\cdot k\right)D_1T_0(s_i)+\left(q_i\cdot\epsilon+\frac{q_i\cdot\epsilon}{q_i\cdot k}p_i\cdot k\right)D_1T_0(s_f)\right]. \qquad (19)$$

Equation (19) is in the same form as Low's result. The correspondence with Ref. 1 [Eq. (2.16)] becomes exact if $D_1T_0(s_i)$ and $D_1T_0(s_f)$ are replaced by $dT_0(\bar{s})/ds$ and $\epsilon\cdot Q_i=\epsilon\cdot Q_f=0$ are used to replace $q_{i(f)}\cdot\epsilon$ by $-p_{i(f)}\cdot\epsilon$.

Now we invoke the argument of Fischer and Minkowski that the differences of $D_1T_0(s_i)$, $D_1T_0(s_f)$ in Eq. (19) and $D_1T_0(\bar{s})$ or $dT_0(\bar{s})/ds$ (as in Ref. 1) may be significant for resonant radiative scattering. These differences induce corrections of order ck which may be of the same size as the b terms in Eq. (1). By expanding the D_1T_0's in Eq. (19) using Eqs. (3) and (4), we find this correction to be

$$(k\cdot Q_i)^2\left[\frac{q_f\cdot\epsilon}{q_f\cdot k}D_1{}^2T_0(s_i)-\frac{q_i\cdot\epsilon}{q_i\cdot k}D_1{}^2T_0(s_f)\right]$$
$$+k\cdot Q_i[\not{\pi}_f D_1B(s_i)-\not{\pi}_i D_1B(s_f)], \qquad (20)$$

where, to be precise, the obvious correction from the second term of the A term of Eq. (17) has been added. Here $D_1{}^2T(s)$ is the analog of the second derivative with respect to s. A similar result obtains for the terms in the B term of Eq. (17). The above result illustrates an advantage of introducing finite-difference ratios, since they replace an infinite Taylor-series expansion of a function. Also, when they are used in expanding a function, the full precision of all remaining terms is retained if the last term is evaluated at the correct s. So, in the above example, starting from $(1/k)$ terms evaluated at the correct s and expanding about $\bar{s}$, both the finite-difference ratio and Low's result agree in form. But the finite-difference ratio result is accurate to all orders of k in the differential form of Low's result. As a result, gauge invariance is imposed to all orders of k. Thus, the most compact expression for the model-independent part of the radiative amplitude has an explicit k dependence of the form $(\alpha/k)+\beta$ as obtained in Ref. 9. Here, α and β are not explicitly gauge-invariant. But after further expansion a result of the form of Eq. (1) is obtained [i.e., $\mathfrak{M}_\mu\epsilon^\mu=(a/k)+b+ck$, where a, b, and c are explicitly gauge-invariant].

The model-dependent part is explicitly of order k for small k. Specifically, we collect together all the terms from Eqs. (9), (13), and (16) which are explicitly of the form ck or higher to which Eq. (18) is to be added, to obtain

$$ck+\cdots=e_\pi\{E\}-\{F\}+\{G\}+\text{Eq. (18)}, \qquad (21)$$

where

$$E=[q_f\cdot k\not{\pi}_f D_2B(s_i,t_p,\Delta_f{}^2)-q_i\cdot k\not{\pi}_i D_2B(s_f,t_p,\Delta_i{}^2)],$$

$$F=\frac{\not{\epsilon}\not{k}}{2p_f\cdot k}\left(\frac{M_f-m}{2M_f}\right)[T_0(s_i,t_\pi,M_f)-T_0'(s_i,t_\pi,M_f)]$$
$$-[T_0(s_f,t_\pi,M_i)-T_0'(s_f,t_\pi,M_i)]\left(\frac{M_i-m}{2M_i}\right)\frac{\not{\epsilon}\not{k}}{2p_i\cdot k},$$

and

$$G=(\not{\epsilon}\not{k}+\not{\phi}_f)D_2F_2^+(M_f)T_0(s_i,t_\pi,M_f)+D_2F_2^+(M_i)T_0(s_f,t_\pi,M_i)(\not{\epsilon}\not{k}-\not{\phi}_i)$$
$$-\frac{\not{\phi}_f}{4m^2}[F_2^+(M_f)T_0(s_i,t_\pi,M_f)-F_2^-(M_f)T_0'(s_i,t_\pi,M_f)]+[F_2^+(M_i)T_0(s_f,t_\pi,M_i)-F_2^-(M_i)T_0'(s_f,t_\pi,M_i)]\frac{\not{\phi}_i}{4m^2},$$

where

$$M_f-m/M_f\simeq p_f\cdot k/m^2,\quad M_i-m/M_i\simeq -p_i\cdot k/m^2.$$

The terms E, F, and G represent contributions from $M_\mu^\pi\epsilon^\mu$, $M_\mu^q\epsilon^\mu$, and $E_\mu^\lambda\epsilon^\mu$, respectively, and Eq. (18) contains terms from $E_\mu^\pi\epsilon^\mu$ and $E_\mu^q\epsilon^\mu$. Equation (21) explicitly exhibits the form imposed by current conservation (excepting the G term) on the unknown in $M_\mu\epsilon^\mu$. Each of the terms E, F, and G is multiplied by a gauge-invariant combination of the four-vectors of the external particles $p_i, p_f, q_i, q_f, k, \epsilon$ ($q\cdot k\not{\pi}=q\cdot\epsilon\not{k}-q\cdot k\not{\epsilon}$ and $m\not{\phi}=p\cdot k\not{\epsilon}-p\cdot\epsilon\not{k}$). Thus, depending on the kinematical conditions of a particular experimental result, Eq. (21) indicates the way in which the unknowns affect the result.

VI. HARD-PHOTON THEOREMS AND CONCLUSIONS

If we define a soft-photon theorem (SPT) as a mathematical description of the consequences of current conservation applied to a given system as $k\to 0$, to which we add the separately gauge-invariant $E_\mu^\lambda\epsilon^\mu$, then Low has shown that SPT's give the functional dependence of $\mathfrak{M}_\mu\epsilon^\mu$ on $T_0(\bar{s})$, $dT_0(\bar{s})/ds$, the static values of the external electromagnetic vertices, and external variables as $k\to 0$. The term "model-independent part of $\mathfrak{M}_\mu\epsilon^\mu$" was introduced to describe this dependence of $\mathfrak{M}_\mu\epsilon^\mu$ on independently measurable or determined quantities. On one hand, SPT's are often considered to be useful in the sense that a statistically significant discrepancy between data and the predictions of an SPT is presumed to be due to off-shell effects or $I_\mu^\lambda\epsilon^\mu$. Thus, the predictions of an SPT permit an experimental evaluation of the onset of contributions due to these unknowns. The term "model-dependent part of $\mathfrak{M}_\mu\epsilon^\mu$" was introduced to describe the dependence of $\mathfrak{M}_\mu\epsilon^\mu$ on such unknowns. On the other hand, $I_\mu^\lambda\epsilon^\mu$ contributes to $ck+\cdots$ terms and hence is only expected to be large at photon energies which may exceed the range of validity of an SPT due to off-shell effects which contribute to $ck+\cdots$ terms also.

In preceding sections, finite-difference ratios were incorporated into Low's prescription to obtain $M_\mu\epsilon^\mu$ for the case of resonant radiative πp scattering. These conclusions seem significant. First, the functional form of the SPT obtained by Fischer and Minkowski [Ref. 9, Eq. (40), or Eq. (17) of this paper when the elastic amplitude is substituted] holds for all orders of k and hence for all k. Second, the off-shell effects appear only in additional terms [Eq. (21)] which are of order k as $k\to 0$. These terms have a more complicated k dependence for $k\gg 0$ due to the implicit k dependence of $\Upsilon(\Delta,M)$ and $F_2^\pm(M)$. Last, the functional dependence of $M_\mu\epsilon^\mu$ on the off-shell effects and the four-vectors of the colliding particles is established as a result of current conservation to all orders of k and hence for all k.

By analogy with an SPT, a hard-photon theorem would be a mathematical description of the consequences of charge conservation for all k to which the separately conserved $E_\mu^\lambda\epsilon^\mu$ is added. If $\Upsilon(\Delta,M)$ and $F_2^\pm(M)$ were known, then a discrepancy between data and the predictions of a hard-photon theorem would be due to $I_\mu^\lambda\epsilon^\mu$. Also, if all other possible unknowns in $E_\mu^\lambda\epsilon^\mu$ were small or computable, then, e.g., an experimental evaluation of $\mu(\Delta^{++})$ could result.[7-9] This goal has not yet been shown to be possible.

These considerations suggest (a) a program for obtaining an experimental evaluation of the unknowns as a function of the four-vectors of the colliding particles, and/or (b) a parametrization of the unknowns in $\mathfrak{M}_\mu\epsilon^\mu$ which allows a convenient theoretical interpretation of a data-fitting procedure. The first objective is satisfied if we modify the result of Fischer and Minkowski to the hard-photon regime. The necessary modification is of a strictly kinematical nature, aggravated by the fact that $\mu\ll m$. For example, for $E_{q_i}=440$ MeV, $\theta_\pi=50°$, with photon emission opposite to the pion direction (all quantities in the lab), then $\Delta_f^2\approx 2\mu^2$ and $\Delta_i^2\approx 0$ when $k_{\rm lab}$ reaches ~15 MeV, while the M^2's are within about a percent of m^2. The point here is that if $\cos\theta$ is calculated from Eq. (2b) as is done in Ref. 9 and in nucleon-nucleon bremsstrahlung, then $|\cos\theta|$ may become >1 for not-too-large k; and when it is inserted into $T_0(s)$ an even more unrealistic amplitude results. This problem has been dealt with by using the virtual masses in evaluating $\cos\theta$ from Eq. (2a) (see Ref. 11, p. 1031). In this case $\cos\theta$ always remains bounded between -1 and 1. For the above example with the initial pion off shell, when $k\gtrsim 100$ MeV, $\cos\theta_{\rm el}<-1$ [Eq. (2b)] ($\Delta_i^2\simeq -6\mu^2$), while $\cos\theta=0.415$ [Eq. (2a)]. When these values are inserted into $1+3\cos^2\theta$ the approximate ratio of elastic cross section for the two values of $\cos\theta$ is $\simeq 2.6$, thereby inducing large corrections to the $(\alpha/k)+\beta$ terms of $\mathfrak{M}_\mu\epsilon^\mu$. We propose that the result of Ref. 9 modified in this way is a "model-independent hard-photon theorem" for resonant radiative πp scattering.

The form of $\Upsilon(\Delta,M)$ has been discussed in connection with quasi-(virtual)-two-body scattering processes such as those encountered in one-particle-exchange reactions.[19] Pion-proton bremsstrahlung may be viewed as involving four such processes, since each of the four particles participating in the two-body scattering process may be virtual, as in Eqs. (5a), (5b), (10a), and (10b). In each case, by treating Δ or M as a parameter, $\Upsilon(\Delta,M)$ is analyzed in a way similar to the elastic case, i.e., by decomposing the amplitude into partial waves. It has been shown that the phases of the off-shell partial-wave amplitudes are the same as the elastic ones when elastic unitarity is valid.[11,14] That is, if $f_{l\pm}^I(s,x)$ is the off-shell amplitude for the partial wave of orbital angular momentum l, total angular momentum $j=l\pm\frac{1}{2}$, isotopic spin I, for virtual masses $x=\Delta$ or M, and

$$f_\pm^I(s,x_0)=(1/|\vec{q}^{\,\rm on}|)\exp[i\delta_\pm^I(s)]\sin\delta_{l\pm}^I(s)$$

is the corresponding elastic partial-wave amplitude with $x=x_0$, for on-shell masses μ or m, then the above result suggests that

$$f^I_{l\pm}(s,x)=K(x)[G_{l\pm}(s,x)/G_{l\pm}(s,x_0)]f^I_{l\pm}(s)\,. \qquad (22)$$

The kinematical quantities $|\vec{q}^{\,\mathrm{on}}|$ (c.m. three-momenta) and s are calculated in the system where both particles are on shell and $\delta^I_{l\pm}(s)$ is the physical phase shift. The ratio in square brackets is sometimes called a kinematical form factor (KFF) and depends on s, l, j, and parity of the π-N state, as well as on x. Because of the smallness of the pion's mass, $f^I_{l\pm}(s,\Delta)$ is expected to differ considerably from $f^I_{l\pm}(s)$ except in phase. The KFF term is written as a ratio to indicate that $G_{l\pm}(s,x)$ is a theoretical or phenomenological prescription for extrapolating $f^I_{l\pm}(s)$ off shell at each s, e.g., a Born approximation extrapolation. The denominator is the same prescription evaluated on shell to give the correct normalization. The term "kinematical" refers to the s and x dependence of this ratio. Several approximate[11,20,21] and one "exact"[22] dispersion-relation calculations of the kinematical form factor for $f_{1+}(1236^2,\Delta)$ agree for $|\Delta^2|<5$–$10\mu^2$ to $\simeq 10\%$. For larger Δ^2, phenomenological models[23–25] based on an analogy to potential scattering give a good account of a number of one-pion-exchange reactions. Dispersion-relation calculations of kinematical form factors for the smaller partial waves with the pion off shell[20] and for $f_{1+}(s,M)$ (Ref. 14) have been performed assuming P_{33} dominance of the dispersive integral. When $f^I_{l\pm}(s,x)$ is obtained in the form of Eq. (22), $K(x)$ is called the vertex or dynamical form factor. $K(\Delta^2)$ is the pionic form factor of the nucleon.[11,26] There are two form factors $K_\pm(M)$ when the nucleon is off shell, related as usual through Lorentz invariance,[15] i.e., $K_-(M)=K_+(-M)$. $K_\pm(M)$ have been calculated[14] assuming unsubtracted dispersion relations dominated by the lowest mass states, and $K_+(m)=1$. $F_2^\pm(M)$ have been calculated[17] under similar assumptions. Comparison with proton-proton bremsstrahlung data was good when the assumption of threshold dominance was relaxed.[18] We conclude:

(1) One-dimensional dispersion-relation calculations offer a means of parametrizing and interpreting low-energy (elastic unitarity regime) πp radiative scattering data for $\Delta^2 \lesssim 10\mu^2$ or $k \lesssim \Gamma_\Delta$ [Γ_Δ is the width of the $\Delta(1236)$ in the case of the above example.] For other conditions the phenomonology of potential scattering appears successful.

(2) The form of the kinematical form factors appears to be nearly model-independent, since both approaches agree that as $x^2 \to \mu^2$ the kinematic form factor reduces to a Born approximation, which in turn goes as $(|\vec{p}^{\,\mathrm{off}}|/|\vec{p}^{\,\mathrm{on}}|)^l$, where $|\vec{p}^{\,\mathrm{off}}|$ is the magnitude of the momentum of the virtual pion in its c.m. system. A Born approximation is also indicated for $f_{l\pm}(s,M)$.

(3) The Δ and M dependence of the $K(\Delta^2)$, $K_\pm(M)$, and $F_2^\pm(M)$ is least well established. We take the point of view that a hard-photon theorem based on Eq. (22) with kinematical form factors given by either of the above approaches and using off-shell kinematics provides a better measure of the remaining unknowns, and have chosen to call it a "kinematically corrected hard-photon theorem." A test of this approach is to determine the range of k over which data can be fitted by functions which depend solely on Δ and M in kinematic regions where $I^\lambda_\mu\epsilon^\mu$ is negligible. If the calculations of $K_\pm(M)$ and $F_2^\pm(M)$ prove reasonably accurate.[14,18] then only $K(\Delta^2)$ remains to be determined by a fitting process.

The purpose of this paper is to consider a procedure for quantitatively establishing the existence of $I^\lambda_\mu\epsilon^\mu$ in resonant radiative πp scattering. Since such radiation is expected to contribute to terms of order $ck+\cdots$ in $\mathfrak{M}_\mu\epsilon^\mu$, it is necessary to consider all contributions to these terms, as was done in preceding sections. Parametrization of $I^\lambda_\mu\epsilon^\mu$ is irrelevant in this context. There are, however, three results which have been argued on general grounds and confirmed by model-dependent calculations. First, $I^\lambda_\mu\epsilon^\mu$ is only expected to be an appreciable part of $\mathfrak{M}_\mu\epsilon^\mu$ under very special kinematic conditions, namely, when the external bremsstrahlung is minimized. This occurs for π^+p bremsstrahlung with photon emission backward to the direction of the scattered pion and proton and the initial pion direction.[7] Second, a parametrization of $I^\lambda_\mu\epsilon^\mu$ depends on s and t, but not to first order on Δ or M. Third, $I^\lambda_\mu\epsilon^\mu$ will depend on the charge of the internal structure. For example, SU(6) symmetry predictions[27] yield $\mu(\Delta^{++})=2\mu_p$ and $\mu(\Delta^0)=0$.

In summary, we have argued that imposition of gauge invariance on M containing $\Upsilon(\Delta,M)$ and $F_2^\pm(M)$ yields the functional dependence of these unknowns on the external variables and charge states of the πp system. Using this formalism under kinematic conditions where the off-shell effects predominate provides a means of confirming and/or improving existing theoretical models of off-shell effects in the hard-photon regime. A significant discrepancy between data[28–30] and the predictions of such a hard-photon theorem under kinematical conditions sensitive to $I^\lambda_\mu\epsilon^\mu$ would establish quantitatively the existence of $I^\lambda_\mu\epsilon^\mu$ in the hard-photon regime where it is expected to contribute. We shall describe the expected sensitivity for this proposal in a future paper.

In the derivation of our result, some of the assumptions deserve further comment. First, we used the free propagator form for the pion and nucleon. These propagators are closely related to the electromagnetic vertex function through the Ward-Takahashi identity, which is based on differential current conservation. As shown in the Appendix, in the absence of the Ward identity three products of vertex and propagator form factor would appear: one for the product of the complete renormalized pion propagator and the pion's electromagnetic vertex function $f(\Delta^2)s(\Delta^2)$, and two for the proton $G_\pm(M)F_1^\pm(M)$. The sole change in our result is that $T_\pm(s,\Delta^2)$ would be multiplied by $f(\Delta^2)s(\Delta^2)$, while $T_0(s,M)$ and $T_0'(s,M)$ would be multiplied by $G_+(M)F_1^+(M)$ and $G_-(M)F_1^-(M)$, respectively. So, in the final result they would show up as modifications of $K(\Delta^2)$ and $K_\pm(M)$. However, the Ward identity requires that the product of these form factors be unity in M_μ^π and M_μ^q. The product of the proton complete renormalized propagator of the proton's magnetic-moment form factors, $G_\pm(M)F_2^\pm(M)$, is not so constrained, and this product should appear in our expression for $E_\mu^\lambda \epsilon^\mu$ [Eqs. (14) and (16)]. Second, we have not considered possible limitations on a hard-photon theorem due to the analytic properties of the off-shell amplitudes. It was assumed that one can extrapolate smoothly off shell without the appearance of new singularities.

ACKNOWLEDGMENTS

We gratefully acknowledge many interesting and rewarding discussions with J. D. Jackson. In particular, our decision to pursue the off-shell effects resulted from such a discussion, as did the particular definition we have called a kinematically corrected hard-photon theorem. We acknowledge the opportunity to discuss aspects of this paper and related topics with numerous members of the theoretical group of the Department of Physics of the University of California, Los Angeles.

APPENDIX

Consider the electromagnetic vertex when the final pion radiates [Fig. 1(c)]. The contribution to E_μ^π was taken to be the product of the electromagnetic vertex function $\epsilon^\mu \Gamma_\mu(\Delta_f, q_f)$, the pion propagator $S(\Delta_f^2)$, and the off-shell πp amplitude $T_+(s_i, \Delta_f^2)$, i.e.,

$$\begin{aligned}\epsilon^\mu \Gamma_\mu(\Delta_f, q_f) S(\Delta_f^2) T_+(s_i, \Delta_f^2) &= \epsilon\cdot(\Delta_f + q_f) T_+(s_i, \Delta_f^2)/(\Delta_f^2 - \mu^2) \\ &= (q_f\cdot\epsilon/q_f\cdot k) T_+(s_i, \Delta_f^2), \qquad \text{(A1)}\end{aligned}$$

without form factors due to higher-order diagrams as a result of the Ward identity, which for this case is

$$(\Delta_f - q_f)_\mu \Gamma^\mu(\Delta_f, q_f) = [S^{-1}(\Delta_f^2) - S^{-1}(q_f^2)]. \qquad \text{(A2)}$$

For completeness we re-prove this statement here. The most general form of $\Gamma_\mu(\Delta_f, q_f)$ is

$$\Gamma_\mu(\Delta_f, q_f) = [(q_f + \Delta_f)_\mu f(\Delta_f^2) + (q_f - \Delta_f)_\mu g(\Delta_f^2)], \qquad \text{(A3)}$$

and the complete renormalized pion propagator is

$$S(\Delta_f^2) = s(\Delta_f^2)/(\Delta_f^2 - \mu^2). \qquad \text{(A4)}$$

Now,

$$\begin{aligned}\epsilon^\mu \Gamma_\mu(\Delta_f, q_f) &= \epsilon^\mu[(q_f + \Delta_f)_\mu f(\Delta_f^2) + (q_f - \Delta_f)_\mu g(\Delta_f^2)] \\ &= \epsilon\cdot(q_f + \Delta_f) f(\Delta_f^2). \qquad \text{(A5)}\end{aligned}$$

Because $(\Delta_f - q_f)_\mu = k_\mu$, $\epsilon\cdot k = 0 = k^2$, the second term of Eq. (A5) does not contribute to $M_\mu^\pi \epsilon^\mu$ or Eq. (A2) and is hereafter neglected. Now, multiply Eq. (A3) by $(\Delta_f - q_f)_\mu S(\Delta_f^2)$; then

$$\begin{aligned}(\Delta_f - q_f)_\mu \Gamma^\mu(\Delta_f, q_f) S(\Delta_f^2) &= \frac{(\Delta_f^2 - \mu^2) f(\Delta_f^2) s(\Delta_f^2)}{\Delta_f^2 - \mu^2} \\ &= f(\Delta_f^2) s(\Delta_f^2), \qquad \text{(A6)}\end{aligned}$$

and multiplying Eq. (A2) by $S(\Delta_f^2)$ one obtains

$$\begin{aligned}(\Delta_f - q_f)_\mu \Gamma^\mu(\Delta_f, q_f) S(\Delta_f^2) &= 1 - \frac{(q_f^2 - \mu^2) s(\Delta_f^2)}{(\Delta_f^2 - \mu^2) s(q_f^2)} \\ &= 1. \qquad \text{(A7)}\end{aligned}$$

Thus, equating Eqs. (A6) and (A7), one gets

$$f(\Delta_f^2) s(\Delta_f^2) = 1, \qquad \text{(A8)}$$

as stated.

When the final proton is off shell [Fig. 1(d)] the situation is similar, e.g., Eq. (A1) becomes $\bar{u}(p_f) \not\epsilon S(M_f) T_f(s_i, M_f)$ without form factors. The equivalent of Eq. (A3) is much more complicated. In general, when a proton is off shell and the photon is virtual ($k^2 \neq 0$, $\epsilon\cdot k \neq 0$), there are six form factors $F_1^\pm$, $F_2^\pm$, and $F_3^\pm$, where, by Lorentz invariance, $F_i^+(-M) = F_i^-(M)$. By the same argument used after Eq. (A5), only four terms survive in $\epsilon^\mu \Gamma_\mu$ or $k^\mu \Gamma_\mu$ when the photon is real. So the equivalent of Eq. (A3) is

$$\begin{aligned}\Gamma_\mu(M_f, p_f) = &[\gamma_\mu F_1^+(M_f) + \sigma_{\mu\nu} k^\nu F_2^+(M_f)/2m]\Lambda_+(M_f) \\ &+ [\gamma_\mu F_1^-(M_f) + \sigma_{\mu\nu} k^\nu F_2^-(M_f)/2m]\Lambda_-(M_f). \qquad \text{(A9)}\end{aligned}$$

The complete renormalized proton propagator becomes

$$S(M_f)=\Lambda_+(M_f)G_+(M_f)/(M_f-m)-\Lambda_-(M_f)G_-(M_f)/(M_f+m), \qquad (A10)$$

where $G_+(-M_f)=G_-(M_f)$. (See Ref. 14 for a discussion of these form factors.) Then Eq. (A6) becomes

$$\bar{u}(p_f)(M_f-p_f)_\mu\Gamma^\mu(M_f,p_f)S(M_f)$$
$$=\bar{u}(p_f)[k_\mu\Gamma^\mu(M_f,p_f)S(M_f)]$$
$$=\bar{u}(p_f)\not{k}[F_1^+(M_f)G_+(M_f)\Lambda_+(M_f)/(M_f-m)-F_1^-(M_f)G_-(M_f)\Lambda_-(M_f)/(M_f+m)]. \qquad (A11)$$

As expected, the gauge-invariant magnetic-moment terms do not contribute. Using Eq. (12) of the text and the application of Dirac's equation following it, Eq. (A11) becomes

$$\bar{u}(p_f)\{(\not{k}+m)[F_1^+(M_f)G_+(M_f)-F_1^-(M_f)G_-(M_f)]/2M_f+\tfrac{1}{2}[F_1^+(M_f)G_+(M_f)+F_1^-(M_f)G_-(M_f)]\}. \qquad (A12)$$

After replacing $q_f(\Delta_f)$ by $p_f(M_f)$ in Eq. (A2), and multiplying it from the right by $S(M_f)$, one obtains

$$\bar{u}(p_f)k_\mu\Gamma^\mu(M_f,p_f)S(M_f)=\bar{u}(p_f)[1-S^{-1}(p_f)S(M_f)]=\bar{u}(p_f), \qquad (A13)$$

because $\bar{u}(p_f)S^{-1}(p_f)=\bar{u}(p_f)(\not{p}_f-m)=0$ is Dirac's equation. Equations (A12) and (A13) are satisfied if $F_1^+(M_f)G_+(M_f)=F_1^-(M_f)G_-(M_f)=1$.

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*Work done under the auspices of the U. S. Atomic Energy Commission.

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