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Pion charge-exchange scattering in the (3, 3)-resonance region in nuclei with a neutron excess

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We solve the multiple-scattering problem describing one-dimensional charge-exchange scattering in nuclear matter with a neutron excess, and use the solution to formulate models for lepton- and proton-induced production of pions in the (3, 3)-resonance region.

I. INTRODUCTION

When pions are produced in a nucleus, either by incident protons or by incident leptons, charge-exchange scattering of the emerging pions within the nuclear target plays an important role in determining the experimentally observed production cross sections. In the particularly interesting case of pions produced in the (3, 3)-resonance region, detailed analyses of nuclear charge-exchange corrections have been given by Sternheim and Silbar[1] (incident protons) and by Adler, Nussinov, and Paschos[2] (incident leptons). Both of these papers are based on a semiclassical picture of pion interaction within the nucleus, in which the nucleus is regarded as a collection of free nucleons from which the pions are multiply scattered. In treating pion multiple scattering, Ref. 1 makes the very simplified approximation of regarding all scattering as forward scattering, which then leads to the easily solved one-dimensional scattering problem of pure forward multicomponent scattering. In Ref. 2 a more accurate approximation to the multiple-scattering problem is used, obtained by projecting all forward-hemisphere scattering onto the forward direction (0°) and all backward-hemisphere scattering onto the backward direction (180°), and then solving the resulting one-dimensional forward-backward multiple-scattering problem. The discussion of Ref. 2 is restricted to nuclei with no neutron excess, for which the dependence of pion multiple scattering on the pion charge is trivially diagonalized, leading to a one-component multiple-scattering problem. The purpose of the present paper is to remove this restriction by solving the one-dimensional forward-backward multiple-scattering problem in the multicomponent case. This will permit the application of the improved scattering approximation of Ref. 2 to heavy nuclei, where the neutron excess cannot be neglected.

Our discussion is organized as follows: In Sec. II we formulate the multiple-scattering problem describing the interaction of pions in a one-dimensional nuclear medium, and work out the geometry needed for applying this one-dimensional approximation to the production of pions in nuclei by incident protons and leptons. In Sec. III we solve the one-dimensional problem in terms of matrix operations and briefly discuss computational aspects of the solution.

II. FORMULATION OF THE ONE-DIMENSIONAL MULTIPLE-SCATTERING PROBLEM AND GEOMETRY FOR PROTON- AND LEPTON-INDUCED PRODUCTION OF PIONS

For nuclei with a neutron excess, the one-dimensional multiple-scattering problem which forms the basis for the scattering approximation of Ref. 2 may be formulated as follows: We consider a uniform one-dimensional nuclear medium extending from $x=0$ to $x=L$, and assume that a pion in charge state i ($i=+,0,-$) is initially produced, moving to the left, with δ-function density distribution $\rho=\delta(x-L_1)$. In the medium pions can scatter either forward or backward, or be absorbed. The total inverse interaction length for a pion in charge state j is κ_j; on interacting this pion scatters forward (backward) into a pion in charge state k with probability $\mu_{kj}^{(+)}$ ($\mu_{kj}^{(-)}$). These parameters will of course be functions of the pion kinetic energy T, which we assume to be fixed throughout the one-dimensional scattering.[3] Given our initial left-moving pion in charge state i, we wish to find the expected numbers

$$M_{fi}^{(+)}(L, L_1, T), \quad M_{fi}^{(-)}(L, L_1, T) \tag{1}$$

of pions in charge state f emerging from the medium respectively without and with a net over-all change in direction (i.e., respectively emerging to the left and to the right).

To identify the parameters appearing in this statement of the problem with physical parameters of the pion-nucleus interaction, we begin by noting that $M_{fi}^{(\pm)}$ can depend on κ_j, L, and L_1 only through the dimensionless combinations $\kappa_j L$, L_1/L. Hence the scale of the κ_j can be readjusted by changing the scale of L; it is convenient to fix the κ_j by taking as the density of the one-dimensional medium the nucleon density[4] $\rho(\vec{0})$ at the geometric center of the nucleus. Letting the numbers of neutrons and protons in the target nucleus be, respectively, N and Z, we define $f_{n,p}$,

$$f_n = \frac{2N}{A}, \quad f_p = \frac{2Z}{A}, \quad A = N + Z \tag{2}$$

as measures of the neutron and proton fractions. (In this notation, the results of Ref. 2 for isotopically neutral nuclei are recovered by setting $f_n = f_p = 1$.) We denote the pion absorption cross section in nuclear matter (assumed to be charge-independent) by $\sigma_{abs}(T)$, and assume that pion charge exchange proceeds entirely through the $I = \frac{3}{2}$ channel, so that the relevant pion-nucleon cross sections can all be expressed in terms of the $\pi^+ p$ cross section $\sigma_{\pi^+ p}(T)$. Following Ref. 2, we include effects of the Pauli principle on forward- and backward-hemisphere charge-exchange scattering through reduction factors $h_+(T)$ and $h_-(T)$. (When Pauli-principle effects on pion scattering are neglected we have $h_+ = h_- = 1$.) Putting all these ingredients together, we then find the following expressions[5] for the interaction parameters κ_j, $\mu_{kj}^{(\pm)}$:

$$\begin{aligned}
&\kappa_j = A\rho(\vec{0})\theta_j\,,\\
&\theta_+ = \sigma_{abs}(T) + \tfrac{1}{4}[h_+(T) + h_-(T)]\,(\tfrac{1}{3}f_n + f_p)\sigma_{\pi^+p}(T)\,,\\
&\theta_0 = \sigma_{abs}(T) + \tfrac{1}{4}[h_+(T) + h_-(T)]\,\tfrac{2}{3}(f_n + f_p)\sigma_{\pi^+p}(T),\\
&\theta_- = \sigma_{abs}(T) + \tfrac{1}{4}[h_+(T) + h_-(T)](f_n + \tfrac{1}{3}f_p)\sigma_{\pi^+p}(T),
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\mu_{kj}^{(\pm)} = \tfrac{1}{4}h_\pm(T)\eta_{kj}\theta_j^{-1}\sigma_{\pi^+p}(T)\,,\\
&\eta_{++} = f_p + \tfrac{1}{9}f_n, \quad \eta_{+0} = \tfrac{2}{9}f_p, \quad \eta_{+-} = 0\,,\\
&\eta_{0+} = \tfrac{2}{9}f_n, \quad \eta_{00} = \tfrac{4}{9}(f_p + f_n), \quad \eta_{0-} = \tfrac{2}{9}f_p\,,\\
&\eta_{-+} = 0, \quad \eta_{-0} = \tfrac{2}{9}f_n, \quad \eta_{--} = f_n + \tfrac{1}{9}f_p\,.
\end{aligned}$$

We see that when $f_n = f_p$ the inverse interaction lengths κ_j all become equal, which is why the isotopically neutral case can be diagonalized to give three independent one-component scattering problems. When $f_n \neq f_p$ this diagonalization is no longer possible, requiring us to solve the one-dimensional multiple-scattering problem which we have just formulated in full multicomponent form. Details of the solution for $M_{fi}^{(\pm)}$ are given in Sec. III below.

Assuming now that the matrices $M_{fi}^{(\pm)}$ have been calculated, we state the geometry needed for using them to determine the production of pions in nuclei by incident protons and leptons, within the semiclassical framework of Refs. 1 and 2.

A. Incident protons

Following Ref. 1, we assume that the incident proton enters the nucleus along a straight-line trajectory, with inverse interaction length [at the standard density $\rho(\vec{0})$] given by κ_p. At a general point $\vec{r}$ in the nucleus, the proton produces an outgoing pion in charge state i at polar angle θ, with the differential production cross sections on free proton and neutron targets given respectively by

$$\frac{d^2\sigma(pp \to NN\pi^i; T\theta)}{dT d\Omega}, \quad \frac{d^2\sigma(pn \to NN\pi^i; T\theta)}{dT d\Omega}. \tag{4}$$

Writing the nuclear density in the form

$$\rho(\vec{r}) = \rho(\vec{0})\bar{\rho}(\vec{r}) \tag{5}$$

and taking the incident proton direction to define the z axis [Fig. 1(a)], we see that the proton flux

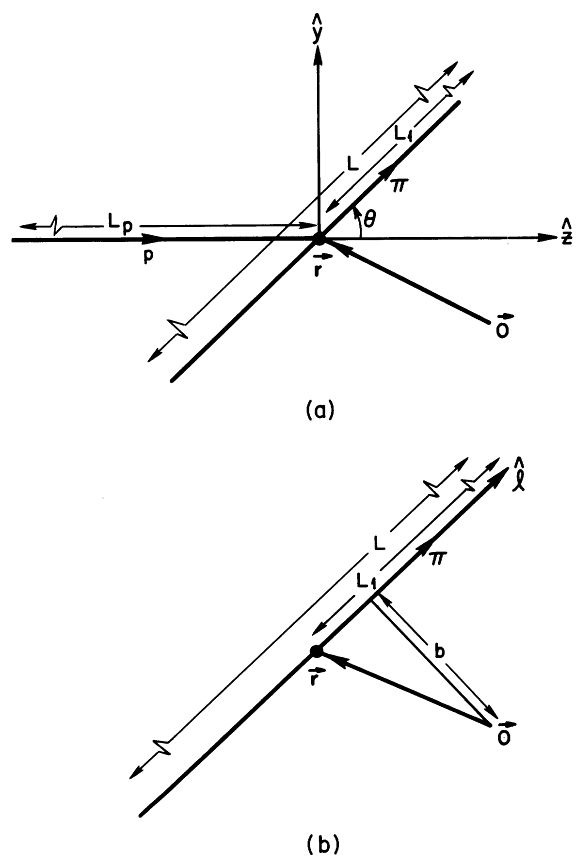

FIG. 1. (a) Geometry for pion production in a nucleus by incident protons. The dot indicates the pion production point. (b) Geometry for pion production in a nucleus by incident leptons. The dot indicates the pion production point.

at the production point is reduced, relative to the incident proton flux, by a factor

$$P(L_p)=\exp(-\kappa_p L_p), \quad L_p=\int_{-\infty}^{0} dl\bar{\rho}(\vec{r}+l\hat{z}) \ . \qquad (6)$$

Taking the πp plane to define the yz plane, and assuming that the pion scatters forward and backward along the initial production direction, we find that the parameters L and L_1 of the one-dimensional multiple-scattering model are given by

$$L_1=\int_0^{\infty} dl\bar{\rho}(\vec{r}+l\cos\theta\hat{z}+l\sin\theta\hat{y}) \ ,$$
$$L=\int_{-\infty}^{\infty} dl\bar{\rho}(\vec{r}+l\cos\theta\hat{z}+l\sin\theta\hat{y}) \ . \qquad (7)$$

Thus, linking together the pion-production and pion charge-exchange-scattering steps, and integrating over the general point $\vec{r}$, we get the following formula for the production cross section for pions in charge state f from the nuclear target:

$$\frac{d^2\sigma}{dTd\Omega}(\pi^f)=\sum_i\left\{I_{fi}^{(+)}(T,\theta)\left[Z\,\frac{d^2\sigma(pp\to NN\pi^i;T\theta)}{dTd\Omega}+N\,\frac{d^2\sigma(pn\to NN\pi^i;T\theta)}{dTd\Omega}\right]\right.$$
$$\left.+I_{fi}^{(-)}(T,\theta)\left[Z\,\frac{d^2\sigma(pp\to NN\pi^i;T\pi-\theta)}{dTd\Omega}+N\,\frac{d^2\sigma(pn\to NN\pi^i;T\pi-\theta)}{dTd\Omega}\right]\right\} , \qquad (8)$$

with

$$I_{fi}^{(\pm)}(T,\theta)=\int d^3r P(L_p)\rho(\vec{r})M_{fi}^{(\pm)}(L,L_1,T) \ . \qquad (9)$$

B. Incident leptons

The geometry in the case of incident leptons is greatly simplified by the fact that leptons are not strongly absorbed, which implies that the nucleus is uniformly illuminated by the incident beam. As a result the analog of Eq. (9) for incident leptons has the factor $P(L_p)$ replaced by 1 and is therefore independent of the pion production angles, giving

$$I_{fi}^{(\pm)}(T)=\int d^3r\rho(\vec{r})M_{fi}^{(\pm)}(L,L_1,T) \ . \qquad (10)$$

Assuming that the nucleon distribution $\bar{\rho}(\vec{r})$ is spherically symmetric,

$$\bar{\rho}(\vec{r})=\bar{\rho}[r] , \qquad (11)$$

it is convenient to change to impact-parameter variables for the $\vec{r}$ integration [Fig. 1(b)]. We write

$$L_1=\int_0^{\infty} dl\bar{\rho}(\vec{r}+l\hat{l}) \ ,$$
$$L=\int_{-\infty}^{\infty} dl\bar{\rho}(\vec{r}+l\hat{l})$$
$$=\int_{-\infty}^{\infty} dl\bar{\rho}[(b^2+l^2)^{1/2}]$$
$$\equiv L(b) \ , \qquad (12)$$

and in terms of these variables we find

$$\int d^3r\rho(\vec{r})M_{fi}^{(\pm)}(L,L_1,T)=2\pi\int_0^{\infty} b\,db\int_0^{L(b)} dL_1 M_{fi}^{(\pm)}(L(b),L_1,T)\rho(\vec{0}) \ . \qquad (13)$$

Introducing the L_1 average of $M_{fi}^{(\pm)}$,

$$\bar{M}_{fi}^{(\pm)}(L(b),T)\equiv\frac{1}{L(b)}\int_0^{L(b)} dL_1 M_{fi}^{(\pm)}(L(b),L_1,T) \ , \qquad (14)$$

and eliminating $\rho(\vec{0})$ through the normalization condition[4]

$$1=\int d^3r\rho(\vec{r})=2\pi\int_0^{\infty} b\,db\,L(b)\rho(\vec{0}) \ , \qquad (15)$$

we get finally a simple expression for the charge-exchange matrices,

$$I_{fi}^{(\pm)}(T)=\frac{\int_0^{\infty} b\,db\,L(b)\bar{M}_{fi}^{(\pm)}(L(b),T)}{\int_0^{\infty} b\,db\,L(b)} \ . \qquad (16)$$

The average over the distribution of optical thickness L appearing in Eq. (16) is the same recipe that appeared repeatedly in Ref. 2. The matrices $I^{(\pm)}$ are related to the matrices $[M_\pm]$ of Eq. (51) of Ref. 2 by the formula

$$[M_\pm]=gI^{(\pm)} \ , \qquad (17)$$

with g a factor which was introduced in Ref. 2 to account for Pauli-principle effects on the cross section for leptonic pion production which are present when the target nucleon is bound in a nucleus.

III. EXPLICIT SOLUTION AND COMPUTATIONAL ASPECTS OF THE ONE-DIMENSIONAL PROBLEM

To solve the one-dimensional multicomponent scattering problem formulated in Sec. II we follow closely the method used in Appendix A of Ref. 2 to treat the one-component case. Although the pion charge-exchange problem specifically involves three components, no formal complexity is added if we treat the general case in which D components are present. We let $P(kxd_k|jyd_j)dx$ be the probability that component j, which emerged from a scattering at y moving in direction d_j, has its next scattering in dx at x and is transformed into component k moving in direction d_k. In terms of the parameters defined in Sec. II, we find[5]

$$\begin{aligned} P(kxr|jyr) &= \mu_{kj}^{(+)}\kappa_j e^{-\kappa_j(x-y)}\theta(x-y) , \\ P(kxl|jyr) &= \mu_{kj}^{(-)}\kappa_j e^{-\kappa_j(x-y)}\theta(x-y) , \\ P(kxl|jyl) &= \mu_{kj}^{(+)}\kappa_j e^{-\kappa_j(y-x)}\theta(y-x) , \\ P(kxr|jyl) &= \mu_{kj}^{(-)}\kappa_j e^{-\kappa_j(y-x)}\theta(y-x) . \end{aligned} \tag{18}$$

Introducing a Dirac-state notation for conditional probabilities,

$$\begin{aligned} \langle kxd_k|P|jyd_j\rangle &= P(kxd_k|jyd_j) , \\ \langle kxd_k|P^n|jyd_j\rangle &= \int_0^L dz \sum_m \sum_{d_m} \langle kxd_k|P|mzd_m\rangle \times \langle mzd_m|P^{n-1}|jyd_j\rangle , \qquad (19) \\ \langle kxd_k|1|jyd_j\rangle &\equiv \delta_{kj}\delta_{d_k d_j}\delta(x-y) , \end{aligned}$$

we see that the density at x of particles of type f moving in direction d_f which have suffered *exactly* n collisions in evolving from an initial δ- function distribution, localized at y, of left-moving particles of type i, is

$$\langle fxd_f|P^n|iyl\rangle . \tag{20}$$

The integrated number of these particles which subsequently escape to the left or right without further interaction is given by

$$\begin{aligned} \text{left-emerging number} &= \int_0^L dx\, e^{-\kappa_f x}\langle fxl|P^n|iyl\rangle , \\ \text{right-emerging number} &= \int_0^L dx\, e^{-\kappa_f(L-x)}\langle fxr|P^n|iyl\rangle . \end{aligned} \tag{21}$$

Summing over all values of n gives the total number of left- and right-emerging particles of type f, and thus we get

$$\begin{aligned} M_{fi}^{(+)}(L, y, T) &= \int_0^L dx\, e^{-\kappa_f x}\langle fxl|(1-P)^{-1}|iyl\rangle , \\ M_{fi}^{(-)}(L, y, T) &= \int_0^L dx\, e^{-\kappa_f(L-x)}\langle fxr|(1-P)^{-1}|iyl\rangle , \end{aligned} \tag{22}$$

with the T dependence arising, of course, from the energy dependence implicit in the parameters $\mu_{kj}^{(\pm)}$ and κ_j.

To evaluate the inverse operator appearing in Eq. (22) we write

$$\langle kxd_k|(1-P)^{-1}|jyd_j\rangle = \delta_{kj}\delta_{d_k d_j}\delta(x-y) + F(kxd_k|jyd_j) \tag{23}$$

and take the $\langle kxd_k|\cdots|jyd_j\rangle$ matrix element of the formal relation $(1-P)^{-1}(1-P)=1$, giving the integral equation

$$F(kxd_k|jyd_j) = P(kxd_k|jyd_j) + \int_0^L dz \sum_m \sum_{d_m} F(kxd_k|mzd_m)P(mzd_m|jyd_y). \tag{24}$$

Because of the reflection symmetry of the one-dimensional medium through the point $x=\frac{1}{2}L$, the kernel $P(kxd_k|jyd_y)$ has the symmetry

$$P(kxd_k|jyd_j) = P(k\, L-x\, \bar{d}_k|j\, L-y\, \bar{d}_j) , \qquad \bar{r}=l, \quad \bar{l}=r , \tag{25}$$

and hence it follows from the integral equation that F has the same symmetry,

$$F(kxd_k|jyd_j) = F(k\, L-x\, \bar{d}_k|j\, L-y\, \bar{d}_j) . \tag{26}$$

Using this symmetry, writing out direction dependences explicitly, and substituting Eq. (18) for the kernel P, we find that the content of Eq. (24) may be written as

$$\begin{aligned} F(kxl|jyl) &= \mu_{kj}^{(+)}\kappa_j e^{-\kappa_j(y-x)}\theta(y-x) + \int_0^y dz \sum_m [F(k\, L-x\, r|m\, L-z\, l)\mu_{mj}^{(-)} + F(kxl|mzl)\mu_{mj}^{(+)}]\kappa_j e^{-\kappa_j(y-z)}, \\ F(kxr|jyl) &= \mu_{kj}^{(-)}\kappa_j e^{-\kappa_j(y-x)}\theta(y-x) + \int_0^y dz \sum_m [F(k\, L-x\, l|m\, L-z\, l)\mu_{mj}^{(-)} + F(kxr|mzl)\mu_{mj}^{(+)}]\kappa_j e^{-\kappa_j(y-z)} . \end{aligned} \tag{27}$$

Substituting Eq. (23) into Eq. (22), we get the following expressions for $\mu_{fi}^{(\pm)}$ in terms of F:

$$M_{fi}^{(+)}(L, y, T) = \delta_{fi}e^{-\kappa_f y} + \int_0^L dx\, e^{-\kappa_f x}F(fxl|iyl)\,,$$
$$M_{fi}^{(-)}(L, y, T) = \int_0^L dx\, e^{-\kappa_f (L-x)}F(fxr|iyl)\,. \tag{28}$$

Since these do not involve the detailed x dependence of F but only definite integrals over x, it is natural to eliminate the variable x from the problem by taking the same definite integrals of the integral equations in Eq. (27). To do this most conveniently we define

$$h_{kj}^{(\epsilon)}(y) = \int_0^L dx\, e^{-\kappa_k x}F(kxl|jyl) + \epsilon\int_0^L dx\, e^{-\kappa_k(L-x)}F(kxr|jyl)\,, \quad \epsilon = \pm 1 \tag{29}$$

in terms of which Eq. (28) becomes

$$M_{fi}^{(+)}(L, y, T) = \delta_{fi}e^{-\kappa_f y} + \tfrac{1}{2}[h_{fi}^{(+)}(y) + h_{fi}^{(-)}(y)]\,,$$
$$M_{fi}^{(-)}(L, y, T) = \tfrac{1}{2}[h_{fi}^{(+)}(y) - h_{fi}^{(-)}(y)]\,, \tag{30}$$

and the integral equation of Eq. (27) takes the form

$$h_{kj}^{(\epsilon)}(y) = g_{kj}^{(\epsilon)}(y) + \int_0^y dz \sum_m [\epsilon h_{km}^{(\epsilon)}(L-z)\mu_{mj}^{(-)} + h_{km}^{(\epsilon)}(z)\mu_{mj}^{(+)}]\kappa_j e^{-\kappa_j(y-z)}\,,$$
$$g_{kj}^{(\epsilon)}(y) = \int_0^L dx[e^{-\kappa_k x}\mu_{kj}^{(+)} + \epsilon\, e^{-\kappa_k(L-x)}\mu_{kj}^{(-)}]\kappa_j e^{-\kappa_j(y-x)}\theta(y-x)\,. \tag{31}$$

Finally, we convert Eq. (31) to a differential equation by multiplying by $\exp(\kappa_j y)$ and differentiating with respect to y, giving

$$\frac{d}{dy}h_{kj}^{(\epsilon)}(y) + \kappa_j h_{kj}^{(\epsilon)}(y) = [e^{-\kappa_k y}\mu_{kj}^{(+)} + \epsilon\, e^{-\kappa_k(L-y)}\mu_{kj}^{(-)}]\kappa_j + \sum_m [\epsilon h_{km}^{(\epsilon)}(L-y)\mu_{mj}^{(-)} + h_{km}^{(\epsilon)}(y)\mu_{mj}^{(+)}]\kappa_j\,, \tag{32a}$$

with the associated boundary condition

$$h_{kj}^{(\epsilon)}(0) = 0\,. \tag{32b}$$

Equations (30) and (32) form our final statement of the one-dimensional, multicomponent scattering problem.

To solve Eq. (32) we make the ansatz

$$h_{kj}^{(\epsilon)}(y) = A_{kj}^{(\epsilon)}e^{-\kappa_k y} + B_{kj}^{(\epsilon)}e^{-\kappa_k(L-y)} + p_{kj}^{(\epsilon)}(y)\,, \tag{33}$$

with $p_{kj}^{(\epsilon)}(y)$ a solution of the homogeneous equation

$$\frac{d}{dy}p_{kj}^{(\epsilon)}(y) + \kappa_j p_{kj}^{(\epsilon)}(y) = \sum_m [\epsilon p_{km}^{(\epsilon)}(L-y)\mu_{mj}^{(-)} + p_{km}^{(\epsilon)}(y)\mu_{mj}^{(+)}]\kappa_j\,. \tag{34}$$

Substituting Eq. (33) into Eq. (32a) and equating coefficients of $\exp(-\kappa_k y)$ and $\exp[-\kappa_k(L-y)]$ gives the equations

$$(\kappa_j - \kappa_k)A_{kj}^{(\epsilon)} = \mu_{kj}^{(+)}\kappa_j + \sum_m [A_{km}^{(\epsilon)}\mu_{mj}^{(+)} + \epsilon B_{km}^{(\epsilon)}\mu_{mj}^{(-)}]\kappa_j\,,$$
$$(\kappa_j + \kappa_k)\epsilon B_{kj}^{(\epsilon)} = \mu_{kj}^{(-)}\kappa_j + \sum_m [\epsilon B_{km}^{(\epsilon)}\mu_{mj}^{(+)} + A_{km}^{(\epsilon)}\mu_{mj}^{(-)}]\kappa_j\,. \tag{35}$$

Introducing two $2D$-component column vectors and a $2D\times 2D$ matrix

$$\left.\begin{aligned} u(k\epsilon)_a &= A_{ka}^{(\epsilon)} \\ u(k\epsilon)_{D+a} &= \epsilon B_{ka}^{(\epsilon)} \\ v(k)_a &= \mu_{ka}^{(+)}\kappa_a \\ v(k)_{D+a} &= \mu_{ka}^{(-)}\kappa_a \end{aligned}\right\}\,, \quad a = 1, \ldots, D$$
$$\left.\begin{aligned} T(k)_{ab} &= \delta_{ab}(\kappa_b - \kappa_k) - \mu_{ba}^{(+)}\kappa_a \\ T(k)_{a\,D+b} &= T(k)_{D+a\,b} = -\mu_{ba}^{(-)}\kappa_a \\ T(k)_{D+a\,D+b} &= \delta_{ab}(\kappa_b + \kappa_k) - \mu_{ba}^{(+)}\kappa_a \end{aligned}\right\}\,, \quad a, b = 1, \ldots, D \tag{36}$$

the two equations of Eq. (35) can be written as a single matrix equation

$$T(k)u(k\epsilon) = v(k)\,, \tag{37a}$$

with the (ϵ-independent) solution

$$u(k\epsilon) = T^{-1}(k)v(k) \equiv u(k)\,. \tag{37b}$$

Thus we have found that

$$h_{kj}^{(\epsilon)}(y) = u(k)_j e^{-\kappa_k y} + \epsilon\, u(k)_{D+j}e^{-\kappa_k(L-y)} + p_{kj}^{(\epsilon)}(y)\,. \tag{38}$$

To find the solution $p_{kj}^{(\epsilon)}(y)$ of the homogeneous equation we make the exponential ansatz

$$p_{kj}^{(\epsilon)}(y) = \sum_q [C_{kj}^{(\epsilon)q}e^{-\sigma_q y} + D_{kj}^{(\epsilon)q}e^{-\sigma_q(L-y)}]\,, \tag{39}$$

with the number of terms appearing in the sum to be determined. Substituting Eq. (39) into Eq. (34) and equating the coefficients of like exponentials we get, for each value of q, the pair of equations

$$(\kappa_j - \sigma_q)C^{(\epsilon)q}_{kj} = \sum_m [C^{(\epsilon)q}_{km}\mu^{(+)}_{mj} + D^{(\epsilon)q}_{km}\mu^{(-)}_{mj}]\kappa_j$$
$$(\kappa_j + \sigma_q)\epsilon D^{(\epsilon)q}_{kj} = \sum_m [\epsilon D^{(\epsilon)q}_{km}\mu^{(+)}_{mj} + C^{(\epsilon)q}_{km}\mu^{(-)}_{mj}]\kappa_j . \qquad (40)$$

Introducing a $2D$-component column vector and a $2D\times 2D$ matrix

$$\left.\begin{aligned} w(k\epsilon q)_a &= C^{(\epsilon)q}_{ka} \\ w(k\epsilon q)_{D+a} &= \epsilon D^{(\epsilon)q}_{ka} \end{aligned}\right\}, \quad a = 1, \ldots, D$$
$$\left.\begin{aligned} S_{ab} &= (\delta_{ab} - \mu^{(+)}_{ba})\kappa_a \\ S_{aD+b} &= -\mu^{(-)}_{ba}\kappa_a \\ S_{D+ab} &= \mu^{(-)}_{ba}\kappa_a \\ S_{D+a\,D+b} &= -[\delta_{ab} - \mu^{(+)}_{ba}]\kappa_a \end{aligned}\right\}, \quad a, b = 1, \ldots, D \qquad (41)$$

we can rewrite Eq. (40) as (1 denotes the $2D\times 2D$ unit matrix)

$$(S - \sigma_q 1)w(k\epsilon q) = 0 , \qquad (42)$$

which has a nonvanishing solution w if and only if

$$\det[S - \sigma_q 1] = 0 . \qquad (43)$$

The eigenvalue condition of Eq. (43) determines $2D$ values of σ_q, which in general will be distinct. Writing S in the compound matrix form

$$S = \begin{pmatrix} S_{ab} & S_{D+a\,b} \\ S_{a\,D+b} & S_{D+a\,D+b} \end{pmatrix} = (\delta_{ab} - \mu^{(+)}_{ba})\kappa_a\tau_3 - \mu^{(-)}_{ba}\kappa_a i\tau_2 , \qquad (44)$$

with τ_3 and τ_2 Pauli matrices, we see that

$$S\tau_1 = -\tau_1 S . \qquad (45)$$

Thus, the eigenvalues of S occur in pairs $\pm\sigma_q$, with D positive and D negative eigenvalues.[6] Since Eq. (39) already contains both positive and negative exponential terms $\exp(\pm\sigma_q y)$, only the D positive eigenvalues need be included in the sum. Denoting the eigenvector corresponding to the eigenvalue σ_q by $z(\sigma_q)$, we have

$$w(k\epsilon q) = E(k\epsilon)_q z(\sigma_q), \quad \sigma_q > 0$$
$$p^{(\epsilon)}_{kj}(y) = \sum_{\sigma_q > 0} E(k\epsilon)_q [z(\sigma_q)_j e^{-\sigma_q y} + \epsilon z(\sigma_q)_{D+j} e^{-\sigma_q(L-y)}] , \qquad (46)$$

with the constants $E(k\epsilon)_q$ to be determined by imposing the boundary condition of Eq. (32b). Setting $y = 0$, the boundary condition gives the relations

$$h(k\epsilon)_j = -\sum_{\sigma_q > 0} E(k\epsilon)_q t(\sigma_q)_j , \qquad (47)$$

with

$$h(k\epsilon)_j = u(k)_j + \epsilon u(k)_{D+j} e^{-\kappa_k L} ,$$
$$t(\sigma_q)_j = z(\sigma_q)_j + \epsilon z(\sigma_q)_{D+j} e^{-\sigma_q L} . \qquad (48)$$

To solve Eq. (47) for $E(k\epsilon)_q$ we multiply by $t(\sigma_{q'})_j$ and sum over j, giving

$$H(k\epsilon)_{q'} = -\sum_{\sigma_q > 0} E(k\epsilon)_q R_{qq'} , \qquad (49)$$

with

$$H(k\epsilon)_{q'} = \sum_j t(\sigma_{q'})_j h(k\epsilon)_j ,$$
$$R_{qq'} = \sum_j t(\sigma_{q'})_j t(\sigma_q)_j, \quad \sigma_{q'} > 0 . \qquad (50)$$

In terms of the inverse matrix R^{-1} we get, finally,

$$E(k\epsilon)_q = -\sum_{\sigma_{q'} > 0} H(k\epsilon)_{q'} R^{-1}_{q'q} , \qquad (51)$$

completing our solution of the one-dimensional, multicomponent scattering problem.[7]

In the form which we have just developed, the solution involves inversion of a $D\times D$ matrix [the matrix R of Eq. (50)], inversion of a $2D\times 2D$ matrix [the matrix T of Eq. (36)], and calculation of the eigenvalues and eigenvectors of a nonsymmetric $2D\times 2D$ matrix [the matrix S of Eq. (41)]. Obviously, a great saving in computation time will result if we can reduce the matrix operations involving $2D\times 2D$ matrices to corresponding operations on $D\times D$ matrices. We will now show that the special form of the matrices T and S makes such a reduction possible. We consider first the inversion problem for the matrix T, which we write in the compound form

$$T(k) = \begin{pmatrix} T_1 - \kappa_k 1 & T_2 \\ T_2 & T_1 + \kappa_k 1 \end{pmatrix} ,$$
$$T_{1ab} = (\delta_{ab} - \mu^{(+)}_{ba})\kappa_a, \quad T_{2ab} = -\mu^{(-)}_{ba}\kappa_a, \quad a, b = 1, \ldots, D \qquad (52)$$

with 1 now the $D\times D$ unit matrix. A straightforward calculation then shows that the inverse matrix $T^{-1}(k)$ is given by

$$T^{-1}(k) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} ,$$
$$t_{11} = [T_1 - \kappa_k 1 - T_2(T_1 + \kappa_k 1)^{-1}T_2]^{-1} ,$$
$$t_{21} = -(T_1 + \kappa_k 1)^{-1}T_2 t_{11} , \qquad (53)$$
$$t_{22} = [T_1 + \kappa_k 1 - T_2(T_1 - \kappa_k 1)^{-1}T_2]^{-1} ,$$
$$t_{12} = -(T_1 - \kappa_k 1)^{-1}T_2 t_{22} ,$$

which involves only the inverses of $D\times D$ matrices. Next we consider the eigenvalue problem for S, which we write in compound form [cf. Eq. (44)] as

$$S = S_1\tau_3 + S_2 i\tau_2 = \begin{pmatrix} S_1 & S_2 \\ -S_2 & -S_1 \end{pmatrix}$$

$$S_1 = T_1, \quad S_2 = T_2 \ . \tag{54}$$

From Eq. (45), we know that if

$$z(\sigma_q) = \begin{pmatrix} z_1(\sigma_q) \\ z_2(\sigma_q) \end{pmatrix} \tag{55}$$

is an eigenvector of S with eigenvalue σ_q, then

$$\tau_1 z(\sigma_q) = \begin{pmatrix} z_2(\sigma_q) \\ z_1(\sigma_q) \end{pmatrix} \tag{56}$$

is an eigenvector of S with eigenvalue $-\sigma_q$. Thus

$$\begin{aligned} z_+(\sigma_q) &= z(\sigma_q) + \tau_1 z(\sigma_q) \\ &= [z_1(\sigma_q) + z_2(\sigma_q)] \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \\ z_-(\sigma_q) &= z(\sigma_q) - \tau_1 z(\sigma_q) \\ &= [z_1(\sigma_q) - z_2(\sigma_q)] \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \tag{57}$$

satisfy the relations

$$\begin{aligned} S z_+(\sigma_q) &= \sigma_q z_-(\sigma_q) , \\ S z_-(\sigma_q) &= \sigma_q z_+(\sigma_q) , \end{aligned} \tag{58}$$

and hence both $z_\pm(\sigma_q)$ are eigenvectors of S^2 with eigenvalue $\sigma_q^{\ 2}$. Given either $z_+(\sigma_q)$ or $z_-(\sigma_q)$, we can recover $z(\sigma_q)$ by using the relations

$$\begin{aligned} z(\sigma_q) &= \tfrac{1}{2}[z_+(\sigma_q) + z_-(\sigma_q)] \\ &= \tfrac{1}{2}(1 + \sigma_q^{\ -1} S) z_+(\sigma_q) \\ &= \tfrac{1}{2}(1 + \sigma_q^{\ -1} S) z_-(\sigma_q) \ . \end{aligned} \tag{59}$$

Multiplying out S^2 in terms of Eq. (54) we find

$$S^2 = \begin{pmatrix} S_1^{\ 2} - S_2^{\ 2} & S_1 S_2 - S_2 S_1 \\ S_1 S_2 - S_2 S_1 & S_1 - S_2^{\ 2} \end{pmatrix} , \tag{60}$$

and substituting Eq. (60) into the relations $S^2 z_\pm(\sigma_q) = \sigma_q^{\ 2} z_\pm(\sigma_q)$ yields the equations

$$\begin{aligned} S_\pm[z_1(\sigma_q) \pm z_2(\sigma_q)] &= \sigma_q^{\ 2}[z_1(\sigma_q) \pm z_2(\sigma_q)] , \\ S_\pm &= S_1^{\ 2} - S_2^{\ 2} \pm (S_1 S_2 - S_2 S_1) \ . \end{aligned} \tag{61}$$

Thus, the D eigenvalues $\sigma_q^{\ 2}$ can be obtained by determining the eigenvalues of *either* of the two $D \times D$ matrices S_+ or S_-. From *either* of the corresponding eigenvectors $z_1(\sigma_q) + z_2(\sigma_q)$ and $z_1(\sigma_q) - z_2(\sigma_q)$, one can determine $z_+(\sigma_q)$ or $z_-(\sigma_q)$ by the direct-product recipe of Eq. (57) and finally get $z(\sigma_q)$ by application of Eq. (59). Hence we have completely reduced the eigenvalue problem for S to a smaller problem involving only $D \times D$ matrices. Substituting Eq. (54) into Eq. (61), we get the following explicit expression for the matrices $S_\pm$:

$$\begin{aligned} (S_\pm)_{ab} = {} & \delta_{ab}\kappa_a\kappa_b - [\mu_{ba}^{(+)} \pm \mu_{ba}^{(-)}]\kappa_a^{\ 2} \\ & - [\mu_{ba}^{(+)} \mp \mu_{ba}^{(-)}]\kappa_a\kappa_b \\ & + \sum_c [\mu_{bc}^{(+)} \pm \mu_{bc}^{(-)}]\kappa_c[\mu_{ca}^{(+)} \mp \mu_{ca}^{(-)}]\kappa_a \ . \end{aligned} \tag{62}$$

In the particular case of pion charge exchange, we have $D=3$ and the eigenvalue problem associated with Eq. (62) leads to a cubic equation which can be solved explicitly; once the eigenvalues are known the corresponding eigenvectors are easily found by solving a pair of coupled linear equations. Thus in this case the computational aspects of the solution are entirely straightforward.[8] For larger values of D, the eigenvalue problem can be solved numerically by use of a computer program[9] which determines the eigenvalues and eigenvectors of a general real matrix.

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[1] M. M. Sternheim and R. R. Silbar, Phys. Rev. D 6, 3117 (1972).

[2] S. L. Adler, S. Nussinov, and E. A. Paschos, preceding paper, Phys. Rev. D 9, 2125 (1974).

[3] In Ref. 2 we expressed the energy dependence of the parameters by writing them as functions of the effective isobar mass W, related to T by $T = [W^2 - (M_N + M_\pi)^2]/(2M_N)$, with M_N and M_π respectively the nucleon and pion masses.

[4] We take the density to be unit normalized, i.e., $\int d^3 r \rho(\vec{r}) = 1$.

[5] Repeated indices *are not* summed unless an explicit summation sign appears. We adhere to this convention throughout.

[6] Because the one-dimensional medium is absorptive, i.e., $\sum_k \mu_{kj}^{(+)} + \mu_{kj}^{(-)} \le 1$, we expect the eigenvalues σ_q to all be real.

[7] For a further discussion of one-dimensional multiple scattering, see G. M. Wing, *An Introduction to Transport Theory* (Wiley, New York, 1962).

[8] We have written and tested a FORTRAN IV program to calculate $M_{fi}^{(\pm)}(L, L_1, T)$ for pion charge exchange. Running time on an IBM 360-91 (compiling with the G-level compiler) is 0.012 sec.

[9] J. Grad and M. A. Brebner, Comm. ACM (Assoc. Comput. Mach.) 11, 820 (1968); H. D. Knoble, *ibid.* 13, 122 (1970). I am grateful to Dr. Knoble for supplying me with a copy of his program.