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PHYSICAL REVIEW D VOLUME 9, NUMBER 7 1 APRIL 1974

$\eta\to 3\pi$ in a renormalizable SU(3) σ model*

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(Received 2 February 1973)

The problem of predicting the $\eta\to 3\pi$ decay rate in terms of the K^+-K^0 mass difference is discussed in the framework of a renormalizable SU(3) σ model. Order-of-magnitude agreement for the process is finally achieved. In addition, some connections with unified weak-electromagnetic gauge schemes are explored.

I. INTRODUCTION

Although the current-algebra or phenomenological Lagrangian technique adequately predicts the energy spectrum of $\eta\to\pi^+\pi^-\pi^0$ to linear order, it has generally given a rate which is an order of magnitude too small. This result follows from relating the π^0-η mixing angle to the "tadpole" part of the K^+-K^0 mass shift. In a rather general form[1] of the SU(3) σ model it was previously shown[2] that the energy spectrum comes out right but that the over-all magnitude involves some constants which cannot be fixed merely by requiring chiral symmetry for the Lagrangian. Thus, although the correct rate could in principle be obtained, more information is needed.

In the present note it is shown that requiring the general SU(3) σ model to be renormalizable fixes these previously unknown constants and, in fact, leads to the correct order of magnitude for the decay rate. What is essentially happening is that in addition to the π^0-η mixing term there is another term resulting from the electromagnetic mixing of the analogous *scalar* mesons which enter the process virtually. This latter term actually *dominates* the former, and together they build up the rate.

We shall continue to use the same notation and formalism as in Refs. (1) and (2), but shall briefly define some relevant quantities again for the sake of coherence. The Lagrangian of the *general* σ model is

$$\mathcal{L}=\tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu\phi\partial_\mu\phi)-\tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu S\partial_\mu S)-V_0-V_{\mathrm{SB}}\,. \tag{1.1}$$

In (1.1) ϕ and S are the 3×3 matrices of pseudoscalar and scalar fields. The members of the scalar nonet are denoted as

$$\{(\kappa^+,\kappa^0),\ (\epsilon^+,\epsilon^0,\epsilon^-),\ (\overline{\kappa^+},\overline{\kappa^0}),\sigma,\sigma'\}\ .$$

V_0 is the most general function of the following chiral $[\mathrm{SU}(3)\times\mathrm{SU}(3)]$-invariants:

$$\begin{aligned}I_1&=\mathrm{Tr}[(S+i\phi)(S-i\phi)]\,,\\ I_2&=\mathrm{Tr}\{[(S+i\phi)(S-i\phi)]^2\}\,,\\ I_3&=\mathrm{Tr}\{[(S+i\phi)(S-i\phi)]^3\}\,,\\ I_4&=6[\det(S+i\phi)+\det(S-i\phi)]\,.\end{aligned} \tag{1.2}$$

It is convenient to use the abbreviations

$$V_i=\left\langle\frac{\partial V_0}{\partial I_i}\right\rangle_0,\qquad V_{ij}=\left\langle\frac{\partial^2 V_0}{\partial I_i\,\partial I_j}\right\rangle_0 \tag{1.3}$$

The symmetry-breaking term V_{SB} will be taken to have the following $[(3,3^*)+(3^*,3)]$ form:

$$V_{\mathrm{SB}}=-2(A_1S_1^1+A_2S_2^2+A_3S_3^3)\ , \tag{1.4}$$

where the A_i are three constants analogous to the three quark masses. A possible origin for these terms will be discussed later. It is crucial to remark that there is not only ordinary symmetry breaking in $\mathcal{L}$, but also "spontaneous" breaking in the ground state. This is described by the three quantities

$$\alpha_i=\langle S_i^i\rangle_0\ . \tag{1.5}$$

Thus, in addition to the V_i, V_{ij}, etc., the system is described by the six parameters $\{A_i,\alpha_i\}$. In the isospin-invariant limit, which we shall take as a first approximation,

$$\begin{aligned}&\alpha_1=\alpha_2\equiv\alpha,\quad A_1=A_2\\&\alpha_3\equiv w\,\alpha\end{aligned}\qquad\text{(isospin limit)}\qquad(1.6)$$

A remarkable feature of the dynamics, discussed in Ref. 1, is that a large number of interesting results can be found *without* specifying the explicit form of V_0 (i.e., the V_i, V_{ij}, ...). This follows because one can derive two generating equations [(2.18) and (2.21) of Ref. 1] which on differentiation yield a chain of Ward-type identities[3] between n- and $(n-1)$-point functions of the theory. In addition, one uses the extremum equations

$$\left\langle\frac{\partial V_0}{\partial S}\right\rangle_0+\left\langle\frac{\partial V_{\mathrm{SB}}}{\partial S}\right\rangle_0=0\ .\qquad(1.7)$$

These take the explicit form

$$\begin{aligned}&\alpha_1[V_1+2V_2(\alpha_1)^2+3V_3(\alpha_1)^4]+6\alpha_2\alpha_3V_4=A_1\ ,\\&\alpha_2[V_1+2V_2(\alpha_2)^2+3V_3(\alpha_2)^4]+6\alpha_1\alpha_3V_4=A_2\ ,\\&\alpha_3[V_1+2V_2(\alpha_3)^2+3V_3(\alpha_3)^4]+6\alpha_1\alpha_2V_4=A_3\ .\end{aligned}\qquad(1.8)$$

From (1.8) we see that if $\alpha_1=\alpha_2=\alpha_3$ [SU(3)-invariant vacuum], we must have $A_1=A_2=A_3$ [SU(3)-invariant Lagrangian], and similarly if $\alpha_1=\alpha_2$ (isospin-invariant vacuum), we must have $A_1=A_2$ (isospin-invariant Lagrangian). Thus the symmetry properties of the vacuum and the Lagrangian are linked in this model. Besides the extremum conditions (1.7) there are stability conditions on the second derivatives of V_0 which lead to certain allowed ranges for the parameters A_i and α_i (see footnote 15 of Ref. 1).

As explained in Ref. 1, it is possible to determine the fundamental parameters of the system in terms of a limited set of input quantities. We initially work in the isospin-invariant limit and choose as *input*

$$\begin{aligned}&\pi^2=1,\quad K^2=13.6\pi^2,\quad \eta^2=16.5\pi^2,\\&\eta'^2=50.3\pi^2,\quad F_\pi=1.01\pi\ ,\end{aligned}\qquad(1.9)$$

where the particle symbol denotes its mass (π^0 mass equals unity) and F_π is the pion decay constant. This choice then enables us to find the basic parameters:

$$\begin{aligned}&\tfrac12(A_1+A_2)=0.25\pi^3,\quad A_3=9.05\pi^3,\quad \alpha=0.5\pi\ ,\\&w=1.71,\quad V_4=-1.85\pi\ .\end{aligned}\qquad(1.10)$$

One then has the predictions

$$\begin{aligned}&\theta_P=0.48^\circ,\quad F_K=1.37\pi\ ,\\&\kappa^2=50.5\pi^2=(959.2\ \mathrm{MeV})^2\ ,\end{aligned}\qquad(1.11)$$

where θ_P is the η-η' mixing angle and F_K is the kaon decay constant. In addition there are many predictions on the (3-, 4-, ...)-point vertices of the theory given in Ref. 1. An interesting feature is that the scalar meson masses (except for the κ) are *free* parameters in the general σ model. Placing restrictions other than chiral SU(3)×SU(3) invariance on V_0 will then relate some of these scalar masses. The additional restriction of scale invariance was investigated in Ref. 1. Here we shall impose the requirement of renormalizability.

II. THE RENORMALIZABLE SU(3) σ MODEL

The renormalizability criterion is that V_0 contain no terms of order higher than 4 in ϕ and S. This gives, in the notation used in Sec. I,

$$\begin{aligned}V_0=&[V_1-V_{11}\,\alpha^2(2+w^2)]I_1+\tfrac12V_{11}(I_1)^2\\&+V_2I_2+V_4I_4\ .\end{aligned}\qquad(2.1)$$

[In (2.1) the quantities α and w are considered to be fixed at the values given in (1.10).] Thus V_0 is described by the four constants V_1, V_{11}, V_2, and V_4. While the motivation for requiring a phenomenological Lagrangian to be renormalizable is certainly not as strong as the corresponding motivation for a "fundamental" Lagrangian, it is, nevertheless, an attractive idea. It is interesting to note that Chan and Haymaker[4] have calculated the one-loop corrections to the tree approximation for the one- and two-point functions in such a model and found them to be only of the order of 5%.

We must now specify the four constants appearing in V_0. These will be determined in the isospin-invariant limit. V_4 occupies a somewhat special position in that it has already been determined from the general σ model [Eq. (1.10)]. V_1 and V_2 can be found from the extremum conditions (1.8). Noting that $V_3=0$ here and that we are working in the isospin-invariant limit gives

$$\begin{aligned}&V_2=\frac{A_3/w-1}{2\alpha^3(w^2-1)}+\frac{3V_4}{\alpha w}=3.75\ ,\\&V_1=\frac{A_1}{\alpha}-2V_2\alpha^2-6V_4\alpha w=8.17\pi^2\ .\end{aligned}\qquad(2.2)$$

Knowing only V_1, V_2, and V_4 is sufficient to give the squared mass of the scalar-isovector particle:

$$\begin{aligned}\epsilon^2=\left\langle\frac{\partial^2V_0}{\partial S_1^2\,\partial S_2^1}\right\rangle_0&=2V_1+12V_2\alpha^2-12V_4\alpha w\\&=46.98\pi^2\\&=(925\ \mathrm{MeV})^2\ .\end{aligned}\qquad(2.3)$$

This result might tempt one to identify the ϵ with the $\delta(960)$.

The remaining constant V_{11} may be determined if the mass of the σ meson is specified. This, however, involves one in the intricacies of the σ-σ' mixing problem. A painless way to proceed is as follows. First define the objects

$$\mathfrak{M}_{ab} = \left\langle \frac{\partial^2 V_0}{\partial S_a^a \partial S_b^b} \right\rangle_0 . \tag{2.4}$$

One then notes that the $\mathfrak{M}_{ab}$ can either be expressed in terms of the squared masses and the scalar mixing angle, θ_S, or in terms of the parameters in V_0. Explicitly, in the isospin-invariant limit,

$$\mathfrak{M}_{11} = \tfrac{1}{2}\epsilon^2 + b'^2\sigma^2 + a'^2\sigma'^2$$
$$= 2V_1 + 12V_2\alpha^2 + 4\alpha^2 V_{11} , \tag{2.5a}$$
$$\mathfrak{M}_{12} = -\tfrac{1}{2}\epsilon^2 + b'^2\sigma^2 + a'^2\sigma'^2$$
$$= 12V_4\alpha w + 4\alpha^2 V_{11} , \tag{2.5b}$$
$$\mathfrak{M}_{13} = \sqrt{2}\, a'b'(\sigma'^2 - \sigma^2)$$
$$= 12V_4\alpha + 4\alpha w V_{11} , \tag{2.5c}$$
$$\mathfrak{M}_{33} = 2a'^2\sigma^2 + 2b'^2\sigma'^2$$
$$= 2V_1 + 12V_2\alpha^2 w^2 + 4\alpha^2 w^2 V_{11} , \tag{2.5d}$$

where a' and b' are certain useful combinations of θ_S:

$$a' = \frac{1}{\sqrt{6}}(\sin\theta_S + \sqrt{2}\cos\theta_S) ,$$
$$b' = \frac{1}{\sqrt{6}}(\cos\theta_S - \sqrt{2}\sin\theta_S) , \tag{2.6}$$
$$a'^2 + b'^2 = \tfrac{1}{2} .$$

We will also need the analogous combinations of the pseudoscalar mixing angle θ_P:

$$a = \frac{1}{\sqrt{6}}(\sin\theta_P + \sqrt{2}\cos\theta_P) ,$$
$$b = \frac{1}{\sqrt{6}}(\cos\theta_P - \sqrt{2}\sin\theta_P) . \tag{2.7}$$

From the set of Eqs. (2.5a)–(2.5d) we may solve for V_{11} in terms of σ^2 and known quantities:

$$V_{11} = \frac{72(V_4)^2 - (4\alpha)^{-2}(2V_1 + 12V_2\alpha^2 + 12V_4\alpha w - \sigma^2)(2V_1 + 12V_2\alpha^2 w^2 - \sigma^2)}{(4 + 2w^2)V_1 + 36\alpha^2 w^2 V_2 + 12\alpha w(w^2 - 4)V_4 - (2 + w^2)\sigma^2} \tag{2.8}$$

Thus, choosing σ^2 gives us V_{11} from (2.8) and completes the specification of V_0. From (2.4) we may then find the σ' squared mass,

$$\sigma'^2 = \mathfrak{M}_{11} + \mathfrak{M}_{12} + \mathfrak{M}_{33} - \sigma^2 , \tag{2.9}$$

and the σ-σ' mixing angle,

$$\tan 2\theta_S = \frac{-2\sqrt{2}(\mathfrak{M}_{11} + \mathfrak{M}_{12} - \mathfrak{M}_{13} - \mathfrak{M}_{33})}{\mathfrak{M}_{33} - \mathfrak{M}_{11} - \mathfrak{M}_{12} - 8\mathfrak{M}_{13}} . \tag{2.10}$$

In solving (2.10) for θ_S a spurious solution will be found in addition to the correct one. The spurious solution may be eliminated by checking to see if it satisfies (2.5c). The actual choice of σ^2 used to be considered controversial, but now π-π phase-shift analyses[5] give an extremely broad object centered around 850 MeV. To be on the safe side we shall investigate this system for a range of σ masses in the correct region. A listing of V_{11}, σ'^2, and θ_S for various values of σ^2 is given in Table I.

TABLE I. Values for the σ-dependent parameters of the renormalizable σ model for various choices of σ (σ in MeV; V_{11}, χ_1, χ_3 unitless; θ_S in degrees; σ'^2, $f_{\sigma\pi\eta}, f_{\sigma'\pi\eta}$ in $\pi=1$ units).

σ (MeV)	V_{11}	σ'^2	θ_S (deg)	χ_1	χ_3	$f_{\sigma\pi\eta}$	$f_{\sigma'\pi\eta}$
600	5.50	66.42	−57.54	−0.0194	0.0211	−0.0434	0.292
650	7.16	71.32	−52.57	−0.0226	0.0212	−0.0327	0.241
700	9.49	79.31	−46.36	−0.0267	0.0206	−0.0123	0.189
750	13.29	94.43	−38.63	−0.0318	0.0189	0.0271	0.135
800	21.89	133.4	−29.26	−0.0388	0.0158	0.108	0.0812
825	33.69	190.4	−23.98	−0.0434	0.0136	0.182	0.0553
845	64.72	344.4	−19.52	−0.0484	0.0116	0.276	0.0357
850	86.21	451.9	−18.38	−0.0499	0.0110	0.307	0.0310
855	131.6	679.4	−17.23	−0.0516	0.0105	0.343	0.0263
860	291.4	1482.0	−16.07	−0.0536	0.0099	0.384	0.0218
863	1186.0	5978.0	−15.37	−0.0548	0.0096	0.412	0.0191

We note that the positivity of the σ, σ' masses does not allow all choices of σ^2 in Eq. (2.9). For m_σ lying between 0 and 863 MeV, $m_{\sigma'}$ lies between 993 MeV and ∞; choosing m_σ between 863 and 993 MeV gives an imaginary $m_{\sigma'}$, therefore ruling out this region. Choosing m_σ between 993 MeV and ∞, $m_{\sigma'}$ lies between 0 and 863 MeV.

III. ISOSPIN VIOLATIONS

Setting $A_1 \neq A_2$ will introduce a violation of isospin invariance into our Lagrangian. By the extremum equations (1.8) this will induce an isospin violation into the vacuum, i.e., $\alpha_1 \neq \alpha_2$. In fact, subtracting the first of Eqs. (1.8) from the second and combining with (2.3) gives

$$\epsilon^2 = \frac{2(A_1 - A_2)}{\alpha_1 - \alpha_2} . \tag{3.1}$$

Since ϵ^2 has already been fixed in terms of the system parameters this means that the two new isospin-violating quantities $(A_1 - A_2)$ and $(\alpha_1 - \alpha_2)$ are linked. It is thus sufficient to find $(\alpha_1 - \alpha_2)$ to specify the system in the presence of a *small* isospin violation. In Ref. 2 it was shown that this quantity is related to the K^+-K^0 mass difference. From (3.5) and (3.6) of Ref. 2, we find

$$(K^+)^2-(K^0)^2=d_K+\frac{\alpha_1-\alpha_2}{\alpha(1+w)}(\epsilon^2-K^2)\,, \tag{3.2}$$

where d_K is the contribution from the electromagnetic self-energy diagrams. Conventionally[6] this is taken to be

$$d_K\simeq 0.15\,\pi^2\,. \tag{3.3}$$

Adopting (3.3) and using (3.1) and (3.2) then gives[7]

$$\alpha_1-\alpha_2=-0.0149\pi,\quad A_1-A_2=-0.352\pi^3\,. \tag{3.4}$$

Now that the system is completely specified we can go on to make predictions about $\eta\to 3\pi$ without introducing any arbitrary parameters. Because of the isospin violation there will be σ-ϵ^0 and σ'-ϵ^0 mixing in addition to the σ-σ' mixing which results from SU(3) violation. The *physical* ϵ^0, σ, and σ' are related to the mathematical quantities S_1^1, S_2^2, and S_3^3 by

$$\begin{bmatrix}\epsilon^0\\ \sigma\\ \sigma'\end{bmatrix}=\begin{bmatrix}(1/\sqrt{2})+\chi_1 b'+\chi_3 a' & -(1/\sqrt{2})+\chi_1 b'+\chi_3 a' & \sqrt{2}(-\chi_1 a'+\chi_3 b')\\ -(1/\sqrt{2})\chi_1+b' & (1/\sqrt{2})\chi_1+b' & -\sqrt{2}\,a'\\ -(1/\sqrt{2})\chi_3+a' & (1/\sqrt{2})\chi_3+a' & \sqrt{2}\,b'\end{bmatrix}\begin{bmatrix}S_1^1\\ S_2^2\\ S_3^3\end{bmatrix}\,. \tag{3.5}$$

In (3.5) χ_1 is the ϵ^0-σ mixing angle and χ_3 is the ϵ^0-σ' mixing angle.[8] A similar formula [Eq. (3.1) of Ref. 2] holds for mixing in the π^0-η-η' complex. There, ψ_1 is the π^0-η mixing angle and ψ_3 is the π^0-η' mixing angle. The pseudoscalar-meson mixing angles have already been determined in the general SU(3) σ model. Taking over (3.13) of Ref. 2 gives

$$\begin{aligned}\psi_1&=\frac{b(\alpha_1-\alpha_2)}{\sqrt{2}\,\alpha(\eta^2-\pi^2)}\\&\quad\times[\pi^2-\epsilon^2+2a(\sqrt{2}\,bw-a)(\eta'^2-\eta^2)]\\&=0.0166\\&=-1.109(\alpha_1-\alpha_2)\,,\\ \psi_3&=\frac{a(\alpha_1-\alpha_2)}{\sqrt{2}\,\alpha(\eta'^2-\pi^2)}\\&\quad\times[\pi^2-\epsilon^2+2b(\sqrt{2}\,aw+b)(\eta'^2-\eta^2)]\\&=-8\times10^{-4}\\&=0.0537(\alpha_1-\alpha_2)\,.\end{aligned} \tag{3.6}$$

A computation analogous to the one leading to (3.6) gives for the ϵ^0-σ and ϵ^0-σ' mixing angles

$$\begin{aligned}\chi_1&=\frac{4(\alpha_1-\alpha_2)}{\epsilon^2-\sigma^2}\\&\quad\times[\sqrt{2}\,\alpha b'(3V_2+V_{11})+a'(3V_4-\alpha wV_{11})]\,,\\ \chi_3&=\frac{4(\alpha_1-\alpha_2)}{\epsilon^2-\sigma'^2}\\&\quad\times[\sqrt{2}\,\alpha a'(3V_2+V_{11})-b'(3V_4-\alpha wV_{11})]\,.\end{aligned} \tag{3.7}$$

Owing to the presence of V_{11} in (3.7), χ_1 and χ_3 depend on our choice of σ^2. Thus, their numerical values are listed in Table I. Besides the isospin-violating two-point functions (mixing angles) we will also need certain isospin-violating three-point vertices for computing $\eta\to 3\pi$ decay. A straightforward differentiation of V_0 [Eq. (2.1)] gives us the needed objects:

$$\begin{aligned}f_{\sigma\pi\eta}&=\left\langle\frac{\partial^3 V_0}{\partial\sigma\partial\pi^0\partial\eta}\right\rangle_0\\&=\sum_{i,j,k}\frac{\partial\phi_i^i}{\partial\eta}\frac{\partial\phi_j^j}{\partial\pi^0}\frac{\partial S_k^k}{\partial\sigma}\left\langle\frac{\partial^3V_0}{\partial S_k^k\partial\phi_j^j\partial\phi_i^i}\right\rangle_0\\&=\frac{8}{\sqrt{2}}V_2bb'(\alpha_1-\alpha_2)+\frac{8\chi_1}{\sqrt{2}}(3aV_4-\sqrt{2}\,\alpha bV_2)+8b'\{2\psi_1(-\alpha a^2V_2+3\sqrt{2}\,abV_4)+\psi_3[2\alpha abV_2-3\sqrt{2}(b^2-a^2)V_4]\}\\&\quad-4\sqrt{2}a'\{\psi_1[4\alpha wa^2V_2-3(1+2b^2)V_4]-2ab\psi_3[2\alpha wV_2+3V_4]\}\,,\end{aligned} \tag{3.8a}$$

$$f_{\sigma'\pi\eta}=\frac{8}{\sqrt{2}}V_2ba'(\alpha_1-\alpha_2)+\frac{8}{\sqrt{2}}\chi_3(3aV_4-\sqrt{2}\,\alpha bV_2)+8a'\{2\psi_1(-\alpha a^2V_2+3\sqrt{2}\,abV_4)+\psi_3[2\alpha abV_2-3\sqrt{2}(b^2-a^2)V_4]\}$$
$$+4\sqrt{2}\,b'\{\psi_1[4\alpha wa^2V_2-3(1+2b^2)V_4]-2ab\psi_3[2\alpha wV_2+3V_4]\}\;. \tag{3.8b}$$

The numerical values of $f_{\sigma\pi\eta}$ and $f_{\sigma'\pi\eta}$ are also listed in Table I for various choices of σ mass.

IV. $\eta\to 3\pi$ DECAY

The details of the calculation of this process for the general σ model are explained in Sec. IV of Ref. 2, the final formula being Eq. (4.6). In its derivation, nontrivial use was made of the Ward-type identities referred to after (1.6). This formula involves the quantities χ_1, χ_3, $f_{\sigma\pi\eta}$, $f_{\sigma'\pi\eta}$, σ'^2, and θ_S, which were not fixed in the general σ model, but which are given in Secs. II and III. For simplicity we write this formula in the (fairly reasonable) approximation where the relevant pseudoscalar squared masses are considered small compared with the scalar squared masses. The approximation is, of course, less good for the first term [σ-pole term], but it turns out that this term is numerically small compared to the others in the interesting range of σ masses. Then

$$T(\eta\to\pi^+\pi^-\pi^0)\simeq\frac{-\eta^2}{\alpha}\left(1-\frac{2\omega_0}{\eta}\right)\left\{\frac{b'f_{\sigma\pi\eta}}{\sigma^2}+\frac{a'f_{\sigma'\pi\eta}}{\sigma'^2}+\frac{b}{\alpha}[(b'\chi_1+a'\chi_3)-(b\psi_1+a\psi_3)]\right\}$$
$$\equiv X\left(1-\frac{2\omega_0}{\eta}\right) \tag{4.1}$$

where ω_0 is the energy of the π^0 in the η rest frame. The spectrum shape[8] represented by the $(1-2\omega_0/\eta)$ factor has also been obtained by the current-algebra technique and is very close to the experimental one. For our present purposes we note that the "experimental" amplitude to linear order looks like[8]

$$T_{\text{exp}}\simeq\pm(0.994\pm0.112)\left(1-\frac{2\omega_0}{\eta}\right) \tag{4.2}$$

The early work using the current-algebra approach essentially neglected all terms in (4.1) except for the ψ_1 term (π^0-η mixing term). Using (3.4) would give in this current-algebra limit

$$T_{\text{CA}}\simeq 0.163\left(1-\frac{2\omega_0}{\eta}\right). \tag{4.3}$$

Thus the predicted width would be about $\frac{1}{25}$ of the experimental one.[9] Now, if all the terms of (4.1) are taken into account we get a much more satisfactory result. For the experimentally plausible value $m_\sigma=850$ MeV, the quantity X in Eq. (4.1) is about 0.64. Since the experimental value may be as low as $X\simeq 0.89$, the agreement is reasonable, considering the simplicity of our model. Reference to Table I shows that the dominant contribution to (4.1) comes from the χ_1 (ϵ^0-σ mixing angle) term. This term is about three times larger than the ψ_1 (π^0-η mixing angle) term, which represents the contribution which originally had been thought to be the most important one. Actually, a similar situation to the present one had already been conjectured in Ref. 2. We furthermore note that X as computed from Eq. (4.1) turns out to be essentially constant in the range $600<m_\sigma<860$ MeV. However, for much lower values of m_σ (which do not seem so reasonable experimentally[5] any more) it is better to use Eq. (4.6) of Ref. 2, rather than its linearized form given by Eq. (4.1). Because of the σ pole, there would, of course, be an enhancement for very low m_σ.

How can the agreement with experiment be improved further? One possibility is that the conventional electromagnetic self-energy contribution to the K^+-K^0 mass shift (d_K) should be modified from the value given in (3.3). In fact, a value

$$d_K=(0.35\pm0.07)\,\pi^2 \tag{4.4}$$

is sufficient to give experimental agreement for $m_\sigma=850$ MeV. Equation (4.4) is easily found by first noting that each term in (4.1) is proportional to $(\alpha_1-\alpha_2)$ and then substituting the required value of $(\alpha_1-\alpha_2)$ into (3.2). Another possibility for improving the agreement is to take more particles into account in our model; including vector and axial-vector mesons[10] would seem to be a logical though complicated first step in this direction.

V. POSSIBLE ORIGIN FOR ISOSPIN VIOLATION

In this section we will make some remarks about the connection of the $\eta\to 3\pi$ problem with unified weak-electromagnetic gauge schemes.

We have correlated the "tadpole" part of the K^+-K^0 mass shift with the $\eta\to 3\pi$ decay rate by using the [SU(3)×SU(3)]-symmetry-breaking interaction (1.4). The chiral SU(3) is broken because not all of the A_i are zero, the ordinary SU(3) is broken because the three A_i are not equal, and, finally, the isospin symmetry is broken because $A_1 \neq A_2$. This type of situation is entirely analogous to the type of symmetry breaking in the quark model. There, one expects the symmetry-breaking term to be

$$\mathcal{L}_{\rm SB} = -m_1\bar{q}_1 q_1 - m_2\bar{q}_2 q_2 - m_3\bar{q}_3 q_3 \ , \tag{5.1}$$

where q_1, q_2, q_3 are the three quarks and m_1, m_2, m_3 are their "masses." Equation (5.1) has the same SU(3)×SU(3) transformation properties as (1.4), so we may identify A_1, A_2, A_3 as being analogous to m_1, m_2, m_3. In fact (see footnote 15 of Ref. 1), if we take A_i proportional to m_i and α_i proportional to $\langle\bar{q}_i q_i\rangle_0$, then the allowed domains for A_3/A_1 and α_3/α_1 are the same as the allowed domains for m_3/m_1 and $\langle\bar{q}_3 q_3\rangle_0/\langle\bar{q}_1 q_1\rangle_0$.

What is the origin of this symmetry breaking? One possibility[11,12] is to consider that the strong interaction is exactly [SU(3)×SU(3)]-invariant and that the symmetry-breaking "masses" arise from a unified weak-electromagnetic gauge theory in much the same way that the electron and muon masses would arise in such a theory. Then the total Lagrangian of the strong, electromagnetic, and weak interactions would be invariant with respect to the weak-electromagnetic symmetry group. However, this symmetry would be spontaneously broken, giving rise to (5.1) (or a similar expression) in the process. The precise implementation of such a mechanism depends on a particular choice of a theory of weak-electromagnetic interactions including the hadrons. Probably all the present theories are, at best, steps in the right direction. Thus we shall content ourselves with illustrating this mechanism in two simple models. Similar approaches can of course be carried out for other models. In a simple theory[13] which contains only three quarks the transformation properties of the quarks with respect to the weak-electromagnetic SU(2) are

doublet:

$$\begin{pmatrix} Q_{1L} \\ Q_{2L}\end{pmatrix} = \tfrac{1}{2}(1+\gamma_5)\begin{pmatrix} q_1 \\ q_2\cos\theta + q_3\sin\theta\end{pmatrix};$$

singlets:

$$\begin{aligned} Q_{1R} &= \tfrac{1}{2}(1-\gamma_5)\,q_1 \ , \\ Q_{2R} &= \tfrac{1}{2}(1-\gamma_5)(q_2\cos\theta + q_3\sin\theta), \\ Q_{3L,R} &= \tfrac{1}{2}(1\pm\gamma_5)(-q_2\sin\theta + q_3\cos\theta) \ . \end{aligned} \tag{5.2}$$

In (5.2) θ is the Cabbibo angle. Then an [SU(2)×U(1)]-invariant form[11] involving a scalar doublet $\binom{\phi^+}{\phi^0}$ is

$$\begin{aligned}(\bar{Q}_{1L}\,\bar{Q}_{2L})\tbinom{\phi^+}{\phi^0}(f_1Q_{2R}+f_2Q_{3R}) + f_3\bar{Q}_{3L}Q_{2R} \\ + f_4\bar{Q}_{3L}Q_{3R} + if_5(\bar{Q}_{1L}\bar{Q}_{2L})\,\tau_2\tbinom{\phi^-}{\bar{\phi}^0}Q_{1R} + \text{H.c.} \ ,\end{aligned} \tag{5.3}$$

where the f_i are some real constants. Equation (5.3) can be identified with (5.1) having *arbitrary* m_1, m_2, and m_3 when we replace $\binom{\phi^+}{\phi^0}$ by its vacuum expectation value $\binom{0}{\lambda}$ and impose[11]

$$\lambda f_2 = f_3 = \tfrac{1}{2}(f_4 - \lambda f_1)\tan 2\theta \ .$$

In such a three-quark model the unwanted semileptonic decays can be eliminated by introducing[13] another U(1) gauge field, thereby extending the weak-electromagnetic group from SU(2)×U(1) to SU(2)×U(1)×U(1). The unwanted nonleptonic processes can be disregarded if we are willing[13] to postulate an exact dynamical suppression of nonoctet components in the effective Hamiltonian.

Another way to eliminate unwanted weak processes is to introduce[14] a fourth quark, q_4. (q_4 has the same charge as q_1 but is distinguished from the other three by having a different "charm" quantum number.) The transformation properties with respect to the weak electromagnetic SU(2) are then

doublets:

$$\begin{pmatrix} Q_{1L} \\ Q_{2L}\end{pmatrix} = \tfrac{1}{2}(1+\gamma_5)\begin{pmatrix} q_1 \\ q_2\cos\theta + q_3\sin\theta\end{pmatrix},$$

$$\begin{pmatrix} Q'_{1L} \\ Q'_{2L}\end{pmatrix} = \tfrac{1}{2}(1+\gamma_5)\begin{pmatrix} q_4 \\ -q_2\sin\theta + q_3\cos\theta\end{pmatrix};$$

singlets:

$$\begin{aligned} Q_{1R} &= \tfrac{1}{2}(1-\gamma_5)\,q_1 \ , \\ Q_{2R} &= \tfrac{1}{2}(1-\gamma_5)(q_2\cos\theta + q_3\sin\theta) \ , \\ Q'_{1R} &= \tfrac{1}{2}(1-\gamma_5)\,q_4 \ , \\ Q'_{2R} &= \tfrac{1}{2}(1-\gamma_5)(-q_2\sin\theta + q_3\cos\theta) \ . \end{aligned} \tag{5.4}$$

An SU(2)-invariant form is similarly

$$\begin{aligned}&(\bar{Q}_{1L}\bar{Q}_{2L})\tbinom{\phi^+}{\phi^0}(h_1Q_{2R}+h_2Q'_{2R}) \\ &\quad + (\bar{Q}'_{1L}\bar{Q}'_{2L})\tbinom{\phi^+}{\phi^0}(h_3Q_{2R}+h_4Q'_{2R}) \\ &\qquad + (\bar{Q}_{1L}\bar{Q}_{2L})(i\tau_2)\tbinom{\phi^-}{\bar{\phi}^0}(h_5Q_{1R}+h_6Q'_{1R}) \\ &\qquad\quad + (\bar{Q}'_{1L}\bar{Q}'_{2L})(i\tau_2)\tbinom{\phi^-}{\bar{\phi}^0}(h_7Q_{1R}+h_8Q'_{1R}) \ ,\end{aligned} \tag{5.5}$$

where the h_i are some real constants. Equation (5.5) will become identical to

$$-m_1\bar{q}_1q_1 - m_2\bar{q}_2q_2 - m_3\bar{q}_3q_3 - m_4\bar{q}_4q_4 \ , \tag{5.6}$$

with *arbitrary* m_i if we replace $\binom{\phi^+}{\phi^0}$ by $\binom{0}{\lambda}$ and in addition impose

$$
\begin{aligned}
h_1 &= \frac{-1}{2\lambda}[(m_2+m_3)+(m_2-m_3)\cos 2\theta], \\
h_2 &= h_3 = \frac{1}{2\lambda}(m_2-m_3)\sin 2\theta, \\
h_4 &= -\frac{1}{2\lambda}[(m_2+m_3)-(m_2-m_3)\cos 2\theta], \\
h_5 &= -\frac{m_1}{\lambda}, \\
h_6 &= h_7 = 0, \\
h_8 &= \frac{-m_4}{\lambda}.
\end{aligned}
\tag{5.7}
$$

Although both the above models apparently do not give us any information on the values of the m_i, they do suggest a heuristic ansatz. Noting that the electron doublet is $\frac{1}{2}(1+\gamma_5)\binom{\nu_e}{e}$ we are tempted to imagine a similarity between the neutrino and q_1 and to therefore postulate[11] $m_1 = 0$. In the case of the σ model this would be $A_1 = 0$. Using (1.10) and (3.4) shows that [with the choice (3.3)] we have, from fitting the spin-zero mass spectrum to experiment,

$$
\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0.08 \\ 0.43 \\ 9.05 \end{pmatrix} \pi^3 . \tag{5.8}
$$

Thus, $A_1 = 0$ does not seem too unreasonable. What would happen if we were to impose this to determine $\alpha_1 - \alpha_2$ instead of relying on (3.2) and (3.3)? Then we would have,[11] from the π^2 and ϵ^2 mass formulas of the general σ model and using the value for ϵ given in (2.3),

$$
(\alpha_1 - \alpha_2) = -2\alpha\left(\frac{\pi}{\epsilon}\right)^2 = -0.0215\pi . \tag{5.9}
$$

This is 1.44 times larger than the value given in (3.4), precisely large enough to bring (4.1) into numerical agreement with experiment [remember that (4.1) is proportional to $(\alpha_1 - \alpha_2)$]. Perhaps this is more than a coincidence.

Note that in the preceding discussion in this section, we have not specified the nature of the strong interaction apart from requiring it to be [SU(3)$\times$SU(3)]-invariant. We emphasize that the "strong" symmetry-breaking terms in (5.1) are actually considered to arise from a weak-electromagnetic gauge theory of some sort. For the strong interaction itself, it is important that it *not* possess the higher symmetry U(3)$\times$U(3), i.e., the so-called axial-vector baryon current must not be conserved. This is a reflection of the necessity of the invariant I_4, which breaks U(3)$\times$U(3) down to SU(3)$\times$SU(3) in the interaction (2.1). The attempt to construct chiral models of the type given in Secs. I and II without such a term has been found[15] to lead to an unsatisfactory pseudoscalar mass spectrum, namely a π (η-type isoscalar) degeneracy, which would make an attempt to calculate $\eta \rightarrow 3\pi$ highly ambiguous.

Another question[16] arises in connection with $\eta \rightarrow 3\pi$ in the presence of a unified weak-electromagnetic gauge scheme. When we calculated $\eta \rightarrow 3\pi$ above, we neglected the contribution from the emission and absorption of a virtual photon. Some justification for doing this is provided by Sutherland's theorem[17] which states that in the current-algebra approximation this contribution vanishes. However, in a unified weak-electromagnetic scheme one should treat the contribution from virtual intermediate vector-meson exchange on the same footing as the virtual-photon contribution. In addition, there may be contributions of various kinds from the auxiliary scalar mesons of the theory. Nevertheless, we will neglect these terms because it is expected[18] that they are of weak rather than electromagnetic order. Actually we may prove a generalization of Sutherland's theorem for the weak current-current terms, but we shall not give details.

After this paper was submitted, we received an interesting report by Weinberg[18] which presents a detailed analysis of second-order corrections to unified weak-electromagnetic gauge schemes. In his language,[19] the models described in this section already break isospin symmetry in zeroth order. This means that the electromagnetic quantity $(m_1 - m_2)$ is not calculable but must be considered as a parameter. It also may be possible to arrange the theory in such a way that isospin is a good zeroth-order symmetry which gets broken in a calculable way in second order. In such an event, he shows that the second-order correction consists of a photon-exchange part plus a term[20] of the form (5.1) with $m_1 \neq m_2$. However, $(m_1 - m_2)$ can then in principle be computed from semileptonic scattering data. Setting $(A_1 - A_2) = (m_1 - m_2)A_3/m_3$ would enable us to use the calculation of the first part of this paper for $\eta \rightarrow 3\pi$, since it was conducted without prejudice as to the origin of $(A_1 - A_2)$.

ACKNOWLEDGMENTS

We would like to thank Professor Y. Ueda for helpful discussions on the subject matter of this paper.

*Work supported by the U. S. Atomic Energy Commission.

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[8]A more detailed discussion is given in Ref. 2. We are using a more recent value for the width here.

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