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Clustering in multiple production*

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The notion of clustering in many-particle production is defined and applied to experimental situations. We argue that the existence of clusters is strongly suggested by a number of correlations which have recently been observed in high-energy experiments, and call attention to strategies for studying the properties of produced clusters.

I. INTRODUCTION

Understanding of the mechanisms of many-particle production, while still at a primitive stage, is developing rapidly under the stimulus of experimental results from the National Accelerator Laboratory and from the CERN Intersecting Storage Rings.[1] The study of inclusive reactions leaves little doubt that a short-range correlation mechanism is responsible for the bulk of particle production at high energies. Although this is a conclusion of considerable importance, it has long been recognized that it is no substitute for a microscopic description in terms of individual channels, i.e., for knowledge and theoretical understanding of the full scattering matrix. Because the multiperipheral model is the prototype for a short-range correlation mechanism, the task ahead is often posed (in terms which may be too narrow) as one of discovering the "correct" multiperipheral model. As presently pursued, even this goal is more modest than it may first appear. One is not yet contemplating the creation of a theory to calculate all scattering amplitudes, but merely the specification of a systematic description of exclusive cross sections. The cluster models presently fashionable in this context have antecedents in cosmic-ray physics and a long history in accelerator physics.[2] In such schemes, groups of hadrons, called clusters, are emitted (either multiperipherally or independently) and subsequently decay into the observed final-state hadrons without any final-state interactions. As microscopic models go, these cluster models—dealing always with cross sections, never with amplitudes—are unquestionably crude, but for the purpose of discovering how inclusive properties arise they are at least instructive. Indeed, impressive fits to a variety of experimental distributions have already been made with elaborate Monte Carlo programs.[3] Such success is always difficult to interpret, and one is bound to wonder whether it is owed to the correctness of the model or to the virtuosity of the programmer. This uncertainty frequently leads experimenters to ask why, if clusters exist, we do not see them directly. In this paper, we wish to explain how clusters manifest themselves in experimental distributions and to show how to study the properties of clusters experimentally. By means of simplified one-dimensional cluster models we illustrate how features present in the data may plausibly arise. Thus our discussion is generally supportive of the idea that the success of multiperipheral cluster models does depend upon the correctness of the model assumptions.

In Sec. II we explain what is meant by clustering in contemporary usage, and why clusters are not observed directly. The circumstantial evidence for the existence of clusters is compiled. Section III is devoted to simple and very schematic one-dimensional cluster models. We comment briefly on the application of such models to charge-transfer observables, and at greater length on their consequences for forward-backward multiplicity correlations. In Sec. IV we address some questions of consistency and recapitulate ways for investigating the properties of clusters.

II. WHY CLUSTERS?

A. The meaning of clustering

Like so many of the *anschaulich* words in common use in high-energy physics, "cluster" or "clustering" means different things to different people, and because each usage is evocative in its own context, it is pointless to attempt to legislate a single meaning. What seems more appropriate is to state carefully the various definitions and to make clear the circumstances under which each is useful. We are aware of three principal interpretations which inevitably overlap considerably.

Berger, Fox, and Krzywicki[4] suggested a dispersion parameter which is in practice a test for the presence of a single cluster, by which they mean a group of particles of which the extension in rapidity is small compared with the available range of rapidity. For each event, given a (sub)set of N particles with rapidities y_i, one may compute the mean rapidity

$$\langle y\rangle \equiv N^{-1}\sum_{i=1}^{N} y_i \tag{2.1}$$

and the dispersion in rapidity

$$\delta \equiv \left[(N-1)^{-1}\sum_{i=1}^{N}(y_i-\langle y\rangle)^2\right]^{1/2}, \tag{2.2}$$

which is a measure of clustering. In the hands of the Pisa–Stony Brook Collaboration,[5] this technique has been shown to be useful for defining two classes of events, extreme examples of which are shown in Figs. 1(a) and 1(b). A tentative identification of the two event types as diffractive and nondiffractive is suggestive. The dispersion method is, however, not well suited for the identification of several clusters within an event, in part because neutrals go undetected in most experiments and in part because the products of different clusters may overlap in rapidity.

Another operational definition of clustering has been proposed by Ludlam and Slansky,[6] who regard clustering as the existence of two or more components in the final-state amplitude which occupy distinct population centers of phase space. Because it refers to the full $(3n-4)$-dimensional phase space for n-particle events, this definition is in many ways the most precise. Ludlam and Slansky define a measure of the average event-to-event fluctuations, the experimental value of which can be compared with a reference value generated by Monte Carlo techniques under the assumption that there is no clustering. Although recourse to a Monte Carlo standard makes the method somewhat opaque, it does appear to be a powerful one for separating reaction mechanisms for fitted events. For inclusive data, on the other hand, the nondetection of neutral secondaries is likely crippling.

The third sort of clustering is that emphasized in the now popular models which conceive particle production to proceed by the emission of clumps of particles. Neither these parents nor their progeny, the observed secondaries, undergo any final-state interactions. If clusters of this description are produced, they may or may not (depending upon their properties) be directly observable in the distribution $d\sigma/dy_1\cdots dy_n$ of secondaries. The two alternatives are depicted in Figs. 1(c) and 1(d). In the first instance, the mobility of progeny from their parent clusters is small compared with the spacing of clusters in rapidity, so that the existence of clusters can be inferred

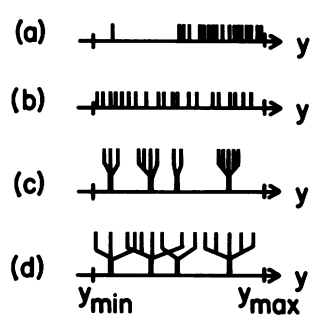

FIG. 1. Various event types which bear on the definitions of clusters. Each stroke represents the rapidity of a charged secondary. (a) A configuration for which the dispersion test indicates the presence of a cluster. (b) A configuration for which the dispersion test indicates no clustering. (c) An event in which several clusters (heavy lines) are produced widely separated in rapidity (compared with the mobility of their products). For this configuration, clustering is apparent to the unaided eye. (d) An event in which the mobility of the decay products is comparable to the spacing of the produced clusters. For such a configuration, $d\sigma/dy_1\cdots dy_n$ supplies no convincing evidence for clustering.

directly from the spectrum of secondaries in the event. (We remark that the mobility is related, *inter alia*, to the mass of the parent cluster and the number of secondaries.) Indeed, it is reasonable to expect that the dispersion method could be applied successfully to the quantitative analysis of such events. If instead the progeny are extremely mobile, direct recognition of clusters is impossible. On present evidence, only the second alternative is worth considering. If it can be demonstrated that clusters of this very mobile variety exist, the question raised in the Introduction can be answered. To begin to suggest how the demonstration might be made, let us review the facts which make it convenient to talk about clusters in the first place.

B. The circumstantial evidence for clusters

Any physical requirement which forces two or more particles to be produced in association with each other can be interpreted in terms of clusters. Quantum-number correlations are among the simplest, and best known, examples. The idea of local (in rapidity) conservation of charge led Wang[7] to suggest that charged pions be produced in $\pi^+\pi^-$ pairs. Independent emission of the clusters gives rise to a multiplicity distribution which is Poisson in the number of pairs (i.e., clusters) and therefore to a broader than Poisson distribution in the number of charged pions. The resulting broadened distribution is in better agreement with the data than a Poisson distribution would be. In much the same way, the local satisfaction of isospin invariance implied by the correlation between the number of neutral and charged pions observed in bubble-chamber events [see Fig. 2 (Ref. 8)] can be interpreted as evidence for the production of clusters made up of both charged and neutral pions.[9]

Related to these effects is the dominantly short-range nature of the correlation function

$$C(y_1, y_2) \equiv \frac{1}{\sigma}\frac{d\sigma}{dy_1 dy_2} - \left(\frac{1}{\sigma}\frac{d\sigma}{dy_1}\right)\left(\frac{1}{\sigma}\frac{d\sigma}{dy_2}\right), \qquad (2.3)$$

and the energy dependence thus implied for the correlation moment

$$f_2 \equiv \int dy_1 dy_2 C(y_1, y_2). \qquad (2.4)$$

That f_2 is positive and growing with energy, as shown in Fig. 3 (see Refs. 10–15), is merely the precise statement of the general remark we have made above that observed multiplicity distributions are broader than Poisson. The more differential correlation function is more sensitive to the detailed properties of clusters. Many authors[3,16–19] have by now carried out the exercise

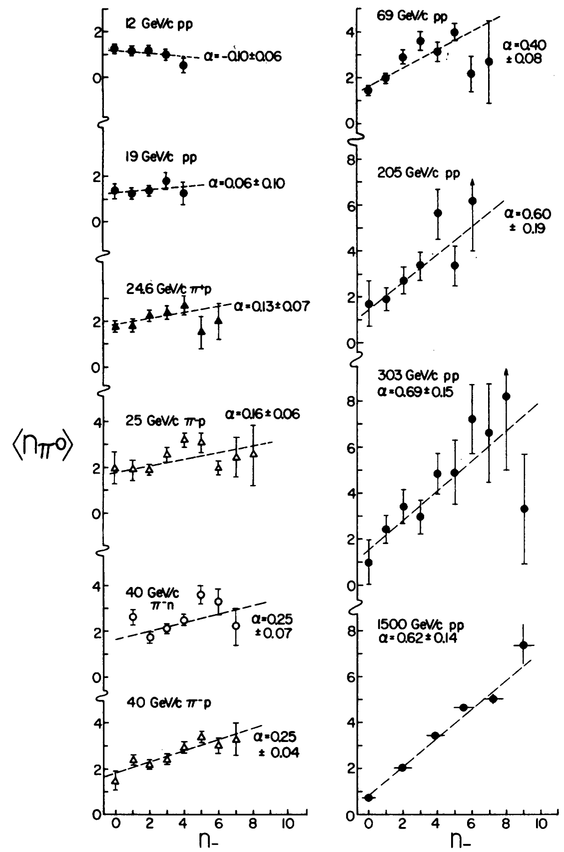

FIG. 2. The mean number of neutral pions as a function of the number of negative tracks in pp and πN collisions. The coefficients α appear in fits of the form $\langle n_{\pi^0}\rangle = \alpha n_- + \beta$. (Data from Ref. 8.)

of adjusting characteristics of independently emitted clusters to fit the observed multiplicity distributions and correlation functions. Good fits can be achieved if clusters have an average mass of 1–2 GeV/c^2 and decay, in the mean, into about

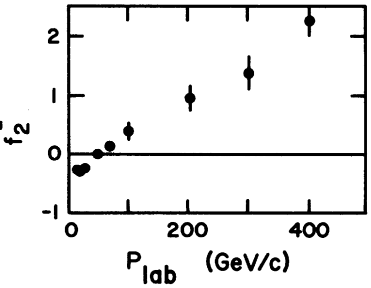

FIG. 3. The correlation moment f_2^- for negative particles produced in pp collisions as a function of the incident beam momentum. The data at 13, 18, 21, 24, and 28.5 GeV/c are from Ref. 10; at 50 and 69 GeV/c they are from Ref. 11, at 102 GeV/c from Ref. 12, at 205 GeV/c from Ref. 13, at 303 GeV/c from Ref. 14, and at 405 GeV/c from Ref. 15.

four pions. Berger and Fox,[3] who have made the most extensive analysis, conclude that the mean spacing of clusters in rapidity is less than the typical mobility of the decay products emerging from a single cluster, which is approximately two units in rapidity. Thus, their model is a realization of the situation suggested by Fig. 1(d).

Finally, and still more indirectly, Hamer and Peierls[20] have noted that the overlap-function connection between the mechanism for particle production and the slope of the diffraction peak in elastic scattering appears to require clusters in a multiperipheral-model framework. Their clusters are prodigious objects with mean masses $\gtrsim 2$ GeV/c^2 and mean multiplicity of decay products on the order of ten to fifteen. We do not know how sensitively these conclusions depend upon the phases of the underlying multiperipheral amplitudes, but other considerations[21] would suggest considerable phase dependence.

III. A ONE-DIMENSIONAL CLUSTER MODEL

To investigate the implications of the multiperipheral picture for charge-transfer fluctuations,[22] Quigg and Thomas[23] wrote down an extremely stylized cluster model in which the transverse momentum degrees of freedom are ignored. They assumed that clusters were emitted multiperipherally, without any correlations. In this case the N-cluster differential cross section in rapidity is

$$\frac{1}{\sigma_N(Y)}\frac{d\sigma}{dy_1\cdots dy_N}=Y^{-N}, \tag{3.1}$$

where $\sigma_N(Y)$ is the N-cluster production cross section, and $Y\equiv\ln(s/M^2)$ in pp collisions. It remains to specify the decay properties of the clusters. The model becomes especially transparent if it is assumed that each cluster decays into the same fixed number of pions at fixed distances in rapidity from the cluster itself. The example discussed most completely in Ref. 23 was that of $(\pi^+\pi^-\pi^0)$ clusters for which a cluster at rapidity y resulted in pions at $y+\Delta$, y, and $y-\Delta$. We shall refer to this as the ω model. The parameter Δ, which is related in an obvious way to the cluster mass and the number of pions into which the cluster decays, is designated the mobility.

A. Charge-transfer observables

Although its simplicity makes a schematic cluster model quite instructive in general, the model holds a special competitive advantage for the analysis of observables which depend upon quantum numbers. This is simply because none of the more ambitiously realistic cluster models has yet treated quantum numbers more than casually. Let us define the net charge transfer between forward and backward c.m. hemispheres as

$$\begin{aligned} u\equiv\tfrac{1}{2}[&(\text{total charge of forward-moving secondaries})\\ &-(\text{charge of beam particle})\\ &-(\text{total charge of backward-moving secondaries})\\ &+(\text{charge of target particle})]. \end{aligned}\tag{3.2}$$

Then it was shown in Ref. 23 that for the ω model the mean squared charge-transfer fluctuation in pp collisions is

$$\langle\langle u^2\rangle\rangle=4\Delta\langle N\rangle/3Y, \tag{3.3}$$

where $\langle N\rangle$ is the mean multiplicity of produced clusters. (For the ω model, $\langle N\rangle=\langle n_{\pi^-}\rangle$.) The mean squared fluctuation has now been measured over a wide range of energies. The data, collected in Fig. 4 (see Refs. 24–26), are in excellent, if schematic, agreement with the trend predicted by (3.3), and may be taken as support for the independent-cluster-emission philosophy over the fragmentation picture.[22]

The comparison can be pushed further.[27] Notice that $\langle N\rangle/Y$ is simply the local density of clusters on the central plateau. By making the replacement

$$\frac{\langle N\rangle}{Y}\to\frac{1}{\sigma}\frac{d\sigma}{dy}, \tag{3.4}$$

where $d\sigma/dy$ is the inclusive cross section for cluster production, one obtains a plausible generalization of (3.3) outside the central region:

$$D^2(\bar{y})=\frac{4\Delta}{3\sigma}\frac{d\sigma}{dy}(\bar{y}). \tag{3.5}$$

Here $D^2(\bar{y})$ is the mean squared fluctuation of charge across a boundary at rapidity $\bar{y}$. Although the coefficient $\frac{4}{3}\Delta$ in Eq. (3.5) is peculiar to the

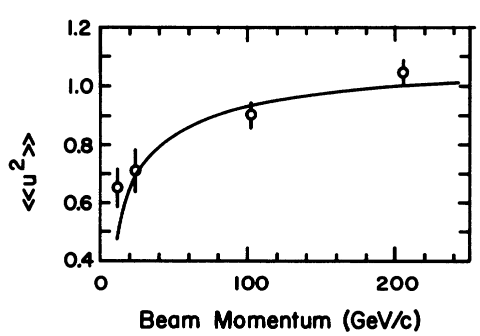

FIG. 4. The mean squared charge fluctuation in pp collisions at 12 and 24 GeV/c (from Ref. 24), at 103 GeV/c (from Ref. 25), and at 205 GeV/c (from Ref. 26). The solid curve is the prediction of the simple model of Ref. 23, normalized to the high-energy data.

ω model, the form

$$D^2(\bar{y}) \propto \frac{1}{\sigma}\frac{d\sigma}{dy}(\bar{y}) \tag{3.6}$$

is not; it is a characteristic result of cluster models. Because the density of clusters is not directly measurable, it is necessary to make a further identification in order to confront (3.5) and (3.6) with experimental results. If $(1/\sigma)d\sigma/dy$ changes slowly over a rapidity interval Δ, we may reliably approximate the cluster density by the observed density of charged pions:

$$\frac{1}{\sigma}\frac{d\sigma}{dy} \propto \frac{1}{\sigma}\frac{d\sigma_{\mathrm{ch}}}{dy}. \tag{3.7}$$

Therefore the cluster-model prediction for the charge fluctuation is

$$D^2(\bar{y}) = \frac{\kappa}{\sigma}\frac{d\sigma_{\mathrm{ch}}}{dy}(\bar{y}), \tag{3.8}$$

where the constant of proportionality κ depends upon the characteristics of the clusters. The most attractive possibility is that κ does not depend upon the incident energy or upon the identity of the incident particles, i.e., that the clusters have the same properties in all reactions at all energies.[28] This is in the spirit of the multiperipheral model, but it is by no means a unique possibility.

Three sets of data on $D^2(\bar{y})$ are displayed in Fig. 5 (see Refs. 27, 29, and 30). In each case the solid line represents the cluster-model prediction (3.8):

$$D^2(\bar{y}) = 0.81\frac{dN_{\mathrm{ch}}}{dy}(\bar{y}), \quad \text{for 16-GeV}/c\ K^-p \tag{3.9a}$$

$$D^2(\bar{y}) = 0.85\frac{dN_{\mathrm{ch}}}{dy}(\bar{y}), \quad \text{for 24-GeV}/c\ pp \tag{3.9b}$$

$$D^2(\bar{y}) = 0.72\frac{dN_{\mathrm{ch}}}{dy}(\bar{y}), \quad \text{for 205-GeV}/c\ pp \tag{3.9c}$$

where dN_{ch}/dy refers to the density of charged particles after elimination of the leading particles. The near equality of the coefficients κ determined in the three experiments suggests that cluster properties do not depend strongly on energy or reaction. As Białas[29] has already remarked, the mobility parameter determined from the charge-fluctuation data by means of (3.5) seems somewhat too large, in comparison with the value required for isotropically decaying clusters to reproduce the observed transverse momentum cutoff. This is not surprising in view of the reports that larger than three-particle clusters are required to reproduce the correlation function.

B. Forward-backward multiplicity correlations

The study of correlations among the number of forward-going and backward-going charged particles in a variety of experimental circumstances provides detailed information about the production mechanism. One-dimensional cluster models again are a source of insight into the physics contained in any particular measurement. In this section we present the expectations of cluster models for a number of measurements which seem practical now or in the immediate future.

1. *Forward-backward fluctuations*

Nussinov, Quigg, and Wang[31] stressed some time ago that in the multiperipheral or independent-emission picture the most likely configuration for

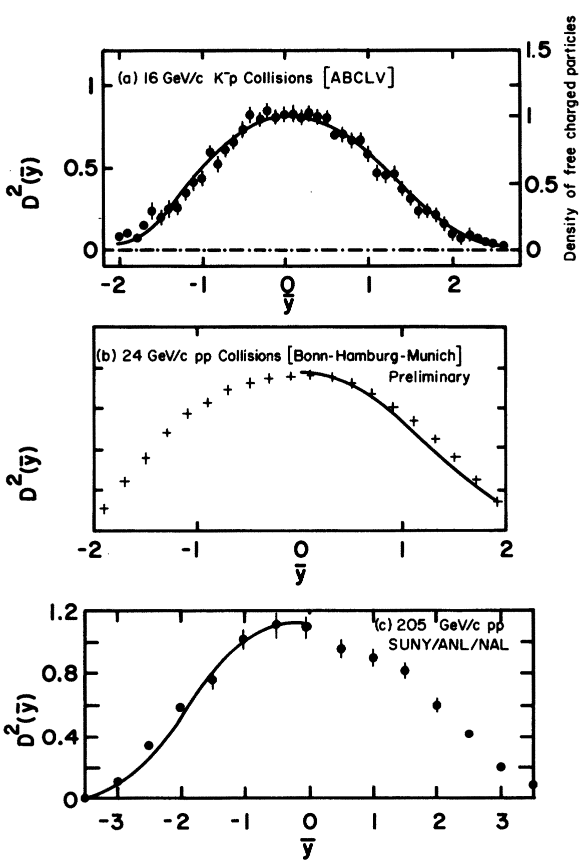

FIG. 5. Comparison of the mean squared fluctuation of charge $D^2(\bar{y})$ with the rapidity distribution of "produced" charged particles: (a) in K^-p collisions at 16 GeV/c, curve is 0.81 $d\sigma_{\mathrm{ch}}/dy$ [from Ref. 27]; (b) in pp collisions at 24 GeV/c, curve is 0.85 $d\sigma_{\mathrm{ch}}/dy$ [from Ref. 29]; (c) in pp collisions at 205 GeV/c, curve is 0.72 $d\sigma_{\mathrm{ch}}/dy$ [from Ref. 30].

an N-particle event is the symmetric one with $\frac{1}{2}N$ particles moving forward and $\frac{1}{2}N$ particles moving backward in the c.m. system. Experiments[25,26] have shown a preference for this configuration over the asymmetric topology favored by the fragmentation picture, so attention has turned[32] to the question of what can be learned about the multiperipheral production mechanism from right-left (i.e., forward-backward) multiplicity distributions. Let us designate the number of forward-going charged secondaries as n_R, and the number of backward-going charged secondaries as n_L. The quantity $\langle (n_R - n_L)^2 \rangle$, averaged over events with fixed $n_{\text{ch}} = n_R + n_L$, is a measure of the fluctuations about the most probable configuration.

It is obvious that $\langle (n_R - n_L)^2 \rangle_{n_{\text{ch}}}$ is sensitive to the existence of clusters. Consider, for example, the extreme case in which particles are produced in pairs so massive that every such cluster decays into one forward-going and one backward-going secondary. Evidently $\langle (n_R - n_L)^2 \rangle_{n_{\text{ch}}} = 0$ in this circumstance. Less extreme sorts of clustering will similarly be reflected in the value of $\langle (n_R - n_L)^2 \rangle_{n_{\text{ch}}}$.

The effect of clustering upon the fluctuation $\langle (n_R - n_L)^2 \rangle_{n_{\text{ch}}}$ will be made more precise if we compute the left-right fluctuation in some simple and explicit models. The generating-function formalism employed in Ref. 23 can be adapted to this problem quite easily. For events in which N clusters are produced, the probability that $(n_R - n_L) = k$ is given by the coefficient $g_{N:k}$ of x^k in the generating function

$$P_N(x) = \sum_{k=-\nu N}^{\nu N} g_{N:k} x^k , \tag{3.10}$$

where ν is the number of charged particles per cluster. The fluctuation of interest is then easy to compute as

$$\langle (n_R - n_L)^2 \rangle_{n_{\text{ch}}} = \left(\frac{x\partial}{\partial x} \right)^2 P_{n_{\text{ch}}/\nu}(x) \bigg|_{x=1} . \tag{3.11}$$

We shall consider three specific models:

(1) Independent emission (one charged particle per cluster),

(2) "ρ" ($\pi^+\pi^-$ cluster, a "ρ" produced with rapidity y results in charged pions at $y \pm \Delta$), and

(3) "ω" ($\pi^+\pi^-\pi^0$ clusters, an "ω" produced with rapidity y results in pions at $y, y \pm \Delta$).

In each case the clusters are emitted independently, according to (3.1). Figure 6 helps one to visualize the construction of the generating functions. Clusters emitted in the interval $L \equiv [-\frac{1}{2}Y, -\Delta]$ will deposit all their charged secondaries in the left hemisphere. Likewise, clusters emitted in the interval $R \equiv [\Delta, \frac{1}{2}Y]$ must deposit all their charged secondaries in the right hemisphere. Clusters emitted in the interval $A \equiv [-\Delta, \Delta]$ have the possibility of depositing charged particles in both hemispheres; such clusters are called "active." The probability that a cluster is active is therefore

$$p = 2\Delta / Y . \tag{3.12}$$

How an active cluster apportions its progeny depends upon the makeup of the cluster. For example, a "ρ" will always distribute one pion in each hemisphere; an "ω" will do so with probability $\frac{2}{3}$, but will place two charged particles in one hemisphere and none in the other with probability $\frac{1}{3}$.

The foregoing analysis allows one to write down at once the generating function appropriate for any model. These are tabulated in Table I, together with the resultant predictions for $\langle (n_R - n_L)^2 \rangle_{n_{\text{ch}}}$. Three features are noteworthy, and more general than the models which help us to recognize them. First, the fluctuation is proportional to the number of produced particles. For extremely high multiplicities, this trend will be opposed by the effect of energy conservation, neglected in these models, which tends to squeeze the secondaries nearer to $y = 0$. Second, at finite energies (for which $p > 0$), clustering reduces the fluctuations between hemispheres. Third, in the high-energy limit in which for clusters of fixed mass $p \to 0$,

$$\langle (n_R - n_L)^2 \rangle_{n_{\text{ch}}} \to 2\nu n_- \tag{3.13}$$

so that a measurement of the left-right multiplicity fluctuation can give a measure of the number of charged particles per cluster.

2. *Forward-backward correlations*

For the subsequent discussion, we restrict our attention to the "ρ" model. None of the qualitative conclusions we wish to draw depends upon this restriction to the simplest kind of cluster, and the expressions we derive will be considerably the simpler for the restriction. In this paragraph we study the influence of clusters on $\langle n_L(n_R) \rangle$, the

FIG. 6. Notations used in the analysis of left-right multiplicity correlations. The horizontal scale is c.m. rapidity. The partitions L, A, and R are described in the text.

TABLE I. Forward-backward multiplicity fluctuations.

Model	Generating function, $P_N(x)$	$\langle (n_L - n_R)^2 \rangle_{n_-}$
Independent emission	$[\frac{1}{2}(x + x^{-1})]^N$	$2n_-$
"ρ" ($\pi^+\pi^-$ clusters)	$[\frac{1}{2}(1-p)(x^2 + x^{-2}) + p]^N$	$4n_-(1-p)$
"ω" ($\pi^+\pi^-\pi^0$ clusters)	$[(\frac{1}{2} - \frac{1}{3}p)(x^2 + x^{-2}) + \frac{2}{3}p]^N$	$4n_-(1 - \frac{2}{3}p)$

mean number of charged left-movers observed in collisions with a fixed number of right-movers. It is once again easy to see that there will be a profound influence. For the extreme case of very massive clusters mentioned above, one would expect

$$\langle\!\langle n_L(n_R)\rangle\!\rangle = n_R\,. \tag{3.14}$$

In the general case, clusters produced in the active region of rapidity will give rise to a positive correlation between $\langle\!\langle n_L(n_R)\rangle\!\rangle$ and n_R. Recourse to a generating-function technique again streamlines the model computation. For an event in which N clusters are produced, the appropriate generating function is

$$\begin{aligned} P_N(x,y) &= \sum_{n_L, n_R} g_{N:n_R,n_L} x^{n_R} y^{n_L} \\ &= [\tfrac{1}{2}(1-p)(x^2+y^2) + pxy]^N \\ &\equiv [\mathcal{P}(x,y)]^N\,, \end{aligned} \tag{3.15}$$

where $g_{N:n_R,n_L}$ is the probability that an N-cluster event is made up of n_R right-movers and n_L left-movers. The quantity of interest is

$$\langle\!\langle n_L(n_R)\rangle\!\rangle = \frac{1}{\sigma}\sum_N \sum_{n_L} \sigma_N n_L g_{N:n_R,n_L}\,. \tag{3.16}$$

We must therefore make an assumption about the multiplicity distribution of produced clusters. For the consistency of the independent cluster-emission picture, we must assume a Poisson distribution, namely

$$\frac{\sigma_N}{\sigma} = e^{-\langle N\rangle}\langle N\rangle^N/N!\,. \tag{3.17}$$

It is advantageous to combine (3.15) and (3.17) into a generating function which represents the probability of a partition n_L, n_R taking account of the relative weights of different cluster multiplicities. It is simply

$$P(x,y) = \sum_{N=0}^{\infty} e^{-\langle N\rangle}[\mathcal{P}(x,y)\langle N\rangle]^N/N! \tag{3.18}$$

$$= \exp[\langle N\rangle(\mathcal{P}(x,y) - 1)] \tag{3.19}$$

$$= \sum_{n_L,n_R} G_{n_R,n_L} x^{n_R} y^{n_L}\,. \tag{3.20}$$

From this generating function it is easy to compute

$$\langle\!\langle n_L(n_R)\rangle\!\rangle = \frac{\partial}{\partial y}\ln\left[\left(\frac{\partial}{\partial x}\right)^{n_R} P(x,y)\bigg|_{x=0}\right]\bigg|_{y=1} \tag{3.21}$$

$$= \frac{\sum_{n_L} n_L G_{n_R,n_L}}{\sum_{n_L} G_{n_R,n_L}}\,. \tag{3.22}$$

The resultant series can be summed to yield

$$\langle\!\langle n_L(n_R)\rangle\!\rangle = \langle N\rangle(1-p) + \frac{\sum_{s=0}^{[n_R/2]} (n_R - 2s)[\frac{1}{2}(1-p)\langle N\rangle]^s [p\langle N\rangle]^{n_L - 2s}/s!\,(n_L - 2s)!}{\sum_{s=0}^{[n_R/2]} [\frac{1}{2}(1-p)\langle N\rangle]^s [p\langle N\rangle]^{n_L - 2s}/s!\,(n_L - 2s)!}\,, \tag{3.23}$$

where $[n_R/2]$ denotes the greatest integer not exceeding $n_R/2$.

Typical results from this model are shown in Fig. 7. The parameters have been chosen to illustrate the effects which may be expected in 100 and 200 GeV/c pp collisions.[33] The odd-even effect evident in the curves is an artifact of the simple $\pi^+\pi^-$ clusters, but a related effect will surely be present experimentally because the total number of charged particles is necessarily even. It needs to be emphasized that n_R and n_L here refer to the number of produced particles, exclusive of the usually present leading protons. Finally, let us note that in the high-energy limit ($p \to 0$), $\langle\!\langle n_L(n_R)\rangle\!\rangle$ becomes independent of n_R and equal to n_-.

3. *Multiplicity correlations across a gap*

A slightly more complicated version of the left-right multiplicity correlation deserves exploration, in part because it corresponds to the experimental conditions of the Pisa–Stony Brook apparatus,[34] and in part because it suggests a direct way

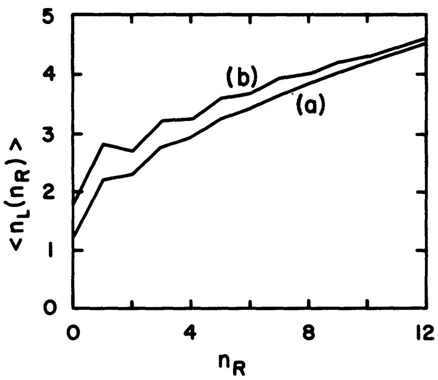

FIG. 7. Left-right multiplicity correlations computed according to Eq. (3.23). Curve (a), symbolic of the situation of 100 GeV/c, is obtained with $\langle N\rangle = 2.4$, $p = 0.5$; curve (b), representative of 200 GeV/c, is calculated assuming $\langle N\rangle = 3.0$, $p = 0.4$.

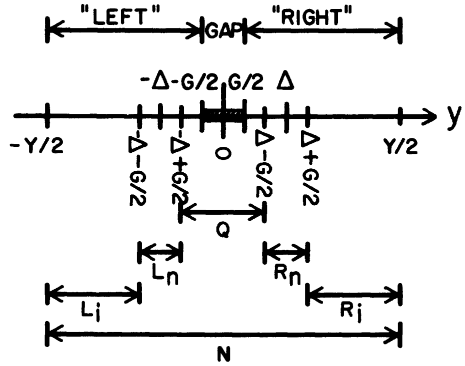

FIG. 8. Notations used in the analysis of left-right multiplicity correlations across a gap. The horizontal scale is c.m. rapidity. The partitions L_i, L_n, Q, R_n, R_i are described in the text.

of measuring the mobility parameter at high energies. The situation is shown schematically in Fig. 8. Left- and right-movers are here defined with respect to a gap of extent G in rapidity centered about the origin, in the c.m. system. At first we shall simply ignore the number of secondaries n_G which fall in the gap. Later we shall study the dependence of the correlation upon n_G. An obvious remark, but one of considerable importance, is that if $G > 2\Delta$, it is not possible for a single cluster to deposit secondaries in both the "right" and "left" regions. Thus one may measure Δ at high energies by increasing the gap size until the correlation between $\langle n_L(n_R)\rangle$ and n_R vanishes.[35] This procedure should provide a direct check of the assumption that cluster properties are energy-independent.

Now let us consider in detail the interesting case $2\Delta > G$. As indicated in Fig. 8, it is relevant to recognize five intervals in rapidity for the emission of clusters. Clusters emitted in the interval $L_i \equiv [-\frac{1}{2}Y, -\Delta - \frac{1}{2}G]$ deposit two charged secondaries in the left bin; those emitted in $L_n \equiv [-\Delta - \frac{1}{2}G, -\Delta + \frac{1}{2}G]$ deposit one charged secondary in the left bin and one in the gap; those emitted in $Q \equiv [-\Delta + \frac{1}{2}G, \Delta - \frac{1}{2}G]$ deposit one charged secondary in the left bin and one in the right bin. The regions R_i, R_n are the right-bin counterparts of L_i, L_n. For an event in which N clusters are produced, the appropriate generating function is

$$\begin{aligned} P_N(x,y) &= \sum_{n_L, n_R} g_{N:n_R,n_L} x^{n_R} y^{n_L} \\ &= [\tfrac{1}{2}(1-p-g)(x^2+y^2) \\ &\quad + g(x+y) + (p-g)xy]^N \\ &\equiv [\mathcal{P}(x,y)]^N, \end{aligned} \tag{3.24}$$

where once again $p \equiv 2\Delta/Y$, and $g \equiv G/Y$. Just as in the no-gap case, we may combine (3.24) with (3.17) to obtain a generating function appropriate for the emission of all numbers of clusters. Indeed, all the results (3.18)–(3.22) carry over to this case if $\mathcal{P}(x,y)$ is defined through (3.24). Once again the computation leads to a closed form for $\langle n_L(n_R)\rangle$, namely

$$\langle n_L(n_R)\rangle_{\text{gap}} = \langle N\rangle(1-p) + (1-g/p)\frac{\sum_{s=0}^{[n_R/2]} (n_R-2s)[\frac{1}{2}(1-p-g)\langle N\rangle]^s[p\langle N\rangle]^{n_R-2s}/s!\,(n_R-2s)!}{\sum_{s=0}^{[n_R/2]} [\frac{1}{2}(1-p-g)\langle N\rangle]^s[p\langle N\rangle]^{n_R-2s}/s!\,(n_R-2s)!}, \tag{3.25}$$

which displays the expected disappearance of correlations as $g \to p$ (i.e., as $G \to 2\Delta$).

Illustrative results are shown in Fig. 9 for the 200-GeV/c situation depicted in Fig. 7. Included are calculations for the no-gap case [same as curve (b) in Fig. 7], for $g/p = \frac{5}{8}$, and for $g/p = 1$.

4. Multiplicity correlations across a gap (gap multiplicity specified)

If the number of charged secondaries n_G produced in the gap can be determined (as it can be in the Pisa–Stony Brook experiment), it is possible to place severe restrictions on the allowed cluster configurations. For example, the requirement that $n_G = 0$ implies that, in the notation of Fig. 8, no clusters be emitted in the intervals L_n and R_n. To obtain the general result we proceed once more by the generating-function technique. This time the generating function appropriate for N cluster events is

$$\begin{aligned} P_N(x,y,z) &= \sum_{n_L, n_R, n_G} g_{N:n_R, n_L, n_G} x^{n_R} y^{n_L} z^{n_G} \\ &= [\tfrac{1}{2}(1-p-g)(x^2+y^2) \\ &\quad + g(xz+yz) + (p-g)xy]^N \\ &\equiv [\mathcal{P}(x,y,z)]^N . \end{aligned} \tag{3.26}$$

A by now familiar procedure leads to the generating function for all events,

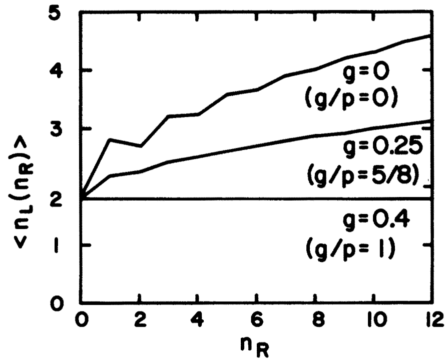

FIG. 9. Dependence of the left-right multiplicity correlation upon the gap size. The parameters used correspond to the 200-GeV/c case of Fig. 7.

$$\begin{aligned} P(x,y,z) &= \exp[\langle N\rangle(\mathcal{P}(x,y,z)-1)] \\ &= \sum_{n_L, n_R, n_G} G_{n_L, n_R, n_G} x^{n_R} y^{n_L} z^{n_G} , \end{aligned} \tag{3.27}$$

from which

$$\begin{aligned} \langle n_L(n_R;n_G)\rangle &= \frac{\partial}{\partial y} \ln\left[\left(\frac{\partial}{\partial x}\right)^{n_R}\left(\frac{\partial}{\partial z}\right)^{n_G} P(x,y,z)\Big|_{x=z=0}\right]\Bigg|_{y=1} \\ &= \langle N\rangle(1-p-g) + \frac{\displaystyle\sum_{s_1=0}^{[n_R/2]}\sum_{s_2=0}^{m} \frac{(n_R+n_G-2s_1-2s_2)[\frac{1}{2}(1-p-g)\langle N\rangle]^{s_1}[(p-g)\langle N\rangle]^{n_R-2s_1-s_2}}{s_2!\,(n_G-s_2)!\,s_1!\,(n_R-2s_1-s_2)!}}{\displaystyle\sum_{s_1=0}^{[n_R/2]}\sum_{s_2=0}^{m} \frac{[\frac{1}{2}(1-p-g)\langle N\rangle]^{s_1}[(p-g)\langle N\rangle]^{n_R-2s_1-s_2}}{s_2!\,(n_G-s_2)!\,s_1!\,(n_R-2s_1-s_2)!}} , \end{aligned} \tag{3.28}$$

where $m \equiv \min(n_G, n_R - 2s_1)$. Some of the properties of (3.28) are illustrated in Fig. 10. The expected behavior appears for $n_G = 0$. Apart from an enhancement of the odd-even effect, the $n_G = 0$ curve is similar to the curve predicted when n_G is unrestricted. For moderate values of n_G, however, $\langle n_L(n_R;n_G)\rangle$ becomes essentially independent of n_R for a wide range of values of n_R. This does not mean that there is no correlation between n_R and n_L. In the absence of any correlation, $\langle n_L\rangle$ would have the value indicated by the broken line in Fig. 10. Understanding the value and flatness of the correlation fully requires attention to numerical details. However, we can see roughly why it is flat by answering the question: "Given n_G, what must be done to increase n_R by one?" There are two possibilities: (i) Add a cluster to region Q; this will tend to increase n_L. (ii) Add a cluster to region R_n and subtract one from L_n; this will tend to decrease $\langle n_L\rangle$. It is the competition between these alternatives which is responsible for the flatness shown in Fig. 10. We believe that although it is the simplicity of our model which permits us to understand how the effect arises, the effect itself will occur in more general cluster models.

IV. DISCUSSION

We end by summarizing the questions we have sought to raise and the partial answers we have given.

(1). Do clusters exist? More properly, is it useful to think in terms of a cluster model? It seems to us that considerable indirect evidence supports the notion of clusters. We claim to have shown, by means of simple, one-dimensional cluster models, that the cluster language allows one to explain—and more importantly, to begin to understand—a wealth of experimental information.

(2). Do clusters really exist? At this time, not only do we not know the answer to this question, which is of a fundamental nature, but we do not know how to go about learning the answer. Some interesting explorations of the very early time structure of the multiple-production mechanism are underway[36] in nuclear targets, but it is hard to see how to probe at intermediate times

when clusters might be "detected" directly.

(3). What are the properties of clusters? A preliminary question is whether the same kinds of clusters are emitted in all reactions at all energies. Scant evidence from charge-transfer measurements gives some hope that this is so. Whereas it is possible to speak in cluster terms even if cluster properties are energy-dependent, the picture is obviously more appealing if the properties are energy-independent. Two properties of basic importance are the (average) cluster mass and the (average) number of charged particles per cluster. Some techniques for studying these characteristics are reviewed in Secs. II and III B.

Useful as they may be for gaining insight into experimental observations, the one-dimensional models discussed here are of little value for addressing questions of consistency. Having seen through the simple models that many observed effects can be understood, one must now ask whether any cluster model can give a consistent, quantitative account of the data. This is a task for the Monte Carlo programs.

Note added in proof. A summary of the evidence for independently emitted clusters, similar to our Sec. II, appears in a recent paper by S. Pokorski and L. Van Hove, CERN Report No. TH-1772, 1973 (unpublished).

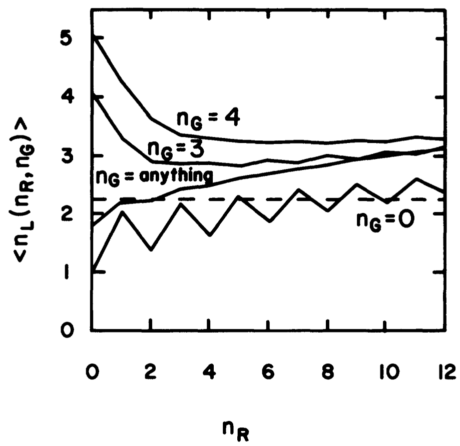

FIG. 10. Dependence of the left-right multiplicity correlation upon the number of secondaries emitted in the gap. The other parameters are $\langle N\rangle = 3.0$, $p = 0.4$, $g = 0.25$. The broken line represents the value of $\langle n_L\rangle$ in the absence of any correlation (i.e., for $p = 0$).

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