s-dependence, sign of the cut, and factorization in a t-channel partial-wave amplitude with Regge cuts*

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Using an effective-range-type expansion in the j plane for the N and D functions, the t-channel partial-wave amplitude $A(j,t)$ $(=N/D)$ is expressed as $[F_1+F_2(j-\alpha_c)+G_1(j-\alpha_c)^{1/2}]/[j-\alpha_0+\epsilon(j-\alpha_c)^{1/2}]$, where a square-root singularity is assumed for simplicity. For α_0 linear in t the following results are obtained: (1) The Regge poles $(=\alpha_\pm)$ are complex below a certain t value. (ii) The amplitude $A(s,t)$ in the scattering region is of the form $a_+(s,t)s^{\alpha_+}+a_-(s,t)s^{\alpha_-}$, with $a_\pm(s,t)$ expressible as a sum of two terms. The first term is one-half the residue of the complex pole, whether the pole be on the physical or unphysical sheet, the second term is a series involving the product $(\alpha_\pm-\alpha_c)\ln s$. It is found by explicit calculation that if ϵ is small (≈ 0.1) then only one or two terms of the series are important up to quite high s ($\approx$200 BeV^2). Only at asymptotic s will the series sum up to give the tip of the cut contribution, $s^{\alpha_c}/(\ln s)^{3/2}$. At presently available energies, therefore, the s dependence is largely given by $s^{\alpha_\pm}$. The results remain unchanged if the pole is on the real axis ($\epsilon=0$). (iii) At the t value where the poles collide, the s dependence is of a typical double pole from $s^{\alpha}\ln s$. (iv) It is observed that the strength of the cut is manifested through $|F_2/F_1|$ and $|G_1/F_1|$ as well as through ϵ. (v) For small, fixed ϵ the F_2 term plays a crucial role in shifting the zeros in t of $A(s,t)$ from their simple pole values. (vi) The sign of the cut is intimately connected with the phase of the complex residues for $t\le 0$, the width of the t channel resonances, and with the question of determining the sheet on which the poles are located for $t\le 0$. (vii) Finally, $A(s,t)$ is in general not factorizable but can be written as a sum of (complex conjugate) factorized quantities.

I. INTRODUCTION

The j-plane singularity structure of t-channel partial-wave amplitudes are believed to play a crucial role in determining the high-energy behavior of the s-channel scattering amplitude. If t-channel unitarity is to be taken seriously then, as emphasized in recent years,[1] the so-called D function of the partial-wave amplitude should inherit the singularity structure of that amplitude. Thus if the amplitude has j-plane cuts then so must the D function.

Let us suppose for simplicity that the cuts are of the square-root type with a branch point at $j=\alpha_c$. One can then expand the D function in a power series around $j=\alpha_c$ in the $(j-\alpha_c)^{1/2}$ plane[2]

$$D(j,t)=d_0+d_1(j-\alpha_c)^{1/2}+d_2(j-\alpha_c)+\cdots .$$

Similarly, one can expand the N function which is the numerator function of the partial-wave amplitude,

$$N(j,t)=n_0+n_1(j-\alpha_c)^{1/2}+n_2(j-\alpha_c)+\cdots .$$

Here the d_i's and n_i's for $i=0,1,\ldots,$ are assumed to be functions of t. The $n_i(t)$ functions are of course not independent of the $d_i(t)$'s, since they are (at least with two-body unitarity), the difference of the $d_i(t)$'s on the two sheets of t. However, for $t\le 0$, this is not a very relevant constraint and we may as well consider the $n_i(t)$'s as independent functions.

In the spirit of the effective-range theory, which has worked so well in the k plane, we keep terms only up to and including the quadratic, $j-\alpha_c$. The partial-wave amplitude $A(j,t)$ $(=N/D)$ can then be expressed, after appropriate readjustment of terms and normalization, as[3]

$$A(j,t)=\frac{F_1+F_2(j-\alpha_c)+G_1(j-\alpha_c)^{1/2}}{j-\alpha_0+\epsilon(j-\alpha_c)^{1/2}} . \qquad (1)$$

One can interpret the numerator above as a product of the factorized form,

$$(\beta_1+\beta_1'(j-\alpha_c)^{1/2}+\cdots)(\beta_2+\beta_2'(j-\alpha_c)^{1/2}+\cdots)$$
$$=F_1+F_2(j-\alpha_c)+G_1(j-\alpha_c)^{1/2}+\cdots .$$

This interpretation would ensure that the residues *at the pole* will be of the factorized form, whether the pole be on the physical or unphysical sheet.

It is implicit in the above expansion that ϵ appearing in the denominator expression is *small*. The phenomenological consideration give $\epsilon \approx 0.1$.[4] Thus the trajectory function obtained from the zero of the denominator will not be too different from α_0. From now on α_0 will be assumed to be *linear* in t, so that the actual trajectory, to a very good approximation, will remain linear in t. The quantities F_2 and G_1 should also be small in absolute magnitude (note that our numerator and denominator functions are dimensionless[3]). However, if F_1 itself happens to be small then the ratio F_2/F_1, G_1/F_1 can be ≈ 1. Obviously, all quantities, in particular F_2, G_1 and ϵ, must be related to each other. In the absence of cuts, one would expect ϵ, F_2, G_1 to vanish identically leaving the amplitude with "unperturbed" pole α_0 and "unperturbed" residue F_1.

The above discussion was based on the square-root model but arguments and formulations remain the same for other types of singularities as well.[2] The square root is among the simpler singularities and, phenomenologically, it appears quite relevant to high-energy scattering.[4] We will assume it to be valid throughout the discussion below.

The expression (1) for the amplitude $A(j, t)$ is a natural extension of the simple pole form, $\beta/(j-\alpha)$, to the case where cuts are present. Its Mellin transform for $t \leq 0$ will be related to the scattering amplitude for the (crossed) s channel. The first question we would like to ask is what kind of high-energy behavior will the partial-wave amplitude (1) predict in the s channel, and to what extent the cuts change the structure of the amplitude from the simple pole case. Of particular relevance is the so-called complex-pole approximation and the related high-energy phenomenology. It is also most interesting to study the possibility of having the crossover zeros and their relation to the strength of the cut. The expression (1) is also useful in the t channel for t values above threshold where the residues will be proportional to the resonance widths. We will investigate the problem of extrapolation from the region $t \leq 0$ to the resonance region as well as the location of poles and the significance of the sign of the cut. Finally, the constraints due to factorization will be investigated.

In Sec. II, we obtain the Mellin transform of (1). In Sec. III we discuss the question of the complex-pole approximation and the tip of the cut contribution. In Sec. IV, we investigate the double pole corresponding to the point at which the complex poles collide. In Sec. V, the poles on the real axis vis-à-vis the absorption model is considered. In Sec. VI the relation of the strength of the cut to the crossover zeros and dips is investigated. In Sec. VII we discuss the location of the poles, extrapolation of the residues, and the sign of the cut. Finally, in Sec. VIII we consider factorization.

II. THE MELLIN TRANSFORM OF $A(j,t)$ AND THE SCATTERING AMPLITUDE

We note that the poles of $A(j, t)$ in (1) are given by the solutions of the equation

$$j-\alpha_0+\epsilon(j-\alpha_c)^{1/2}=0\,.$$

If α_1 and α_2 are the two solutions, then

$$(\alpha_k-\alpha_c)^{1/2}=-\tfrac{1}{2}\epsilon \pm (\tfrac{1}{4}\epsilon^2+\alpha_0-\alpha_c)^{1/2}\,. \qquad (2)$$

The two poles collide when the term inside the above square root vanishes. Suppose this happens at $t=t_0$; then one can write

$$\tfrac{1}{4}\epsilon^2+\alpha_0-\alpha_c=c(t-t_0), \quad c>0$$

where we assume that α_0 and α_c are linear in t. We also assume, following standard theoretical arguments, that α_c has a *smaller* slope than α_0. The expression (2) can then be written as

$$(\alpha_k-\alpha_c)^{1/2}=-\tfrac{1}{2}\epsilon \pm [c(t-t_0)]^{1/2} \qquad (3)$$

and taking the square on both sides, we obtain the two solutions

$$\alpha_k=\alpha_c+\tfrac{1}{4}\epsilon^2+c(t-t_0) \mp \epsilon[c(t-t_0)]^{1/2}$$
$$=\alpha_0+\tfrac{1}{2}\epsilon^2 \mp \epsilon[c(t-t_0)]^{1/2}\,. \qquad (4)$$

For $t \gg t_0$ the two poles are on different sheets, one physical $[\mathrm{Re}(j-\alpha_c)^{1/2}>0]$ and the other unphysical. The two poles collide at $t=t_0$. For $t<t_0$ the poles are complex conjugates of each other. They will both lie on the unphysical sheet, for $t<t_0$, if $\epsilon>0$ and on the physical sheet if $\epsilon<0$. In Fig. 1, we have sketched a diagram for the unphysical sheet case. The real part of the trajectory, α_R, for $t<t_0$ is given by

$$\alpha_R=\alpha_c+\tfrac{1}{4}\epsilon^2+c(t-t_0)$$
$$=\alpha_0+\tfrac{1}{2}\epsilon^2\,.$$

Thus α_R differs from α_0 only by $\frac{1}{2}\epsilon^2$.

Let us designate α_+ to be that pole, either on the physical or unphysical sheet, which develops a *positive* imaginary part α_I whenever $t<t_0$. Then α_- will be the other pole. Thus for $t<t_0$, and $\alpha_I=|\epsilon|[c(t-t_0)]^{1/2}$

$$\alpha_+ = \alpha_2 = \alpha_c + \tfrac{1}{4}\epsilon^2 + c(t-t_0) + i\epsilon[c(t_0-t)]^{1/2} \qquad (\epsilon > 0),$$

$$\alpha_+ = \alpha_1 = \alpha_c + \tfrac{1}{4}\epsilon^2 + c(t-t_0) - i\epsilon[c(t_0-t)]^{1/2} \qquad (\epsilon < 0). \quad (5)$$

From now until Sec. VII we will consider the case where the poles are on the unphysical sheet ($\epsilon > 0$). In Sec. VII we will compare the results with the physical-sheet situation. We now consider the Mellin transform of $A(j, t)$ for $t \leqslant 0$. This Mellin transform is the imaginary part, $A(s, t)$, of the scattering amplitude, $T(s, t)$, above the s-channel threshold.

$$A(s, t) = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} dj\, s^j A(j, t) = \mathrm{Im}\, T(s, t),$$

where γ is to the right of all the j-plane singularities of $A(j, t)$ and where from now on we employ BeV units with the scale factor $s_0 = 1$.

For the case where the poles are on the *unphysical* sheet ($\epsilon > 0$),

$$A(s, t) = -\frac{1}{\pi}\int_{-\infty}^{\alpha_c} dj\, s^j\, \mathrm{disc}A(j, t), \qquad (6)$$

where $\mathrm{disc}A(j, t)$ is the discontinuity of $A(j, t)$ across the j-plane cut. In order to evaluate $\mathrm{disc}A$ from the expression (1) we note that one can write

$$A(j, t) = \frac{f_1 + g_1(j-\alpha_c)^{1/2}}{j - \alpha_0 + \epsilon(j-\alpha_c)^{1/2}} + F_2, \qquad (7)$$

where

$$f_1 = F_1 + F_2(\alpha_0 - \alpha_c) = F_1 + F_2[c(t-t_0) - \tfrac{1}{4}\epsilon^2], \qquad (8)$$

$$g_1 = G_1 - \epsilon F_2,$$

then because F_2 is independent of j the second term in (7) will not contribute to $\mathrm{disc}A$. Therefore,

$$\mathrm{disc}A(j, t) = \frac{[g_1(j-\alpha_0) - \epsilon f_1](\alpha_c - j)^{1/2}}{(j-\alpha_+)(j-\alpha_-)}, \qquad (9)$$

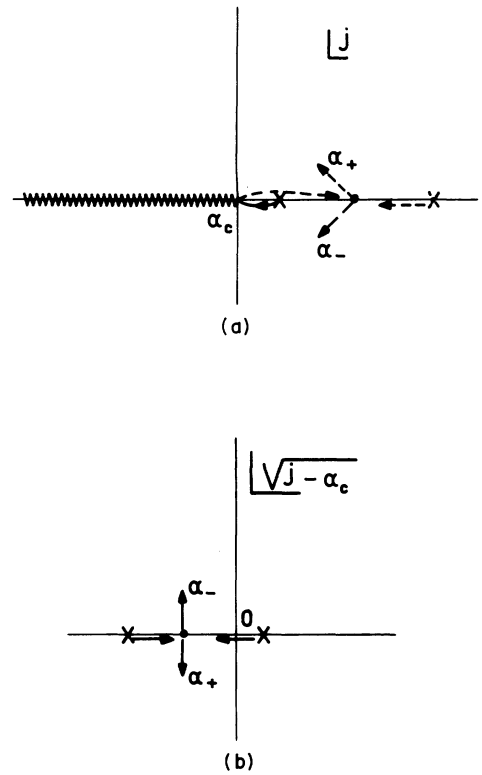

FIG. 1. A sketch of the motion of the poles for $\epsilon > 0$ in (a) the $(j-\alpha_c)^{1/2}$ plane and in (b) the j plane. This is the unphysical-sheet case.

where in the denominator $|j - \alpha_0 + \epsilon(j-\alpha_c)^{1/2}|^2$ is replaced by the products involving the two roots α_+ and α_-.

From (4) we note that

$$\alpha_0 = \tfrac{1}{2}(\alpha_+ + \alpha_-) - \tfrac{1}{2}\epsilon^2.$$

Also for the unphysical sheet case ($\epsilon > 0$) under discussion where α_+ corresponds to the positive sign in (4) we have for $t < t_0$

$$\tfrac{1}{2}(\alpha_+ - \alpha_-) = i\epsilon[c(t_0-t)]^{1/2} = i\alpha_I.$$

The expression (9) can then be written as

$$\mathrm{disc}A(j, t) = \frac{1}{2}\left\{g_1 + \frac{i}{[c(t_0-t)]^{1/2}}(f_1 - \tfrac{1}{2}\epsilon g_1)\right\}\frac{(\alpha_c - j)^{1/2}}{j - \alpha_+} + \text{c.c.},$$

where c.c. means complex conjugation. The Mellin transform integral (6) is then reduced to a single integral which is the following[5]:

$$\int_{-\infty}^{\alpha_c} dj\, s^j \frac{(\alpha_c-j)^{1/2}}{j-\alpha_+} = -\pi(\alpha_+-\alpha_c)^{1/2} s^{\alpha_+} + \tfrac{1}{2}\left(\frac{\pi}{\ln s}\right)^{1/2} \phi_+(s,t)s^{\alpha_+}, \tag{10}$$

where

$$\phi_+(s,t)=\sum_{n=0}^{\infty}\frac{(\alpha_c-\alpha_+)^n(\ln s)^n}{(n-\frac{1}{2})n!}\,. \tag{11}$$

The amplitude $A(s,t)$ is then, using (3)

$$A(s,t)=\tfrac{1}{2}\bar{\gamma}_+(t)s^{\alpha_+} - \frac{1}{4\sqrt{\pi}}\left(g_1+\frac{i}{[c(t_0-t)]^{1/2}}(f_1-\tfrac{1}{2}\epsilon g_1)\right) \times \frac{1}{(\ln s)^{1/2}}\phi_+(s,t)s^{\alpha_+} + \text{c.c.}, \tag{12}$$

where $\bar{\gamma}_+$ is given by

$$\bar{\gamma}_+(t)=\left((f_1-\epsilon g_1)-ig_1[c(t_0-t)]^{1/2} - \frac{i\epsilon}{2[c(t_0-t)]^{1/2}}(f_1-\tfrac{1}{2}\epsilon g_1)\right). \tag{13}$$

We can also write

$$A(s,t)=a_+(s,t)s^{\alpha_+}+a_-(s,t)s^{\alpha_-}\,.$$

We note a very important point about (12) above. It is expressed in terms of the complex poles α_+ and α_- and the energy dependence is in terms of s^{α_+} and s^{α_-}. The first term in the expression corresponds to the contribution from the semicircle around the pole α_+ with $\bar{\gamma}_+$ the full residue, the second (series) term involving φ_+ corresponds to the rest of the contour (see Fig. 2), with a contribution that depends on both s and t. As we shall see later, only a few terms in the series are found to be significant if ϵ is small. One can consider (12) as a complex-pole expression with *energy-dependent residues* $a_\pm(s,t)$. In other words, the energy dependence is largely determined by $s^{\alpha_\pm}$ though with residues that may have slow $\ln s$ dependence. Of course, if the energy variable s is extremely large then the higher-order terms will be important and the infinite series in $\varphi_+(s,t)$ will add up to give

$$\phi_+(s,t)s^{\alpha_+} \sim \frac{s^{\alpha_c}}{(\alpha_c-\alpha_+)\ln s} \quad \text{as } s\to\infty\,.$$

$$A(s,t)=\tfrac{1}{2}g_1\left(-\epsilon-i[c(t_0-t)]^{1/2}+\frac{i\epsilon^2}{4[c(t_0-t)]^{1/2}}\right)s^{\alpha_+}-\frac{g_1}{4\sqrt{\pi}}\left(1-\frac{i\epsilon}{2[c(t_0-t)]^{1/2}}\right)\frac{1}{(\ln s)^{1/2}}\phi_+(s,t)s^{\alpha_+}+\text{c.c.}$$

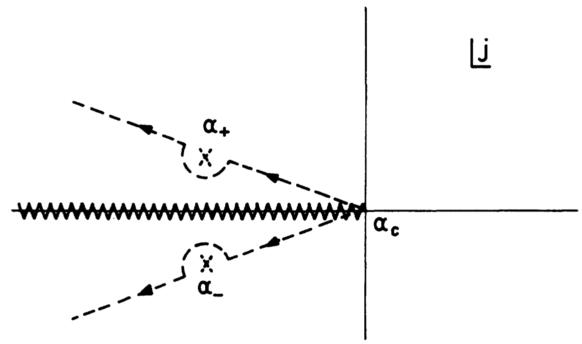

FIG. 2. The contour of integration for the case when the poles are on the unphysical sheet.

We will see an explicit demonstration of this in the next section. We will also see, however, that for s as high 200 BeV^2 only a few terms in the series need be important.

The expression for $A(j,t)$ can be divided into two distinct parts. One coming from the term

$$\frac{f_1}{j-\alpha_0+\epsilon(j-\alpha_c)^{1/2}}$$

which we call the "pole-type" term. The other coming from the term

$$\frac{g_1(j-\alpha_c)^{1/2}}{j-\alpha_0+\epsilon(j-\alpha_c)^{1/2}}$$

which we call the "cut-type" term.

The $A(s,t)$ corresponding to the "pole-type" term is obtained from (12) by putting $g_1=0$

$$A(s,t)=\tfrac{1}{2}f_1\left(1-i\frac{\epsilon}{2[c(t_0-t)]^{1/2}}\right)s^{\alpha_+} -i\frac{f_1}{4\sqrt{\pi}[c(t_0-t)]^{1/2}}\frac{1}{\sqrt{\ln s}}\phi_+(s,t)s^{\alpha_+} + \text{c.c.}$$

As $\epsilon\to 0$, α_+ and α_- each become real ($=\alpha_0$). The second term and its complex conjugate mutually cancel each other so that we have for the above case

$$\left.\begin{array}{l} A(s,t)\to f_1 s^{\alpha_0} \\ A(j,t)\to \dfrac{f_1}{j-\alpha_0}\end{array}\right\} \text{as } \epsilon\to 0$$

which is simply the usual (real) pole behavior. Hence we designate the f_1 term in $A(j,t)$ as the "pole-type" term.

The $A(s,t)$ corresponding to the "cut-type" term is obtained from (12) by putting $f_1=0$

As $\epsilon\to 0$, only the *second-term* and its complex conjugate will survive so that for the above case

$$\left.\begin{aligned} A(s,t) &\to -\frac{g_1}{2\sqrt{\pi}}\frac{1}{(\ln s)^{1/2}}\phi_0(s,t)s^{\alpha_0} \\ A(j,t) &\to \frac{g_1(j-\alpha_c)^{1/2}}{j-\alpha_0} \end{aligned}\right\} \text{as } \epsilon\to 0,$$

where $\varphi_0(s,t)$ corresponds to replacing α_+ by α_0 in the series expansion (11) of $\varphi_+(s,t)$. The interesting point to note is that these expressions do not give the simple pole behavior as did the "pole-type" term, discussed earlier. Eventhough $A'(j,t)$ has the pole at $j=\alpha_0$, the residue at the pole for $\alpha_0\leq\alpha_c$ gets different contributions from the upper and lower half of the semicircles. In the limit $\epsilon\to 0$ (but $g_1\neq 0$) the presence of the branch cut makes the contribution from the upper half of the semicircle around $j=\alpha_0$ to be of equal but *opposite* sign compared to the lower half. Thus the two contributions exactly cancel for $\epsilon=0$ instead of add as in the usual case. The leading behavior of $A(s,t)$ has, therefore, $\ln s$ dependence in it with an overall power dependence of the pole form, s^{α_0}. Hence we designate the g_1 term in $A(j,t)$ as the "cut-type" term. We remind ourselves here that the limit $\epsilon\to 0$, with other parameters fixed, is a mathematical limit. *Physically,* we anticipate g_1 to depend on ϵ through unitarity etc. so that physically, of course, as $\epsilon\to 0$ one should expect also $g_1\to 0$.

In general the amplitude $A(s,t)$ will be some mixture of the "pole-type" and the "cut-type" terms. In the next section we will investigate, for both cases, the validity of approximating $\varphi_\pm(s,t)$ by just a few terms.

III. THE COMPLEX-POLE APPROXIMATION AND THE TIP OF THE CUT CONTRIBUTION

A. Complex-pole approximation

In order to facilitate our discussion let us make the following observations: (i) phenomenologically, the analyses of high-energy data with complex poles yield, generally, a very small value of ϵ, roughly ~0.1.[4] (ii) Also, phenomenologically, α_I is consistent with a $\sqrt{-t}$ behavior so that the collision of poles seem to take place at $t=0$.[4] (iii) Theoretically, as well as phenomenologically, α_c is found to have a smaller slope than the pole trajectory. We will assume the slope of α_c to be identically zero. Thus we will take

$$t_0=0,\quad \alpha_c=a\ (=\text{constant}).$$

For simplicity, we will also take the slope of α_0 to be 1 BeV^2 so that $c=1$, and

$$\alpha_0=a-\tfrac{1}{4}\epsilon^2+t\,.$$

For the unphysical sheet case ($\epsilon>0$) for $t\leq 0$

$$\begin{aligned}\alpha_+ &= a+\tfrac{1}{4}\epsilon^2+t+i\epsilon\sqrt{-t}\\ &=\alpha_-^*\,. \end{aligned}\tag{14}$$

We consider the following case first, "pole-type":

$$A^P(s,t)=\tfrac{1}{2}f_1\left(1-\frac{i\epsilon}{\sqrt{-t}}\right)s^{\alpha_+} - \frac{if_1}{4\sqrt{\pi}}\frac{1}{(-t\ln s)^{1/2}}\phi_+(s,t)s^{\alpha_+}+\text{c.c.},\tag{15}$$

where

$$\phi_+(s,t)=\sum_{n=0}^{\infty}\frac{(a-\alpha_+)^n(\ln s)^n}{(n-\frac{1}{2})n!}\,.$$

The first term in (15) and its complex conjugate behave like complex conjugate poles with s-independent residues. We call the two terms CP. The second term in (15) and its complex conjugate behave like complex conjugate poles but with s-dependent residues. These terms involve a summation. The contribution of the Nth term in the summation we call Σ_N. Thus

$$A^P(s,t)=\text{CP}+\Sigma_0+\Sigma_1+\Sigma_2+\cdots\,.$$

In Figs. 3(a), 3(b), 3(c) we have plotted the *exact* value (the solid line) of $A^P(s,t)$ with $f_1=1$ and with $s^{\bar{a}}$ factored out ($\bar{a}=a+\frac{1}{4}\epsilon^2$). The plots are for three specific s values $s=5$, 20, and 200, each over a t range from 0 to -0.5. We have chosen $\epsilon=0.1$ (note that the specific value of a never enters). We note that CP is reasonably close to the exact value even up to $s=200$, the error being 10–15%. $\text{CP}+\Sigma_0$ is even closer and $\text{CP}+\Sigma_0+\Sigma_1$ is almost identical to the exact value. The higher series terms are, therefore, negligible and we can write the approximation

$$A^P(s,t)\approx\text{CP}+\Sigma_0+\Sigma_1\,.$$

Let us point out a few salient facts. The series terms above and in $\varphi_\pm(s,t)$ involve powers of $\ln s$. These terms, however, are multiplied by powers of $(a-\alpha_\pm)$ which is the vector distance of $\alpha_\pm$ from α_c. For t's not too large they are effectively proportional to α_I and, therefore, to ϵ. Thus if ϵ is small the higher terms will get smaller. To this fact we add that the factor $1/n!$ also greatly reduces the contribution of the higher terms. The above approximation will break down if ϵ and t are large and if s is extremely large. Nevertheless, what is interesting is that we have chosen a very realistic example ($\epsilon=0.1$, $c=1$, $t_0=0$) and have found the approximation to work very well even up to $s = 200$.

We next consider the case "cut-type":

$$A^c(s,t) = \tfrac{1}{2} g_1\left(-\epsilon - i\sqrt{-t} + i\frac{\epsilon^2}{4\sqrt{-t}}\right)s^{\alpha_+} - \frac{g_1}{4\sqrt{\pi}}\left(1 - \frac{i\epsilon}{2\sqrt{-t}}\right)\frac{1}{(\ln s)^{1/2}}\phi_+(s,t)s^{\alpha_+} + \text{c.c.} \tag{16}$$

In Figs. 3(a′), 3(b′), and 3(c′) we have plotted the *exact* value (the solid line) of $A^c(s,t)$ with $g_1 = 1$, $\epsilon = 0.1$ and $s^{\bar{\alpha}}$ factored out. We again find, with the same s, t, and ϵ values as in the previous case, that

$$A^c(s,t) \approx \text{CP} + \Sigma_0 + \Sigma_1$$

even at $s = 200$.

We thus conclude that the approximation of keeping only a few terms (in particular, just two) in the series is very good for the "pole-type" term as well as the "cut-type" term. Thus for the general amplitude $A(s,t)$ $(= A^p + A^c)$ we expect

$$A(s,t) \approx \text{CP} + \Sigma_0 + \Sigma_1$$

to hold. That is, a complex-pole approximation with energy-dependent residues should hold. The energy dependence of the residues is, however, of the $\ln s$ type and will *not* change rapidly with energy.

The above discussions strongly indicate that the s dependence of the scattering amplitudes is largely determined by $s^{\alpha_\pm}$ even in the presence of cuts. It also indicates that the complex-pole analyses[4] of different high-energy data carried out in recent years is quite consistent with the model of the partial-wave amplitude given by (1). In these analyses the residues were assumed to be s-independent. This effectively means that the $\ln s$ terms are approximated as constants. One can improve upon the phenomenological formulation by explicitly including $\ln s$ terms given by our model. The inclusion of additional $\ln s$ terms, however, will *not* involve additional parameters. The same parameters that appeared earlier, e.g., f_1, g_1, and ϵ will appear as coefficients of the $\ln s$ terms.

In the paper of Ball, Marchesini, and Zachariasen[6] where the feasibility of the complex-pole approximation was first pointed out, it was the "pole-type" case that was considered as their example. We have found here that even in the "cut-type" case the dominant energy dependence is $s^{\alpha_\pm}$. Furthermore, we have shown that any corrections to the residues due to $\ln s$ terms can be explicitly taken into account without adding new parameters.

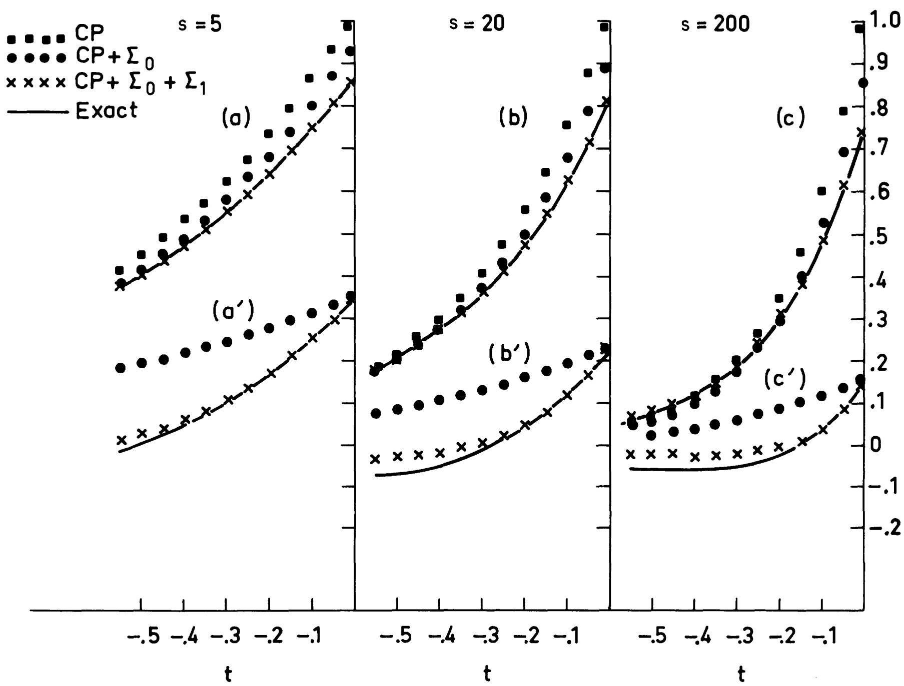

FIG. 3. Comparison of the exact result (solid line) with the approximations mentioned in the text for $\epsilon = 0.1$ and $s = 5$, 20, and 200. Both the pole-type and the cut-type cases are considered. The quantity $s^{\bar{\alpha}}$ is factored out. (For units see Ref. 3.)

B. Tip of the cut contribution

In the above section we have found that for small ϵ (0.1) and t not too large only a few terms of $\varphi_\pm(s,t)$ are important even up to $s=200$—the energy dependence being largely determined by $s^{\alpha_\pm}$. As s is increased, more and more terms of the series will become important. At extremely high energies higher terms in the series will dominate. As we shall see below it is at this stage, well beyond the energies of the current measurements, that the tip of the cut will begin to dominate giving an $s^{\alpha_c}/(\ln s)^{3/2}$ dependence rather than $s^{\alpha_\pm}$. We note,

$$\phi_+(s,t)=\sum_{n=0}^{\infty}\frac{(\alpha_c-\alpha_+)^n(\ln s)^n}{(n-\frac{1}{2})n!}$$

$$\xrightarrow[s\to\infty]{}\sim\sum_{N}^{\infty}\frac{(\alpha_c-\alpha_+)^n(\ln s)^n}{(n+1)!}\ ,$$

where lower-order terms are neglected for large s and, for the higher-order terms, $n-\frac{1}{2}$ replaced by $n+1$. Now

$$\sum_{N}^{\infty}\frac{(\alpha_c-\alpha_+)^n(\ln s)^n}{(n+1)!}\approx\frac{1}{(\alpha_c-\alpha_+)\ln s}\,e^{(\alpha_c-\alpha_+)\ln s}=\frac{s^{\alpha_c-\alpha_+}}{(\alpha_c-\alpha_+)\ln s}$$

and, therefore,

$$\phi_+(s,t)\underset{s\to\infty}{\sim}\frac{s^{\alpha_c-\alpha_+}}{(\alpha_c-\alpha_+)\ln s}\ . \tag{17}$$

Thus from (10) substituting the above expression for $\varphi_+(s,t)$ we get

$$\int_{-\infty}^{\alpha_c}dj\,\frac{s^j(\alpha_c-j)^{1/2}}{j-\alpha_+}\underset{s\to\infty}{\sim}\frac{\sqrt{\pi}}{2(\alpha_c-\alpha_+)}\,\frac{s^{\alpha_c}}{(\ln s)^{3/2}}\ . \tag{18}$$

The result on the right-hand side can be confirmed by explicitly doing the integral on the left for asymptotic s (when the poles are on the physical sheet there would be an added pole contribution to the above expression).

The amplitude $A(s,t)$ for asymptotic s is given by

$$A(s,t)\underset{s\to\infty}{\sim}-\frac{1}{2\sqrt{\pi}}\left(\frac{g_1(\alpha_c-\alpha_0)-\epsilon f_1}{(\alpha_c-\alpha_+)(\alpha_c-\alpha_-)}\right)\frac{s^{\alpha_c}}{(\ln s)^{3/2}}$$

$$=\frac{1}{2\sqrt{\pi}}\left(\frac{\epsilon f_1-g_1(\frac{1}{4}\epsilon^2-t)}{(\frac{1}{4}\epsilon^2+t)^2-\epsilon^2 t}\right)\frac{s^{\alpha_c}}{(\ln s)^{3/2}}\ . \tag{19}$$

In Fig. 4 we have plotted the above (tip of the cut) contribution and compared with the exact result (solid line). Here again we divide out $s^{\overline{a}}$ ($\overline{a}=a+\frac{1}{4}\epsilon^2$). As before, we consider two cases, the "pole-type" ($f_1=1$, $g_1=0$), and the "cut-type"

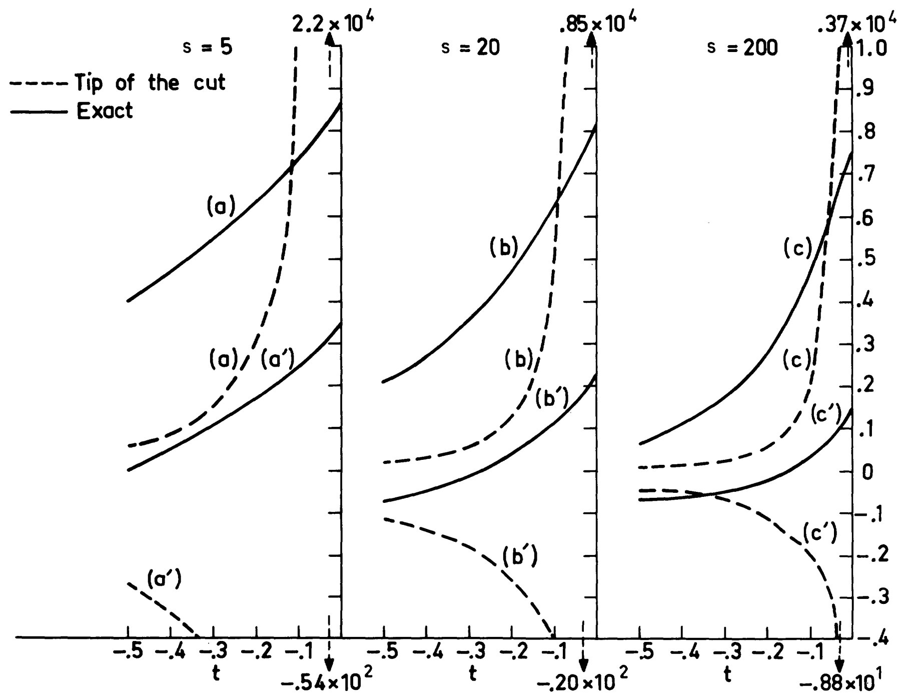

FIG. 4. Comparison of the exact result (solid line) with the tip of the cut contribution given by (19) for $\epsilon=0.1$ and $s=5$, 20, and 200. Both the pole-type and the cut-type case are considered. The quantity $s^{\overline{a}}$ is factored out.

($f_1 = 0$, $g_1 = 1$). We note, for both cases, an enormous difference between the exact result and the above tip of the cut contribution. For the "pole-type" case the denominator $(\alpha_c - \alpha_+)(\alpha_c - \alpha_-)$ gives rise to a sharp peak at $t = 0$ with the value four orders of magnitude larger than the exact result. Away from $t = 0$, the contribution falls off very quickly, going below the exact result around $t = -0.1$. In the "cut-type" case the contribution of the tip of the cut is negative at $t = 0$. The reason for this is the following: From the exact results at $s = 5$, 20, and 200 we note that $A(s,t)$ is positive at $t = 0$ but it has a zero at a t value which moves towards $t = 0$ as s is increased. For asymptotic s this zero moves past the point $t = 0$ so that $A(s,t)$ now starts out negative. As far as magnitudes are concerned we find here again as in the "pole-type" case that the tip of the cut contribution is very different from the exact result.

At what s will the tip of the cut dominate? To answer this question we look at the series expansion of $\varphi_+(s,t)$ and note that to achieve the dependence $s^{\alpha_c}/(\ln s)^{3/2}$ the higher-order terms must be large. Thus for n sufficiently large,

$$\frac{|\alpha_c - \alpha_+|\ln s}{n} \gtrsim 1 .$$

If we take $n = 5$ then for $t = 0$ and $\epsilon \sim 0.1$ this will give $\ln s > 2000$, whereas for $t = -0.5$ it will give $\ln s > 10$. These are extremely large energies, well beyond the range of our present interest. We also remark that the dominance of the tip of the cut depends on both s and t and that the correct asymptotic limit should be $|\alpha_c - \alpha_+|\ln s \gg 1$.

One can improve upon the above asymptotic expression and include higher-order terms. The integral in (10) can be expressed using the expansion

$$\frac{1}{j-\alpha_+} = \frac{1}{\alpha_c - \alpha_+}\sum_{n=0}^{\infty}\left(\frac{j-\alpha_c}{\alpha_c - \alpha_+}\right)^n$$

to give

$$\int_{-\infty}^{\alpha_c} dj\ \frac{s^j(\alpha_c - j)^{1/2}}{j-\alpha_+} = \frac{s^{\alpha_c}}{(\alpha_c-\alpha_+)(\ln s)^{3/2}} \times \sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{2})}{(\alpha_c-\alpha_+)^n(\ln s)^n} .$$

Thus adding higher-order terms (i.e., lower powers in $\ln s$) in the sense of an asymptotic series one can achieve the exact result.

One may wish to express the integral in terms of the above series involving explicitly the contribution of the tip of the cut. However, we prefer to write the expression in terms of $s^{\alpha_\pm}$ as in (10) because it seems more meaningful to do so. First of all, as we have seen, by doing so one needs only very few terms in the series for the s values of interest. Secondly, there is a direct connection between the expansion involving in $s^{\alpha_\pm}$ and ordinary Regge poles, especially when the cuts are weak.

IV. THE DOUBLE POLE AT $t = t_0$

In the expressions written down earlier for $A(s,t)$ there were terms proportional to $1/[c(t_0-t)]^{1/2}$. At first glance these terms appear to give an infinite contribution to $A(s,t)$. As we shall soon see they actually correspond to a double-pole contribution at $t = t_0$.[7] The two poles α_+ and α_- collide at $t = t_0$ and, hence, the amplitude there has a double pole. As a consequence, $A(s,t)$ will have the behavior $s^\alpha \ln s$ at that point. Let us express $A(s,t)$ as a sum of two terms, $A_1(s,t)$ which is regular at $t = t_0$, and $A_2(s,t)$ which has the factor $1/[c(t_0-t)]^{1/2}$.

$$A(s,t) = A_1(s,t) + A_2(s,t),$$

$$A_2(s,t) = \frac{-i}{4[c(t_0-t)]^{1/2}}(f_1 - \tfrac{1}{2}\epsilon g_1) \times\left(\epsilon(s^{\alpha_+} - s^{\alpha_-}) + \frac{1}{(\pi \ln s)^{1/2}}(\phi_+ s^{\alpha_+} - \phi_- s^{\alpha_-})\right).$$

The above case is for the poles on the unphysical sheet ($\epsilon > 0$). Corresponding expressions for poles on the physical sheet can be also written.

Now let us look at $t = t_0$. Since $\alpha_I = \epsilon[c(t_0-t)]^{1/2}$, we find at $t = t_0$,

$$A_2(s,t) = \tfrac{1}{2}(f_1 - \tfrac{1}{2}\epsilon g_1)s^{\alpha_R}\ln s \times\left(\epsilon^2 + \frac{\epsilon}{(\pi\ln s)^{1/2}}(\phi - \bar{\phi})\right), \qquad (20)$$

where

$$\phi(s,t_0) = \sum_{n=0}^{\infty}\frac{(\alpha_c - \alpha_R)^n(\ln s)^n}{(n-\frac{1}{2})n!} ,$$

$$\bar{\phi}(s,t_0) = \sum_{n=0}^{\infty}\frac{(\alpha_c - \alpha_R)^n(\ln s)^n}{(n+\frac{1}{2})n!} .$$

We have thus exhibited explicitly that precisely at $t = t_0$ there is a different s dependence in $A(s,t)$ than at other t's and this dependence corresponds to the double-pole contribution $s^\alpha \ln s$. The above case was for the poles on the unphysical sheet for which eventually, at asymptotic s, the tip of the cut will dominate at all t and the double-pole contribution will be negligible. However, at present energies its contribution need not be negligible. If the poles collide at a point on the physical sheet then at all energies, including asymptotic, one will have the double-pole contribution.

An interesting question is whether such a double-pole effect can be observed experimentally. One area where one may look for this effect involves the ρ exchange in πN scattering. From the $d\sigma/dt$ data the trajectory intercept for ρ seems to be lower than that obtained from the cross-section difference, $\Delta\sigma$, where only the value of the trajectory at $t=0$ is involved. An intriguing possibility[7] is that the poles collide at $t=0$ giving an effectively larger s-dependence, $s^{\alpha}\ln s$, there than at other t values where the dependence is s^{α} This possibility is being investigated presently.[8] In the previous phenomenological analyses such an effect was ignored[4].

V. POLE ON THE REAL AXIS, THE ABSORPTION MODEL, AND THE SIGN OF THE CUT

Suppose, for some reason, the poles are not complex for $t<0$ (we consider $t_0=0$ in this section). That is suppose $\epsilon=0$ in (1) without, at the same time, having F_2 and G_1 (or f_1 and g_1) vanishing. The amplitude we will have then is

$$\frac{f_1+g_1(j-\alpha_c)^{1/2}}{j-\alpha_0}=\frac{f_1}{j-\alpha_0}+\frac{g_1(j-\alpha_c)^{1/2}}{j-\alpha_0}\ .$$

The first term on the right is a pure pole. The second (the "cut-type" term) has a pole but, as explained in Sec. III, the contributions to the residue from the upper and lower half semicircles cancel each other due to the $(j-\alpha_c)^{1/2}$ term. The second term, therefore, is a pure cut term and we have

$$A(s,t)=f_1 s^{\alpha_0}-\frac{P}{\pi}\int_{-\infty}^{\alpha_c} dj\,\frac{s^j g_1(\alpha_c-j)^{1/2}}{j-\alpha_0}\ ,$$

where P means principal part. To obtain this principal part we simply go back to (12) and put $\epsilon=0$. We then obtain

$$A(s,t)=f_1 s^{\alpha_0}-\frac{g_1}{2\sqrt{\pi}}\,\frac{1}{(\ln s)^{1/2}}\,\phi_0(s,t)\,s^{\alpha_0}\ ,$$

$$\phi_0(s,t)=\sum_{n=0}^{\infty}\frac{(\alpha_c-\alpha_0)^n(\ln s)^n}{(n-\frac{1}{2})\,n\,!}\ . \qquad (21)$$

Here again as in the previous case of complex poles only the first two terms in the series of $\varphi_0(s,t)$ will be important. The demonstration of this fact is no different from that in Sec. III. In other words, *even when the poles are on the real axis the power dependence of $A(s,t)$ is largely given by s^{α_0} for s as high as 200* BeV^2. *It is not at all crucial for our arguments about s dependence that the poles be complex.*

Let us now consider the absorption model.[9] This model is not expressed in terms of the language which is inherent in our formulation. In the absorption model the cut term is a separate, independent, entity not sharing a common D function with the pole as is the case in our model. According to the model (appropriately modified for the square-root situation) the pole is on the real axis with the amplitude given by

$$\frac{f_1}{j-\alpha_0}+\bar{g}_1(j-\alpha_c)^{1/2}\ .$$

Note the absence of the denominator, $j-\alpha_0$, in the second term. The amplitude $A(s,t)$ is then given by.

$$A(s,t)=f_1 s^{\alpha_0}-\frac{\bar{g}_1}{2\sqrt{\pi}}\,\frac{s^{\alpha_c}}{(\ln s)^{3/2}}\ . \qquad (22)$$

One major difference between expression (22) of the absorption model and the expression (21) predicted by our model is, of course, the s dependence of the second term. In our case the second term has the basic power dependence s^{α_0} whereas in the absorption model the tip of the cut is the only contributor.

The other major difference is that $\bar{g}_1$ in the absorption model must be positive. This is necessary because the zeros in the amplitude are assumed to arise from the *cancellation* between the first and the second terms in (22), with f_1 and $\bar{g}_1$ having no zeros in t. In our case the sign of g_1 in (21) does not necessarily play a crucial role in giving the zeros of $A(s,t)$. It is the zeros in t of f_1 ($\approx F_1+F_2 t$) which are responsible for the zeros of $A(s,t)$ (see Sec. VI.).

In order to discuss the question of the sign of the cut let us use the traditional definition, i.e., the definition in terms of the sign of the asymptotic cut contribution at $t=0$. The absorption sign then corresponds to a negative relative cut contribution and hence positive $\bar{g}_1$. To discuss our model let us consider (21) and put $t=0$, where we also have $\alpha_0=\alpha_c$. From $\varphi_0(s,t)$ above we obtain

$$\phi_0(s,t=0)=-2.$$

Thus at $t=0$,

$$A(s,t)=f_1 s^{\alpha_0}+\frac{g_1}{\sqrt{\pi}}\,\frac{s^{\alpha_0}}{(\ln s)^{1/2}}\ . \qquad (23)$$

We note that the power of the second term is $s^{\alpha}/(\ln s)^{1/2}$ rather than $s^{\alpha}/(\ln s)^{3/2}$. This is a consequence of the fact that at $t=0$ the pole lies on top of the cut.[10] Now, for our model the absorption sign will correspond to a *negative* g_1. As it will be clear in the next sections the sign of g_1 is not too crucial in determining the zeros of $A(s,t)$ but that it contributes to the phase of the complex residue and can be determined phenomenologically.

VI. THE RELATION OF THE STRENGTH OF THE CUT TO THE CROSSOVER ZEROS AND DIPS

In order to discuss the problem of the zeros of $A(s,t)$ let us make the following simplifications which allow us to bring out the most important points. We will ignore the $\ln s$ terms, and, except in the powers α_+ and α_-, we will also ignore ϵ. We get for the unphysical sheet case ($\epsilon>0$), and $t_0=0$, $c=1$, the following:

$$A(s,t)\sim\tfrac{1}{2}(f_1-ig_1\sqrt{-t})\,s^{\alpha_+}+\tfrac{1}{2}(f_1+ig_1\sqrt{-t})\,s^{\alpha_-}$$
$$\sim\tfrac{1}{2}(F_1+F_2t-iG_1\sqrt{-t})\,s^{\alpha_+}+\tfrac{1}{2}(F_1+F_2t+iG_1\sqrt{-t})\,s^{\alpha_-}\,. \tag{24}$$

We might call $(F_1+F_2t\mp iG_1\sqrt{-t})$ the "residues" of the pole. Combining appropriate terms in the above expression we can write

$$A(s,t)\sim(F_1+F_2t)\,s^{\alpha_R}\cos(\alpha_I\ln s)+G_1\sqrt{-t}\;s^{\alpha_R}\sin(\alpha_I\ln s)\,.$$

For small α_I (and small $\alpha_I\ln s$) the zeros of $A(s,t)$ are determined, to a good approximation, by the solutions of

$$F_1+F_2t=0\,, \tag{25}$$

where, we remind ourselves, that *F_1 and F_2, in general, are functions of t*. The contribution of the G_1 term will be proportional to $(G_1\epsilon\ln s)t$ which will be small compared to F_2t unless G_1 is extremely large. Of course, the presence of the G_1 term will give a correction to the position of the zero which will depend on s. Thus the zero will shift as a function of s.

Now the strength of the cut is manifested in two different ways. One through the magnitude of ϵ and other through the *ratios* $|F_2/F_1|$ and $|G_1/F_1|$ at some t value, say $t=0$. It is entirely possible that two different amplitudes (e.g., spin flip and spin nonflip) have the same ϵ but different $|F_2/F_1|$ and $|G_2/F_1|$. In other words, it is possible that F_2 and G_1 (as well as ϵ) are small but the two F_1's are very different. In what follows we will keep ϵ small and constant but designate by the strength of the cut the magnitudes of $|F_2/F_1|$, $|G_2/F_1|$ at $t=0$.

We return to the factored form discussed in the Introduction. We can divide the cut strength in three parts:

(i) Weak cut. This corresponds, in the factored form, to the situation where $\beta_i'\ll\beta_i$ ($i=1,2$) so that $F_2\ll F_1$ and $G_1\ll F_1$, and, therefore, $g_1\ll f_1$. This will then correspond to the situation where the "pole-type" term is the dominant term.

As discussed earlier, this "pole-type" case would resemble very closely the simple pole situation. The F_1 will presumably be given by the dual-resonance model,

$$F_1(t)\approx\frac{1}{\Gamma(\alpha_0)}$$

and consequently the zeros of $A(s,t)$ will be given $\alpha_0=0$, etc. The spin-flip amplitude will come under this category. The position of dips in $d\sigma/dt$ corresponding to the appropriate zeros of $1/\Gamma(\alpha_0)$ will remain unchanged from the usual simple-pole prescription. This is what is observed experimentally and hence cuts here are weak.

(ii) Strong cut. This corresponds in the factored form to $\beta_i'\sim\beta_i$ ($i=1,2$) so that $F_2\sim F_1\sim G_1$ and $f_1\sim g_1$. This will involve an approximately equal mixture of "pole-type" and "cut-type" terms. Here we expect

$$F_1\approx\frac{1}{\Gamma(\alpha_0)}+\text{perturbations}\,,$$

where "perturbations" refer to any modification from the dual-resonance picture due to cuts.

If we ignore, for illustrative purposes, this perturbation and take $F_1\approx\alpha_0$, the first term in $1/\Gamma(\alpha_0)$, and also take F_2 and G_1 to have comparable magnitude to F_1 at $t=0$ (e.g., $F_2\approx1$, $G_1\approx1$) then the zeros will be given by

$$\alpha_0+t\approx0\,.$$

Clearly the zero will be moved considerably from the prediction of the dual resonance model. The nonflip amplitude will come under this category. Here we have the so-called crossover zeros which are at much smaller t values than predicted by the dual-resonance model. Hence the cuts here are strong.

(iii) Extremely strong cut. This is an interesting extreme case where $\beta_i'\gg\beta_i$ ($i=1,2$) so that $F_1\ll F_2$, $G_1\ll F_2$. We are then back to the "pole-type" situation with the numerator of $A(j,t)$ being essentially $F_2(j-\alpha_c)$. The zero is moved closer and will lie at $t=0$. This case has not been observed experimentally.

A word of caution about some of the assumptions made above which are not related to the basic results of our model. First of all, in this section we have neglected the ln$s$ terms. Secondly, in the strong cut example above we neglected the effect of perturbations on F_1 due to cuts which may well be important. Finally the G_1 term, neglected in the residues above, will also contribute to the displacement of the zeros. The point we are trying to emphasize all along is that the stronger the cut, the further the zero moves from the dual-resonance prediction. Furthermore in nature there seem to be two types of amplitudes. One which is very much like simple Regge poles and, therefore, has small corrections to the Regge residues. The other, which is very different, has a structure which can be explained only in terms of strong cuts. We remind ourselves that by strong cuts we do not necessarily mean large ϵ but rather large corrections to the residues coming from F_2 and G_1.

VII. LOCATION OF POLES, EXTRAPOLATION OF RESIDUES, AND THE SIGN OF THE CUT

A. The case of poles on physical sheet for $t<t_0$

Up to now we confined our attention only to the case where the $\alpha_\pm$ poles for $t\leq t_0$ were on the unphysical sheet ($\epsilon>0$). Let us quickly summarize the situation when the poles are on *the physical sheet* corresponding to $\epsilon<0$. We have for this case

$$(\alpha_\pm-\alpha_c)^{1/2}=-\tfrac{1}{2}\epsilon\pm[c(t-t_0)]^{1/2},\quad t>t_0$$
$$=-\tfrac{1}{2}\epsilon\pm i[c(t_0-t)]^{1/2},\quad t<t_0 .$$

In particular for $t\leq t_0$

$$\alpha_\pm=\alpha_c+\tfrac{1}{4}\epsilon^2+c(t-t_0)\mp i\epsilon(t_0-t)^{1/2} .$$

For $\epsilon<0$, both α_+ and α_- are on the physical sheet when $t\leq t_0$. The residue of the α_+ pole is then

$$\hat\gamma_+=\frac{f_1+g_1(\alpha_+-\alpha_c)^{1/2}}{1+\epsilon/2(\alpha_+-\alpha_c)^{1/2}}$$
$$=\left((f_2-\epsilon g_1)+ig_1[c(t_0-t)]^{1/2}+\frac{i\epsilon}{2[c(t_0-t)]^{1/2}}(f_1-\tfrac{1}{2}\epsilon g_1)\right)$$

and, also

$$A(s,t)=\hat\gamma_+s^{\alpha_+}+\hat\gamma_-s^{\alpha_-}-\frac{1}{\pi}\int_{-\infty}^{\alpha_c}dj\,s^j\,\mathrm{disc}\,A(j,t) .$$

Using the fact that for this case

$$\int_{-\infty}^{\alpha_c}dj\,\frac{s^j(\alpha_c-j)^{1/2}}{j-\alpha_+}=\pi(\alpha_+-\alpha_c)^{1/2}+\tfrac{1}{2}\left(\frac{\pi}{\ln s}\right)^{1/2}\phi_+(s,t)\,s^{\alpha_+} ,$$

we obtain ($\epsilon<0$)

$$A(s,t)=\tfrac{1}{2}\hat\gamma_+(t)\,s^{\alpha_+}-\frac{i}{4\sqrt{\pi}}\left(g_1-\frac{i}{[c(t_0-t)]^{1/2}}(f_1-\tfrac{1}{2}\epsilon g_1)\right)\times\frac{1}{(\ln s)^{1/2}}\phi_+(s,t)\,s^{\alpha_+}+\text{c.c.}$$

We notice that $\hat\gamma_+$ here is the same quantity as the one obtained by taking the complex conjugate of $\bar\gamma_+$ defined in (13) for the unphysical sheet case, remembering that ϵ in $\hat\gamma_+$ above is a negative quantity.[11] In fact $A(s,t)$ for the *physical-sheet case* above is itself the *complex conjugate of $A(s,t)$ in (13) for the unphysical-sheet case, again remembering that ϵ in $A(s,t)$ above is negative.* Thus all previous discussions on double pole, asymptotic behavior, zeros etc. can be simply carried over to the physical sheet case by this prescription.

B. The region above threshold

The discussions so far have been confined to the region $t\leq t_0$. We now consider the region above the t-channel threshold. In particular, we will be interested in the residues and poles in the neighborhood of resonances.

Let us consider the simplified but realistic case treated in the last few sections, i.e., $t_0=0$, $c=1$, $\alpha_c=a$. We first consider the continuation of the poles which, for $t\leq0$, are on the unphysical sheet ($\epsilon>0$). The two roots of the denominator in the partial-wave amplitude above threshold then are

$$(\alpha_k-\alpha_c)^{1/2}=-\tfrac{1}{2}\epsilon\pm\sqrt{t}\quad(k=1,2) .$$

Thus for $\epsilon>0$, α_1 $(=\alpha_-)$ will be on the physical sheet and α_2 $(=\alpha_+)$ on unphysical sheet for t above threshold. Hence the α_+ pole which was on the unphysical sheet for $t\leq0$ will remain in the unphysical sheet for t above threshold, whereas the α_- pole which was also on the unphysical sheet for $t\leq0$, will, however, go into the physical sheet for t above threshold. The residue $\hat\gamma_-$ for α_- above threshold can be obtained in the usual manner,

$$\hat\gamma_-=[(f_1-\epsilon g_1)+g_1\sqrt{t}-\frac{\epsilon}{2\sqrt{t}}(f_1-\tfrac{1}{2}\epsilon g_1)] .$$

For $\epsilon<0$, both the poles α_+ and α_- are on the physical sheet for $t\leq0$. However, just as before, for t above threshold α_1 $(=\alpha_+)$ will be on the physical sheet and α_2 $(=\alpha_-)$ will be on the unphysical sheet. The residue of α_+ pole for t above threshold is then

$$\hat{\gamma}_+ = [(f_1-\epsilon g_1)+g_1\sqrt{t} - \frac{\epsilon}{2\sqrt{t}}(f_1-\tfrac{1}{2}\epsilon g_1)].$$

Note that this is the *same* expression as $\hat{\gamma}_-$, obtained above for the $\epsilon>0$ case, except that the sign of ϵ is now negative. The fact that the expressions for the residues are the same is not surprising since we are considering for both cases the residue of the same pole, α_1, which is on the physical sheet above threshold.

Below we summarize the total situation:

(i) Poles on unphysical sheet for $t\leq 0$ $(\epsilon>0)$.

$$A(s,t) = \text{cut } (\approx \text{complex poles}), \quad t\leq 0$$
$$= \hat{\gamma}_- s^{\alpha-} + \text{cut}, \quad t>0.$$

(ii) Poles on physical sheet for $t\leq 0$ $(\epsilon<0)$.

$$A(s,t) = \hat{\gamma}_+ s^{\alpha+} + \hat{\gamma}_- s^{\alpha-} + \text{cut } (\approx \text{complex poles}), \quad t\leq 0$$
$$= \hat{\gamma}_+ s^{\alpha+} + \text{cut}, \quad t\geq 0.$$

Note that in the limit

$$\epsilon \to 0, \quad g_1 \to 0,$$

we have $\hat{\gamma}_+ = \hat{\gamma}_- = \gamma$ and $\alpha_+ = \alpha_- = \alpha$ so that for the two cases

$$\text{(i)} \quad \text{cut} \to \tfrac{1}{2}(\hat{\gamma}_+ s^{\alpha+} + \hat{\gamma}_- s^{\alpha-}) \to \gamma s^{\alpha}, \quad t\leq 0$$
$$\to 0, \quad t\geq 0$$
$$\text{(ii)} \quad \text{cut} \to -\tfrac{1}{2}(\hat{\gamma}_+ s^{\alpha+} + \hat{\gamma}_- s^{\alpha-}) \to -\gamma s^{\alpha}, \quad t\leq 0$$
$$\to 0, \quad t\geq 0.$$

Thus, as expected, when the cut strength vanishes, we recover, for both cases, the simple pole result,

$$A(s,t) \to \gamma s^{\alpha}.$$

C. Determination of the sign of the cut and location of poles

Let us continue to consider the very simplified case of neglecting the contributions of the $\ln s$ terms and ϵ in the residues. In other words, we will keep the poles complex but in the approximation of the discontinuity integral neglect the $\ln s$ and ϵ dependent terms. We find the following two cases:

(i) $t\leq 0$.

$$A(s,t) \sim \tfrac{1}{2}(f_1 - g_1\sqrt{-t})s^{\alpha+} + \tfrac{1}{2}(f_1 + ig\sqrt{-t})s^{\alpha-} \quad (\epsilon>0 \text{ unphys.})$$
$$A(s,t) \sim \tfrac{1}{2}(f_1 + ig_1\sqrt{-t})s^{\alpha+} + \tfrac{1}{2}(f_1 - ig_1\sqrt{-t})s^{\alpha-} \quad (\epsilon<0 \text{ phys.}).$$

Here "unphys." and "phys." indicate the sheet on which the poles are located for $t\leq 0$. For the resonance widths we have

(ii) $t>0$.

$$\text{Resonance width} \approx \hat{\gamma}_-(t) \sim f_1 + g_1\sqrt{t} \quad (\epsilon>0)$$
$$\text{Resonance width} \approx \hat{\gamma}_+(t) \sim f_1 + g_1\sqrt{t} \quad (\epsilon<0).$$

As previously pointed out the formula for the width remains unchanged irrespective of where the poles are located for $t\leq 0$, whereas the residues for $t\leq 0$ have opposite signs for the imaginary part (i.e., opposite phase) for the two cases. On the basis of our simplified example above we can determine the sign of the cut (i.e., the sign of g_1) by a careful analysis of the resonance data. To determine the location of the poles (for $t\leq 0$) we now have to go to the high-s and $t\leq 0$ scattering region. For instance, if we assume f_1 positive, then if one finds that g_1 is positive from the resonance data and the phase of the residue of the α_+ term negative from the scattering data then the poles must be on the unphysical sheet for $t\leq 0$.

Recent phenomenological analyses which ignore the $\ln s$ terms do, indeed, show that the phase is negative from the scattering data.[4] An analysis of the resonance data is presently being carried out.[7] In this, as in any careful analysis, the $\ln s$ and the ϵ terms will be kept. Whether these terms will change the above discussion remains to be seen. But in principle, it is clear, that the sign of the cut and the location of the poles can be determined on the basis of our partial wave model.

VIII. FACTORIZATION

In the Introduction we had indicated that one possible way of expressing the numerator function in the partial-wave amplitude $A(j,t)$ is to write it in a factorized form. Such an assumption, though perhaps *ad hoc* in nature, insures that the residues at the pole are factorizable. The question we would now like to consider is whether $A(s,t)$ itself is factorizable.

Let us return to the factored form mentioned in the Introduction,

$$[\beta_1 + \beta_1'(j-\alpha_c)^{1/2}][\beta_2 + \beta_2'(j-\alpha_c)^{1/2}] = F_1 + F_2(j-\alpha_c) + G_1(j-\alpha_c)^{1/2}.$$

We consider, specifically, the point $j=\alpha_+$ where it is easy to show that

$$F_1+F_2(\alpha_+-\alpha_c)+G_1(\alpha_+-\alpha_c)^{1/2}=f_1+g_1(\alpha_+-\alpha_c)^{1/2}.$$

Thus the above factored form allows us to write

$$\begin{aligned} f_1+g_1(\alpha_+-\alpha_c)^{1/2} &= [\beta_1+\beta_1'(\alpha_+-\alpha_c)^{1/2}] \\ &\quad\times[\beta_2+\beta_2'(\alpha_+-\alpha_c)^{1/2}] \\ &= (\gamma_1 e^{i\phi_1})(\gamma_2 e^{i\phi_2}) \\ &= \gamma_1\gamma_2 e^{i(\phi_1+\phi_2)}, \end{aligned}$$

where the γ_i's are positive quantities and the φ_i's are the phases. Both are functions of t only. The indices on the right, on γ_i and φ_i correspond to the different vertices for a given residue. Now the residue $\bar{\gamma}_+$ defined in (13) is also given by

$$\begin{aligned} \bar{\gamma}_+ &= \frac{f_1+g_1(\alpha_+-\alpha_c)^{1/2}}{1+\epsilon/2(\alpha_+-\alpha_c)^{1/2}} \\ &= \frac{\gamma_1\gamma_2 e^{i(\phi_1+\phi_2)}}{1+\epsilon/2(\alpha_+-\alpha_c)^{1/2}}. \end{aligned}$$

We note that the denominator depends only on the trajectory function and not on the vertices. It can, therefore, be easily absorbed (and, say, equally divided) into the numerator. We can then write

$$\bar{\gamma}_+ = \bar{\gamma}_1\bar{\gamma}_2 e^{i(\bar{\phi}_1+\bar{\phi}_2)}$$

which is in an explicit factorized form.

The amplitude $A(s,t)$ given in (12) can also be written as

$$A(s,t)=\tfrac{1}{2}\bar{\gamma}_+(t)\left[1-\frac{1}{2\sqrt{\pi}}\,\frac{1}{(\alpha_+-\alpha_c)^{1/2}}\,\frac{\phi_+(s,t)}{(\ln s)^{1/2}}\right]s^{\alpha_+} + \text{c.c.}$$

Again the terms in the square bracket depend only on the trajectory function and not on the vertices. They can, therefore, be easily absorbed into the factored function $\bar{\gamma}_+(t)$. One can then write

$$A(s,t)=\tfrac{1}{2}r_1r_2e^{i(\theta_1+\theta_2)}s^{\alpha_+}+\tfrac{1}{2}r_1r_2e^{-i(\theta_1+\theta_2)}s^{\alpha_-},$$

where r_i and θ_i are functions of t and s. We notice that each individual term in the right-hand side above is of the factored form but the sum is not. To be more explicit

$$A(s,t)=r_1r_2\cos(\theta_1+\theta_2+\alpha_I\ln s)\,s^{\alpha_R}.$$

Thus because of the cosine term $A(s,t)$ is *not* of the factored form. That is it cannot be written as a product of two functions, each depending only on a single vertex.

To understand the deviation from the predictions of exact factorization let us introduce explicit indices in $A(s,t)$,

$$A_{ij}(s,t)=r_ir_j\cos(\theta_i+\theta_j+\alpha_I\ln s)s^{\alpha_R}.$$

We then have for the difference, $A_{12}{}^2-A_{11}A_{22}$, the following:

$$A_{12}{}^2-A_{11}A_{22}=r_1{}^2r_2{}^2\sin^2(\theta_1-\theta_2).$$

For exact factorization to hold the left-hand side should vanish ($A_{12}{}^2=A_{11}A_{22}$). Such a factorization is known to hold for simple Regge poles. In the presence of cuts, as our model indicates above, explicit factorization is not possible *unless* the phases θ_1 and θ_2 are equal or more precisely the phases φ_1 and φ_2, defined earlier in this section, are equal.

In general, φ_1 and φ_2 will be different. If so then, for instance, a zero (e.g., crossover zero) in A_{12} at a negative t value ($t=t_0$) will imply that at $t=t_0$

$$A_{11}A_{22}<0.$$

Hence, if all A_{ij}'s are positive at $t=0$, then one of the two, A_{11} or A_{22}, will have a zero for $t<t_0$. It would be most interesting to carefully analyze the high-energy data and to check whether, in terms of our model, any deviations exist from exact factorization. (All the above results can be carried through to the physical sheet case, $\epsilon<0$).

IX. CONCLUSION

Incorporating the constraints due to unitarity in the t channel and using effective-range-type expansion in the j plane we wrote down, for the square-root singularity, a simple expression for $A(j,t)$ given by (1). For linear α_0 we found, not surprisingly, that there are two Regge poles which are complex conjugates of each other below a certain value $t=t_0$. At the point t_0 the poles collide giving rise to a double pole. The s dependence of the amplitude, $A(s,t)$, can be obtained from the Mellin transform of $A(j,t)$. This $A(s,t)$ is really the s-channel imaginary part of the total scattering amplitude $T(s,t)$. We found that $A(s,t)$ can be written as a sum of two terms each having a multiplicative factor $s^{\alpha_\pm}$. The coefficient of $s^{\alpha_\pm}$ in the first term is one-half of the residue of the complex pole, whether the pole be on the physical or unphysical sheet. The coefficient of $s^{\alpha_\pm}$ in the second term involves the series $\varphi_\pm(s,t)$ in terms of the product $(\alpha_\pm-\alpha_c)\ln s$. If ϵ is small (≈ 0.1) then for s as high as 200 BeV^2 and t not too large we found that at most one or two terms are needed to reproduce the exact result. Thus the s-dependence is largely determined by $s^{\alpha_\pm}$. As s becomes larger more and more terms of $\varphi_{\pm}(s,t)$ will be important and at asymptotic s the terms will add up to give the tip of the cut contribution, $s^{\alpha_c}/(\ln s)^{3/2}$. However, at present s values this behavior is inconsistent with the exact result. All the above results remain unchanged even if poles happen to be on the real axis ($\epsilon = 0$).

It is observed that the strength of the cut is manifested through two sources, one from ϵ and other from the ratios $|F_2/F_1|$ and $|G_1/F_1|$. For a fixed, small ϵ it is found that when the cuts are strong, i.e., when F_2 is of the order of F_1 the zeros in $A(s, t)$ can appear at a value different from the "unperturbed" situation. For instance, the zeros could be the solutions of $F_1 + F_2 t \approx 0$. Such a circumstance would correspond to the existence of the so-called crossover zeros. When $|F_2/F_1|$ is small we revert to the simple-pole-type situation with zeros given by $F_1 \approx 0$ which presumably are the same as the zeros of $1/\Gamma(\alpha_0)$ predicted by the dual resonance model. The strength of G_1 does not appear to play a crucial role in developing zeros. However, it appears in the phase of the residues $(F_1 + F_2 t \mp i G_1 \sqrt{-t})$. The sign of G_1 is the same as the so-called sign of the cut.[12] This sign enters in a crucial way in determining the resonance widths as well as in determining the sheet in which the complex poles lie for $t \leq 0$. Some simplified examples have been given to illustrate the way in which the sign of G_1 can be determined. As far as factorization is concerned, we found that $A(s, t)$ can be written as a sum of two complex conjugate terms each of which is factorizable but the sum, in general, need not.

We used a specific model for $A(j, t)$ and hence were able to obtain several specific conclusions. The expression we used for $A(j, t)$ as well as the type of parameters we chose were, we feel, quite realistic. The results we have obtained should be most useful in any future theoretical and phenomenological work.

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[1]P. Kaus and F. Zachariasen, Phys. Rev. D 1, 2962 (1970); R. Oehme, Phys. Lett. 20B, 414 (1969); Phys. Rev. D 4, 1485 (1971). Also see Ref. 6 below.

[2]For a general cut one can write, in the j plane, $D = f_0 + f_1 D_c + f_2 D_c{}^2 + \cdots$, where D_c corresponds to the cut [e.g., $(j - \alpha_c)^{1/2}$, $\ln(j - \alpha_c)$ etc.] and f_i are polynominals in $(j - \alpha_c)$.

[3]We will use BeV units throughout the paper. The partial-wave amplitude $A(j,t)$ is dimensionless, however. We normalize it so that it is proportional to $[t/(t - 4m^2)]^{1/2} e^{i\delta} \sin\delta$. The quantities F_1, F_2, G_1, and ϵ will also be taken to be dimensionless. Furthermore, when we write s^{α} we mean $(s/s_0)^{\alpha}$ with $s_0 = 1$ BeV^2.

[4]B. R. Desai and P. R. Stevens, Phys. Lett. 45B, 497 (1973). Other relevant references are given here.

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[7]We are most grateful to Dr. G. Cohen-Tannoudji for pointing out the importance of the double pole in our formalism and its relevance to high energy data.

[8]B. R. Desai and P. R. Stevens, report (unpublished).

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[10]For $\epsilon = 0$ we find that the asymptotic expression (19) blows up at $t = 0$, reflecting the fact that the pole lies on top of the cut. Expression (23) is the correct one to use at this point.

[11]Eventhough $\bar{\gamma}$ and $\hat{\gamma}$ both correspond to the residue at the pole, the former is reserved for the pole on the unphysical sheet, the latter for the physical sheet pole.

[12]We have talked interchangeably about g_1 and G_1 as if they were the same quantities. They are, of course, not. But for small ϵ and F_2 not too large they are almost the same.