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Quantum corrections to the Schwarzschild solution

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The quantum field theory of gravitation is used to compute quantum corrections to the classical Schwarzschild solution of the Einstein equations. The event horizon at $r = 2M$ must now be modified, and there are indications that quantum effects may prevent the appearance of the intrinsic singularity at the origin.

I. QUANTUM CORRECTIONS

In a recent paper by the author[1] it was shown, using the techniques of quantum field theory, how the classical Schwarzschild solution of the Einstein equations, which in familiar coordinates is given by[2]

$$ds^2 = \left(1 - \frac{L_s}{r}\right) dt^2 - \left(1 - \frac{L_s}{r}\right)^{-1} dr^2 - r^2\, d\Omega\ ,$$
$$d\Omega = d\theta^2 + \sin^2\theta\, d\phi^2,\quad L_s = 2Gm\ , \tag{1.1}$$

could be generated by the tree diagrams of Fig. 1. The single-graviton exchange diagram [Fig. 1(a)] provided the linearized line element

$$ds^2 = \left(1 - \frac{L_s}{r}\right) dt^2 - \left(1 + \frac{L_s}{r}\right) dr^2 - r^2\, d\Omega\ , \tag{1.2}$$

while the remaining diagrams yielded the nonlinear contributions. The motivation for this work was to pave the way for genuine quantum corrections by including the closed-loop diagrams, some of which are shown in Fig. 2.

The purpose of this paper is to find out how Eq. (1.2) is modified by including the single-closed-loop insertion of Fig. 2(a), which to order G^2 is the only quantum correction. We begin by writing the classical linearized metric as

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa\, \phi_c^{\mu\nu},\quad \tilde{g}^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu}\ , \tag{1.3}$$

where $\phi_c^{\mu\nu}$ is the classical field evaluated from Fig. 1(a). To perform the calculation we employ the harmonic gauge $\tilde{g}^{\mu\nu}{}_{,\nu} = 0$, and explicitly we have

$$\kappa\, \phi_c^{00} = \frac{2L_s}{r},\quad \kappa\, \phi_c^{ij} = 0\ , \tag{1.4}$$

or, in momentum space,

$$\kappa\, \phi_c^{00} = 16\pi^2 L_s \frac{\delta(p_0)}{|\vec{p}|^2},\quad \kappa\, \phi_c^{ij} = 0,\quad \text{with } p_\mu \phi_c^{\mu\nu} = 0\ . \tag{1.5}$$

Here p_μ is the momentum transfer of the virtual graviton and is spacelike, $p^2 < 0$. Now, by including Fig. 2(a), Eq. (1.3) becomes instead

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa(\phi_c^{\mu\nu} + \phi_q^{\mu\nu})\ , \tag{1.6}$$

where the quantum correction $\phi_q^{\mu\nu}$ is given by

$$\phi_q^{\mu\nu} = D^{\mu\nu\alpha\beta}\, \Pi_{\alpha\beta\gamma\delta}\, \phi_c^{\gamma\delta}\ . \tag{1.7}$$

$D^{\mu\nu\alpha\beta}$ is the graviton propagator

$$D^{\mu\nu\alpha\beta}(p^2) = \frac{1}{2p^2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta})\ , \tag{1.8}$$

and $\Pi_{\alpha\beta\gamma\delta}$ is the self-energy insertion which, by symmetry and Lorentz invariance, must be of the general form

$$\begin{aligned}\Pi_{\alpha\beta\gamma\delta}(p^2) = {}& \Pi_1(p^2)\, p^4\, \eta_{\alpha\beta}\eta_{\gamma\delta} + \Pi_2(p^2)\, p^4 (\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}) + \Pi_3(p^2)\, p^2 (\eta_{\alpha\beta} p_\gamma p_\delta + \eta_{\gamma\delta} p_\alpha p_\beta) \\ & + \Pi_4(p^2)\, p^2 (\eta_{\alpha\gamma} p_\beta p_\delta + \eta_{\alpha\delta} p_\beta p_\gamma + \eta_{\beta\gamma} p_\alpha p_\delta + \eta_{\beta\delta} p_\alpha p_\gamma) + \Pi_5(p^2)\, p_\alpha p_\beta p_\gamma p_\delta\ .\end{aligned} \tag{1.9}$$

The immediate difficulty, of course, is that $\Pi_{\alpha\beta\gamma\delta}$ is highly divergent. Recently, however, Capper, Leibbrandt, and Ramón Medrano[3] have shown, using dimensional regularization, how one may extract the finite part of $\Pi_{\alpha\beta\gamma\delta}$ in a manner consistent with the Slavnov-Ward identity[4]

$$p_\mu p_\rho\, D^{\mu\nu\alpha\beta}\, \Pi_{\alpha\beta\gamma\delta}\, D^{\gamma\delta\rho\sigma} = 0\ , \tag{1.10}$$

which imposes the restrictions

$$\Pi_2 + \Pi_4 = 0,\quad 4(\Pi_1 + \Pi_2 - \Pi_3) + \Pi_5 = 0\ . \tag{1.11}$$

The calculation is in fact much more complicated than Fig. 2(a) would indicate; one must also include the "fictitious-particle" contributions and possible "tadpole" diagrams. We refer the reader to Ref. 3 for details. Splitting up $\Pi_{\alpha\beta\gamma\delta}$ into an infinite piece and a finite piece, with real and imaginary parts, these authors find that the real

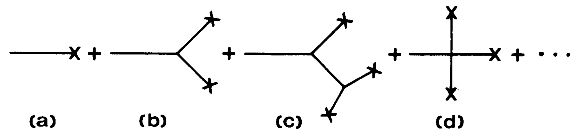

FIG. 1. Feynman diagrams for the classical gravitational field produced by a c-number source (denoted by the crosses). The solid lines represent the virtual gravitons.

finite part is given by Eq. (1.9) with

$$\Pi_R = \kappa^2\left(a_R \ln\left|\frac{p^2}{\mu^2}\right| + b_R\right), \quad R = 1, 2, 3, 4, 5 \quad (1.12)$$

where a_R and b_R are purely numerical coefficients, of order unity, which do indeed satisfy

$$a_2 + a_4 = 0, \quad 4(a_1 + a_2 - a_3) + a_5 = 0$$

and (1.13)

$$b_2 + b_4 = 0, \quad 4(b_1 + b_2 - b_3) + b_5 = 0\ ,$$

and μ is an arbitrary subtraction constant having the dimensions of mass. The significance of μ is intimately connected with the dimensional-regularization program, but, as we shall see, it need not concern us here. The infinite part of $\Pi_{\alpha\beta\gamma\delta}$ will presumably be canceled by appropriate counterterms in the Lagrangian, though one must await a complete theory of renormalization for the gravitational field before this question is satisfactorily settled and before one can be precise about the absolute magnitude of the coefficients a_R and b_R.[5]

Since p^2 is spacelike, there will be no contribution from the imaginary part of $\Pi_{\alpha\beta\gamma\delta}$, so we may insert Eq. (1.9) back into Eq. (1.7) using the Π_R of Eq. (1.12). We find, with $p^\mu \equiv \eta^{\mu\alpha} p_\alpha$, that

$$\phi_q^{\mu\nu} = [2\Pi_2 p^2 \delta^\mu_\alpha \delta^\nu_\beta - (\Pi_1 + \Pi_2 + \tfrac{1}{2}\Pi_3) p^2 \eta^{\mu\nu}\eta_{\alpha\beta} + \Pi_3 p^\mu p^\nu \eta_{\alpha\beta}]\, \phi_c^{\alpha\beta}\ . \quad (1.14)$$

Although we have focused our attention on the Schwarzschild solution, the above equation equally describes the first quantum correction to all solutions $\phi_c^{\alpha\beta}$ of the classical Einstein equations.

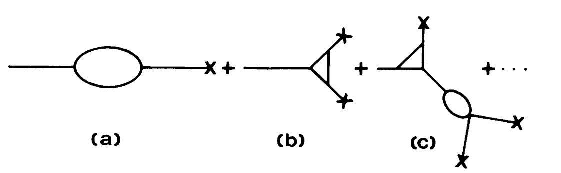

FIG. 2. Some typical quantum corrections. There will be a factor $\hbar$ associated with each closed loop. The above diagrams are schematic only; in practice one must include the fictitious-particle contribution and possible tadpole diagrams.

II. THE SCHWARZSCHILD METRIC

In the Schwarzschild case, using Eq. (1.5), and then Fourier-transforming back to x space, we find that there are three types of contribution to the $\phi_q^{\mu\nu}$ of Eq. (1.14). They are of the forms $\delta^3(\vec{r})$, $r^{-3}\ln\mu r$, and r^{-3}. We shall now discuss each in turn. The $\delta^3(\vec{r})$ terms are analogous to those which are found in quantum electrodynamics when one computes the vacuum polarization correction to the Coulomb potential and which are partly responsible for the observed Lamb-shift splitting in the hydrogen atom.[6] Their significance in gravity theory is not so clear. In any case they arise only if the r^{-1} behavior of ϕ_c^{00} is continued all the way down to the origin, i.e., for a strictly pointlike mass. For a mass of finite extension, they leave the exterior solution unaltered. Next we look at the logarithmic terms. The appearance of the constant μ in metric (1.6) may, at first glance, seem disturbing. Fortunately, we can trace the emergence of these logarithms to the $p^\mu p^\nu$ terms in Eq. (1.14). When the "loose end" of Fig. 2(a) is attached to a source $T_{\mu\nu}$ which is conserved,

$$p^\mu T_{\mu\nu} = p^\nu T_{\mu\nu} = 0\ , \quad (2.1)$$

these "gauge" pieces make no contribution and so, for practical purposes, may be ignored.[7] Thus we are left with the r^{-3} terms.

Mapping back to familiar coordinates, and relaxing the condition $\hbar = c = 1$, we find that the line element of Eq. (1.2) becomes

$$ds^2 = \left(1 - \frac{L_s}{r} - \alpha\frac{L_s L_p^2}{r^3}\right)c^2\, dt^2 - \left(1 + \frac{L_s}{r} + \beta\frac{L_s L_p^2}{r^3}\right)dr^2 - r^2\left(1 + \beta\frac{L_s L_p^2}{r^3}\right)d\Omega\ , \quad (2.2)$$

where

$$L_s = \frac{2Gm}{c^2}$$

is the Schwarzschild radius and

$$L_p = \left(\frac{G\hbar}{c^3}\right)^{1/2} \sim 10^{-33}\ \mathrm{cm}$$

is the Planck length. α and β are numerical constants which emerge from the details of the calculation.[8]

To obtain the complete quantum-gravity Schwarzschild metric we must add to the line element (2.2) the known higher-order classical terms of Eq. (1.1) which are of the form $L_s{}^n r^{-n}$, where n is the number of crosses which appear in any given diagram, and the higher-order quantum corrections which are as yet unknown but which, on dimensional grounds and ignoring possible logarithms, will be of the form $L_s{}^n L_p{}^{2m} r^{-(n+2m)}$, where m is the number of closed loops in any given diagram.

So we have shown that the r^{-1} and r^{-2} behavior of the Schwarzschild solution is left unaltered by quantum corrections, which serve to modify only the classical r^{-n} $(n \geq 3)$ terms, becoming comparable in size with them only for masses whose Schwarzschild radius is as small as the Planck length. This occurs when

$$m \sim \left(\frac{\hbar c}{G}\right)^{1/2} \sim 10^{-5}\ \text{g} .$$

Since r^{-3} terms are beyond the range of experimental test, even for masses as great as the sun's, the chances of directly observing these quantum corrections are, unfortunately, exceedingly remote.

It is amusing the examine the case of elementary particles where $L_p \gg L_s \sim 10^{-55}$ cm; then the r^{-l} $(l \geq 3)$ behavior is completely dominated by the quantum effects. In fact, it may be that at very small distances the gravitational interaction between elementary particles is almost exclusively quantum in nature, though this is a matter for speculation.

Without summing up the entire perturbation series, we can talk with confidence about the metric only at large values of r, but it is really just the short-distance behavior that makes quantum gravity interesting. Qualitatively, at least, the following comments would seem to be in order. Firstly, the event horizon obtained by setting $g_{00} = 0$, which occurs at $r = L_s$ in the classical theory, should now be modified. We are not yet in a position to say where the new horizon will be located, nor indeed if the number of such horizons should be restricted to only one. Secondly, the scalar J formed from the Riemann tensor,

$$J = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \sim \frac{L_s{}^2}{g_{\theta\theta}{}^3} ,$$

blows up at the origin like r^{-6} in the classical theory since $g_{\theta\theta} = r^2$. Now, however, $g_{\theta\theta}$ becomes

$$g_{\theta\theta} = r^2\left(1 + \beta\frac{L_s L_p{}^2}{r^3} + \cdots\right)$$

and there are indications that J might now remain finite in the quantum theory and vanish as $r \to 0$. The infinity would reappear of course in the $\hbar \to 0$ limit. Thus, the hope that quantum gravity might somehow come to the rescue and avoid the appearance of intrinsic space-time singularities may yet be realized.

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[1] M. J. Duff, Phys. Rev. D 7, 2317 (1973). (The problems of mass renormalization discussed in this paper are not directly relevant to our present purposes and will be ignored.) See also M. J. Duff, Ann. Phys. (N.Y.) 79, 261 (1973).

[2] Natural units are used with $\hbar = c = 1$ and $\kappa^2 = 32\pi G$, where G is the Newtonian gravitational constant. $\eta^{\mu\nu}$ denotes the Minkowski metric with signature (1, −1, −1, −1). Greek indices run over 0, 1, 2, 3, and latin over 1, 2, 3.

[3] D. M. Capper, G. Leibbrandt, and M. Ramón Medrano, Phys. Rev. D 8, 4320 (1973).

[4] D. M. Capper and M. Ramón Medrano, this issue, Phys. Rev. D 9, 1641 (1974).

[5] In Ref. 3 they are $a_1 = 59a$, $a_2 = -81a$, $a_3 = -104a$, $a_4 = 81a$, $a_5 = -328a$, where $a = 1/(4\pi)^2 \times 240$, $b_1 = 1517b$, $b_2 = -1143b$, $b_3 = 598b$, $b_4 = 1143b$, $b_5 = 896b$, where $b = 1/(4\pi)^2 \times 7200$. However, one cannot rule out the possibility that the counterterms might also change the finite part.

[6] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[7] Actually one must first compute $\kappa\chi^{\mu\nu} \equiv g^{\mu\nu} - \eta^{\mu\nu}$ rather than $\kappa\phi^{\mu\nu} = \tilde{g}^{\mu\nu} - \eta^{\mu\nu}$, and then discard the $p^\mu p^\nu$ terms since the coupling will be of the form $\kappa\chi^{\mu\nu} T_{\mu\nu}$.

[8] $\alpha = -4 \times 32\pi\,(a_1 + 2a_2)$ and $\beta = 4 \times 32\pi a_1$, so from the values of a_1 and a_2 in Ref. 5, both α and β are positive.