Comments and Addenda

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Class of solutions of axial-symmetric Einstein-Maxwell equations

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(Received 14 December 1973)

A class of solutions of the coupled Einstein-Maxwell field equations is found.

Recently, Misra *et al.*[1] considered the solution of the axial-symmetric Einstein-Maxwell equations. They showed that the problem of obtaining a solution of these equations can be reduced to that of finding solutions to the Ernst[2] equation, which was originally proposed in order to find a solution of the vacuum Einstein equations. It therefore follows that if one knows a certain solution of vacuum Einstein equations one can then construct a corresponding solution of the coupled Einstein-Maxwell equations. In their paper Misra *et al.*[1] constructed only one such solution, namely that which formally corresponds to the Kerr metric of the vacuum Einstein equations. The resulting solution was then shown to represent the static field of a massive body carrying an electric or magnetic dipole moment.

Meanwhile, Sato and Tomimatsu[3] (called TS hereafter) have constructed a series of solutions of the Ernst equation, from which they obtained the fields due to appropriate spinning masses. What are the corresponding solutions of the Einstein-Maxwell equations? This paper is devoted to answering this last question.

We first recapitulate the principal results of Ref. 1. Consider the metric

$$ds^2 = e^{2u}dt^2 - e^{2k-2u}(d\rho^2 + dz^2) - \rho^2 e^{-2u}d\phi^2 , \tag{1}$$

and write the electromagnetic fields as

$$F^{31} = \frac{1}{\rho}e^{2u-2k}A_{,2} ,$$
$$F^{23} = \frac{1}{\rho}e^{2u-2k}A_{,1} , \tag{2}$$
$$F_{01} = B_{,1} , \quad F_{02} = B_{,2} ,$$

where $A(\rho, z)$ and $B(\rho, z)$ are the magnetic and the electric potential, respectively. After a duality rotation $A = C\cos\beta$, $B = C\sin\beta$, the Einstein-Maxwell equations become

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*\nabla\xi^*\cdot\nabla\xi , \tag{3}$$

$$\frac{\partial k}{\partial \rho} = \frac{4\rho}{(\xi\xi^* - 1)^2}\left(\frac{\partial \xi}{\partial \rho}\frac{\partial \xi^*}{\partial \rho} - \frac{\partial \xi}{\partial z}\frac{\partial \xi^*}{\partial z}\right) , \tag{4}$$

$$\frac{\partial k}{\partial z} = \frac{8\rho}{(\xi\xi^* - 1)^2}\mathrm{Re}\left(\frac{\partial \xi}{\partial \rho}\frac{\partial \xi^*}{\partial z}\right) , \tag{5}$$

where

$$\xi = \frac{1 + e^u + iC}{1 - e^u - iC} . \tag{6}$$

C is a new potential and β is a constant. Equation (3) is the Ernst equation. Once this has been solved the function $k(\rho, z)$ can be obtained by solving Eqs. (4) and (5). This then yields the metric as well as the electromagnetic fields. We now proceed to solve Eqs. (3)-(5). Following TS, the solution of ξ can be written as

$$\xi = \alpha/\beta , \tag{7}$$

where α and β are complex polynomials of prolate spherical coordinates λ and μ. The metric function e^u and potential C become

$$e^u = A/B , \tag{8}$$

$$C = D/B , \tag{9}$$

where

$$\begin{aligned} \alpha &= W + iv\,, \\ \beta &= m + in\,, \\ A &= W^2 + v^2 - m^2 - n^2\,, \\ B &= (W+m)^2 + (v+n)^2\,, \\ D &= 2(vm - nW)\,. \end{aligned} \tag{10}$$

From (4) and (5) we can obtain the metric function e^{2k} as

$$e^{2k} = \frac{A^4}{p^{8\delta}(\lambda^2 - \mu^2)^{\delta^2}}\,.$$

p will be defined later and δ is an integer. For $\delta = 1-3$, the expressions for A, B, D are as follows.

$\delta = 1$:

$$\begin{aligned} A &= p^2(\lambda^2 - 1) - q^2(1 - \mu^2)\,, \\ B &= (p\lambda + 1)^2 + q^2\mu^2\,, \\ D &= -2q\mu\,. \end{aligned} \tag{11}$$

$\delta = 2$:

$$\begin{aligned} A &= p^4(\lambda^2-1)^4 + q^4(1-\mu^2)^4 - 2p^2q^2(\lambda^2-1)(1-\mu^2)[2(\lambda^2-1)^2 + 2(1-\mu^2)^2 + 3(\lambda^2-1)(1-\mu^2)]\,, \\ B &= [p^2(\lambda^2+1)(\lambda^2-1) - q^2(\mu^2+1)(\mu^2-1) + 2p\lambda(\lambda^2-1)]^2 + 4q^2\mu^2[p\lambda(\lambda^2-1) + (p\lambda+1)(1-\mu^2)]^2\,, \\ D &= -4q\mu(1-\mu^2) - 4p^2q[\lambda^4\mu(1-\mu^2) + 2\lambda^2\mu(\lambda^2-1)(\lambda^2-\mu^2)] + 4q^3\mu^5(1-\mu^2)\,. \end{aligned} \tag{12}$$

$\delta = 3$:

$$\begin{aligned} A &= p^6(\lambda^2-1)^9 - 4p^4q^2(\lambda^2-1)^4(1-\mu^2) \\ &\qquad \times [3(\lambda^2-1)^4 + 12(\lambda^2-1)^3(1-\mu^2) + 28(\lambda^2-1)^2(1-\mu^2)^2 + 30(\lambda^2-1)(1-\mu^2)^3 + 12(1-\mu^2)^4] \\ &\quad + 3p^2q^4(\lambda^2-1)(1-\mu^2)^4[12(\lambda^2-1)^4 + 30(\lambda^2-1)^3(1-\mu^2) + 28(\lambda^2-1)^2(1-\mu^2)^2 + 12(\lambda^2-1)(1-\mu^2)^3 + 3(1-\mu^2)^4] \\ &\quad - q^6(1-\mu^2)^6\,, \\ B &= [p(\lambda^3+3\lambda)(\lambda^2-1)^3 - pq^2(\lambda^3+3\lambda\mu^2)(\lambda^2-\mu^2)^3 + p^2(3\lambda^2+1)(\lambda^2-1)^3 - q^2(3\mu^2+1)(1-\mu^2)^3]^2 \\ &\quad + q^2\mu^2[p^2(\lambda^2-\mu^2)^3(3\lambda^2+\mu^2) - (1-\mu^2)^3(\mu^2+3) + 12p\lambda(\lambda^2-\mu^2)(\lambda^2-1)(1-\mu^2)]^2\,, \\ D &= p^2q[2(1-\mu^2)^3(\mu^3+3\mu)(\lambda^2-1)^3(3\lambda^2+1) + 24\lambda\mu(\lambda^2-\mu^2)(\lambda^2-1)^4(1-\mu^2)(\lambda^3+3\lambda)] \\ &\quad - 2q^3(1-\mu^2)^3(\mu^3+3\mu)(3\mu^2+1) \\ &\quad - p^2q^3[2(\lambda^2-\mu^2)^3(\mu^3+3\lambda^2\mu)(1-\mu^2)^3(3\mu^2+1) + 24\lambda\mu(\lambda^2-\mu^2)^4(\lambda^2-1)(1-\mu^2)(\lambda^3+3\lambda\mu^2)] \\ &\quad - 2p^4q(\lambda^2-\mu^2)^3(\mu^3+3\lambda^2\mu)(\lambda^2-1)^3(3\lambda^2+1)\,, \end{aligned} \tag{13}$$

where $p^2 + q^2 = 1$ and q can be identified as charge per unit mass $-e/m$. The prolate spheroidal coordinates (λ, μ) are related to spherical coordinates r, θ and cylindrical coordinates ρz by

$$\begin{aligned} \rho &= (mp/\delta)(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2} \\ &= (r^2 - 2mr + m^2q^2)^{1/2}\sin\theta\,, \\ z &= (mp/\delta)\lambda\mu = (r-m)\cos\theta\,. \end{aligned}$$

The following remarks about the above solutions are in order.

(a) When $\delta = 1$, the result is the same as that of Misra *et al.*

(b) The asymptotic form of electromagnetic potential is (for $\delta = 1-3$)

$$C \approx \frac{2\delta^2 q\mu}{p^2\lambda^2} \approx \frac{2me\cos\theta}{r^2}\,,$$

which is independent of δ. This can be regarded as the static field of an electric and magnetic dipole.

(c) Just as in the TS metric, the solutions have a singularity wherever $r^2 + e^2\cos^2\theta = 0$.

(d) In the $q = 0$ limit, the solutions reduce to the Weyl metric with $\delta' = 2\delta$, where δ' is the Weyl parameter.[4]

(e) The solutions have asymptotic flatness.

The author wishes to thank Dr. S. K. Bose for his encouragement and discussions.

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[3] A. Tomimatsu and H. Sato, Prog. Theor. Phys. 50, 95 (1973).
[4] H. Weyl, Ann. Phys. (Leipz.) 54, 117 (1917).