directly to variational approximations of the Rayleigh-Ritz type.

[13]A similar result holds when long-ranged Coulomb interactions are present; see L. Rosenberg, Phys. Rev. D 8, 1833 (1973). In Sec. IV of this reference an effective-potential theory for three-body charged-particle scattering is worked out. This formalism can be extended to the general multichannel case using the methods of the present paper. We have ignored long-ranged Coulomb effects here to simplify the discussion.

[14]We are chiefly motivated here by considerations of unitarity-preserving approximations of the type described by J. Carew and L. Rosenberg, Phys. Rev. D 7, 1113 (1973). Construction of unitary variational approximations based on the effective-potential formalism is complicated by the appearance of such spurious cuts.

[15]See, e.g., M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), Chap. 8.

[16]W. Zimmermann, Nuovo Cimento 21, 249 (1961).

[17]L. Rosenberg, Phys. Rev. D 1, 1019 (1970).

[18]In this case one is led to linear integral equations for the scattering matrix which, due to the appearance of disconnected parts, require further analysis of the type used by Faddeev for the three-body system. We shall not take up this analysis here, however.

PHYSICAL REVIEW D VOLUME 9, NUMBER 6 15 MARCH 1974

Bounds on multiplicities and polynomial boundedness*

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(Received 21 November 1973)

We prove that in any "model" or "theory" in which (a) perturbation expansions have a finite radius of convergence and (b) the cross sections for producing n hadrons, $\sigma_n(s,\lambda)$, are polynomially bounded, $|\sigma_n(s,\lambda)| \le s^N$, for all values of the coupling constant λ inside the circle of convergence, with N independent of n, the following bounds on the average multiplicities hold for large s: $\langle n\rangle \le C\ln s$ and $\langle n^p\rangle \le C^p(\ln s)^p$. In proving this result we also use the Jin-Martin lower bound for physical λ, $\sigma_{\text{tot}}(s,\lambda) \ge (\text{const})\times s^{-6}$, which follows rigorously from analyticity and unitarity only. We discuss the possible validity of the above bounds in axiomatic field theory and show that our bounds also follow from alternative conditions which, unlike (a), have a chance of being true in field theory. It is also shown that the behavior $\langle n\rangle \sim s^a$ necessarily leads to an exponential behavior in s for $\text{Max}_n|\sigma_n(s,\lambda)|$ for some complex λ whenever (a) holds.

I. INTRODUCTION

In addition to unitarity and analyticity, the main input that leads to general asymptotic bounds, like the Froissart bound, is polynomial boundedness.[1] In the framework of the Wightman axioms, this follows from the fact that the Wightman functions and hence the Green's functions are taken to be tempered distributions. Their Fourier transforms are polynomially bounded in momentum space.

Thus far, to the best of our knowledge, there exist no rigorous bounds on the average multiplicity $\langle n\rangle$, except for the trivial one: $\langle n\rangle \le s^{1/2}$ for large s. Logunov and co-workers[2] have derived interesting bounds that apply to certain limited regions in phase space, but no bound of the general nature of the Froissart bound exists for the full $\langle n\rangle$. In fact, it has not been clear if general properties such as analyticity and polynomial boundedness control the growth of the average multiplicity.

In this paper we first consider "theories" or "models" in which the cross sections, $\sigma_n(s)$, for producing n hadrons in the final state have a convergent power-series expansion in some coupling constant. We prove that if these $\sigma_n(s)$ are polynomially bounded in s for all values of the coupling constant, real or complex, inside the circle of convergence, then a powerful bound on the average multiplicity holds, namely $\langle n\rangle \le C\ln s$, and $\langle n^p\rangle \le C^p \ln^p s$. Thus, these two simple and general conditions restrict $\langle n\rangle$ in a strong way.

The conditions we start with, convergence of the perturbation series and extended temperedness, are features of many models in the literature, especially those of the multiperipheral and multi-Regge types. Our result points out the fact that the behavior $\langle n\rangle \le \ln s$ in these models is a consequence of general properties, not dynamical details. In the situation of axiomatic field theory, the status of our two assumptions is quite different. First, it is exceedingly unlikely that perturbation expansions are anything more than asymptotic expansions in an actual field theory.[3] Second, very little is known about temperedness properties for complex values of the coupling constant. However, some progress along this direction has been achieved recently by Glimm, Jaffe, and Spencer[4] for two-dimensional Euclidean field theories. The following question arises: Are the conditions we use necessary for our result, and if not, will the alternative conditions be provable within a general axiomatic framework? The answer to the first part of the question is negative. The same bounds could follow from weaker conditions. In Sec. IV we discuss this point and the results of Ref. 4 briefly. We also point out several alternative conditions which will lead to the same bounds and are weaker than the two used in our main theorem. From the discussion of Sec. IV it becomes clear that the important property one needs is some extended form of temperedness, and nothing need be assumed about the convergence of perturbation theory.

The most interesting model that violates our bound and gives a behavior for $\langle n\rangle$ increasing like a power of s is that given recently by Cheng and Wu.[5] We discuss a simplified form of that model in Sec. III. The ansatz we use for $\sigma_n(s)$ in that case has a convergent expansion in the coupling constant. However, one can see explicitly that for complex values of the coupling the expression for $\sigma_n(s)$ can blow up exponentially in s and thus violates our second condition. Unfortunately, the Cheng-Wu model does not provide a counterexample to the temperedness condition. It has already been shown by Cheng and Wu and others that summing leading $\ln s$ and restricting oneself to a subset of diagrams can easily lead to answers that violate unitarity bounds. It is quite possible in this way to also violate some temperedness bounds that might be established in a general framework.

Finally, we stress the fact that in this paper we always use the lower bound $\sigma_{\rm tot}(s) \geqslant (\text{const})\times s^{-6}$ for large s as shown in general by Jin and Martin.[6] This lower bound follows from the cut-plane analyticity of the forward amplitude and the positivity of its imaginary part. Thus it is not only a consequence of axiomatic field theory, but must also be a feature of any sane model. We use this lower bound *only* for real physical values of the coupling constant. We do not need it for complex values of the coupling. The exact power s^{-6} is not crucial for our results. We only need a lower bound of the form $\sigma_{\rm tot}(s) \geqslant (\text{const})\times s^{-\beta}$, where β is any finite real positive number.

In some papers $\langle n\rangle_i$ is defined without including $\sigma_{\rm el}$, as $\langle n\rangle_i \sigma_{\rm inel} = \sum_{n=3}^{\sqrt{s}} n\sigma_n$. With this definition of $\langle n\rangle_i$ we need to assume the lower bound $\sigma_{\rm inel}(s) \geqslant Cs^{-\beta}$ in order to get the results of this paper. Even though this is a weak assumption, it has not been proved rigorously in field theory like the lower bound for $\sigma_{\rm tot}(s)$. *Thus in this paper we are excluding models in which* $\sigma_{\rm inel}(s)$ *decrease faster than any large inverse power of* s.

II. BOUNDS ON $\langle n^p\rangle$

In this section we shall consider "theories" or "models" which have a convergent perturbation expansion, and for which temperedness still holds even for complex coupling constant λ as long as $|\lambda|$ is inside the circle of convergence. We shall argue in a later section that the important input is the temperedness and that the results we get might still be true even if perturbation theory is only asymptotic.

We start by defining the different quantities we are going to use and explicitly stating our *two* main assumptions.

The forward scattering amplitude $T(s,\lambda)$ is considered as a function of s and some renormalized coupling constant λ. One has the expansion

$$T(s,\lambda)=\sum_{k=0}^{\infty}\lambda^k t^{(k)}(s)\ , \qquad (2.1)$$

where for ϕ^3 or Yukawa-type couplings λ should be replaced by g^2. We shall assume that the above series is absolutely convergent for all λ such that

$$|\lambda|<\tilde{R}\ , \qquad (2.2)$$

with $\tilde{R}$ independent of s for large s. Later we shall discuss the possibility of weakening this assumption.

Since Im T, for real λ, is proportional to $\sigma_{\rm tot}(s,\lambda)$ we can also write

$$\sigma_{\rm tot}(s,\lambda)=\sum_{k=0}^{\infty}\lambda^k\alpha^{(k)}(s)\ . \qquad (2.3)$$

This series we shall assume also has some radius of convergence R. We now write the identity

$$\sigma_{\rm tot}(s,\lambda)=\sum_{n=2}^{\sqrt{s}}\sigma_n(s,\lambda)\ , \qquad (2.4)$$

where σ_n is the cross section for producing n hadrons in the final state. For physical λ, both $\sigma_{\rm tot}$ and σ_n obey the Froissart bound

$$\sigma_n(s,\lambda)\leqslant\sigma_{\rm tot}(s,\lambda)\leqslant\frac{\pi}{m_\pi^{\,2}}\ln^2 s\ . \qquad (2.5)$$

Even if for some real values of λ unitarity is violated, we will still have, provided we maintain temperedness and positivity,

$$\sigma_n(s,\lambda)\leqslant\sigma_{\rm tot}(s,\lambda)\leqslant s^N\ . \qquad (2.6)$$

One can expand $\sigma_n(s,\lambda)$ also in power series in λ, and get

$$\sigma_n(s,\lambda)\equiv\sum_{k=n}^{\infty}\lambda^k\alpha_n^{(k)}(s) \qquad (2.7)$$

It is crucial to notice that in a ϕ^4-type theory the lowest order in (2.7) is $k=n$. In Yukawa-type or ϕ^3-type theories we replace λ by g^2.

We are now in a position to state precisely our two assumptions:

(a) The series in (2.7) for large s is absolutely convergent for $|\lambda|<R_1$ with R_1 independent of s and n.

(b) For all $|\lambda|<R_1$, the $\sigma_n(s,\lambda)$ defined by the right-hand side of (2.7) are still tempered,

$$|\sigma_n(s,\lambda)|\leqslant s^N\,, \tag{2.8}$$

where $N>0$ and is independent of n.

Our assumption (a) is clear even though it is almost certain to be not true in realistic field theories while it is true in most models. We shall later show that (a) may not be crucial for our results. Assumption (b) is motivated by the results of Glimm, Jaffe, and Spencer[4] and all we are assuming is that given some analyticity in λ, we impose the same temperedness assumption on the σ_n's for complex λ that we know holds for real λ. For complex λ unitarity certainly will not hold and $|\sigma_n|$ will not obey the Froissart bound, but if vacuum expectation values of time-ordered products remain tempered for complex λ it is quite possible that the σ_n's will also be tempered and bounded by some possibly large power of s which is independent of n. In Sec. III assumption (b) is restated in terms of more physical quantities.

The average multiplicity is defined by

$$\langle n\rangle\sigma_{\rm tot}(s,\lambda)\equiv\sum_{n=2}^{\sqrt{s}}n\sigma_n(s,\lambda)\,. \tag{2.9}$$

We also need the higher moments,

$$\langle n^p\rangle\sigma_{\rm tot}(s,\lambda)\equiv\sum_{n=2}^{\sqrt{s}}n^p\sigma_n(s,\lambda)\,. \tag{2.10}$$

It is clear that the assumption (b) by itself for real λ provides no restriction at all on $\langle n\rangle$. If $\sigma_n(s,\lambda)\leqslant s^N$ for all n, this by itself does not tell us anything about $\langle n\rangle$. However, we shall see below that if $|\sigma_n(s,\lambda)|\leqslant s^N$ for complex λ, $|\lambda|<R_1$, and if perturbation theory converges, then remarkable bounds on $\langle n\rangle$ and $\langle n^p\rangle$ follow. Indeed we shall prove the following theorem:

Theorem. In any "theory" or "model" in which (a) and (b) hold, and in which $\sigma_{\rm tot}(s,\lambda)$ (for real physical λ) satisfies the Jin-Martin lower bound, the following bounds hold for large s:

$$\langle n\rangle\leqslant C\ln s\,, \tag{2.11}$$

$$\langle n^p\rangle\leqslant C^p\ln^p s,\quad p\ \text{fixed.} \tag{2.12}$$

Proof. We start by defining "reduced" cross sections $\tilde\sigma_n(s,\lambda)$, where

$$\sigma_n(s,\lambda)\equiv\lambda^n\tilde\sigma_n(s,\lambda)\,. \tag{2.13a}$$

From Eq. (2.7) it follows that $\tilde\sigma_n$ has the power-series expansion

$$\tilde\sigma_n(s,\lambda)\equiv\sum_{j=0}^{\infty}\lambda^j\alpha_n^{(j+n)}(s)\,. \tag{2.13b}$$

This series obviously must also converge for $|\lambda|<R_1$ just as the one in Eq. (2.7) does. By definition the first term of the series in Eq. (2.13b) is $\alpha_n^{(n)}(s)\equiv\tilde\sigma_n(s,0)$, and $\tilde\sigma_n(s,0)$ has only contributions from the tree diagrams shown in Fig. 1 with the coupling at the vertices set at unity. This contribution we are assuming to be finite.

From (a) it follows that $\sigma_n(s,\lambda)$ is analytic in λ for all λ such that $|\lambda|\leqslant R<R_1$. By definition also it is evident from Eq. (2.13b) that $\tilde\sigma_n(s,\lambda)$ is also analytic in the same region. From (b) we have for $R<R_1$

$$\operatorname*{Max}_{\phi}|\tilde\sigma_n(s,Re^{i\phi})|\equiv\operatorname*{Max}_{\phi}\frac{|\sigma_n(s,Re^{i\phi})|}{R^n}\leqslant\frac{s^N}{R^n}\,. \tag{2.14}$$

However, since $\tilde\sigma_n(s,\lambda)$ is analytic for $|\lambda|\leqslant R$, it takes its maximum at the boundary, and hence

$$|\tilde\sigma_n(s,\lambda)|\leqslant\frac{s^N}{R^n}\,,\quad\text{for all }|\lambda|\leqslant R\,. \tag{2.15}$$

The bound (2.15) is, of course, also true for real positive λ as long as $\lambda\leqslant R<R_1$.

Hence for any positive real λ, $\lambda<R$, we get

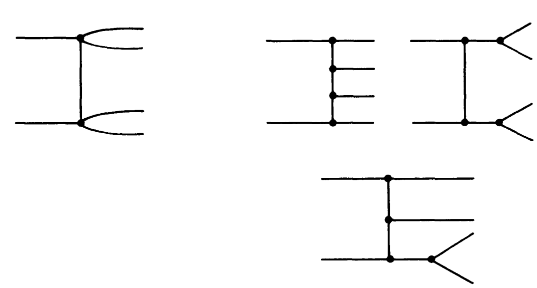
$$\sigma_n(s,\lambda)\leqslant\left(\frac{\lambda}{R}\right)^n s^N\,. \tag{2.16}$$

This last bound means that the generating function $\sum_n z^n\sigma_n(s,\lambda)$ is tempered for $z=1+\delta$. The situation is analogous to the starting point of the proof of the Froissart bound where one has the fact that for some positive $t>0$, t inside the Lehmann-Martin ellipse,

$$A(s,t)=\sum(2l+1)a_l(s)P_l(z(t))$$

is bounded by s^N for large s.[7]

The rest of our proof is almost identical to the



(a) (b)

FIG. 1. Diagrams that contribute to $\tilde\sigma_n(s,\lambda)$ for $\lambda=0$; (a) ϕ^4 coupling, (b) ϕ^3 coupling, and $n=4$.

proof of the Froissart bound. First, it is evident from (2.16) that for all n and real λ, $\lambda < R$,

$$\sigma_n(s,\lambda) \leq s^N e^{-\gamma n} \,, \tag{2.17}$$

with

$$\gamma \equiv \ln\left(\frac{R}{\lambda}\right) > 0 \,. \tag{2.18}$$

We also have the bound

$$\sigma_n(s,\lambda) \leq \sigma_{\text{tot}}(s,\lambda) \leq s^N \,. \tag{2.19}$$

Actually, if the Froissart bound is respected we should have $(\pi/m_\pi^2)\ln^2 s$ on the right in this last inequality, but to include in our arguments models like the multiperipheral model we use Eq. (2.19). It is enough for our purposes.

The bound (2.17) tells us that for $n \gg [(N/\gamma)\ln s]$, $\sigma_n(s,\lambda)$ will become smaller than a large inverse power of s. Thus we can always find a positive constant C,

$$C \gg \frac{N}{\gamma} \,, \tag{2.20}$$

such that the effective cutoff in Eq. (2.4) is at $n = L$, where

$$L = C\ln s \,. \tag{2.21}$$

More precisely we can write Eq. (2.4) as

$$\sigma_{\text{tot}}(s,\lambda) = \sum_{n=2}^{L} \sigma_n(s,\lambda) + \sum_{n>L}^{\sqrt{s}} \sigma_n(s,\lambda) \,. \tag{2.22}$$

By choosing C large enough, we get

$$\sum_{n>L}^{\sqrt{s}} \sigma_n(s,\lambda) \leq \int_{C\ln s}^{\sqrt{s}} s^N e^{-\gamma n}\, dn \leq \frac{1}{s^M} \,,$$

where $M > 0$, and M can be made as large as we please by choosing C large enough. Therefore we obtain for large s the estimate

$$\sigma_{\text{tot}}(s,\lambda) = \sum_{n=2}^{L} \sigma_n(s,\lambda) + O\left(\frac{1}{s^M}\right). \tag{2.23}$$

We now use the Jin-Martin lower bound which follows only from positivity and analyticity,

$$\sigma_{\text{tot}}(s,\lambda) \geq \frac{\text{const}}{s^6} \,. \tag{2.24}$$

This guarantees that by choosing $M \gg 6$, Eq. (2.23) will always give us a good estimate of σ_{tot}, i.e.,

$$\sum_{n=2}^{L} \sigma_n(s,\lambda) = \sigma_{\text{tot}}(s,\lambda)\left[1 + O\left(\frac{1}{s^{M-6}}\right)\right] . \tag{2.25}$$

The role of the Jin-Martin lower bound is crucial at this stage since if $\sigma_{\text{tot}} \sim e^{-s^\alpha}$, Eq. (2.25) does not follow from Eq. (2.23) and the first L terms of Eq. (2.4) will not give a good approximation to σ_{tot}.

The assertions of our theorem are now obvious:

$$\begin{aligned}\langle n\rangle \sigma_{\text{tot}}(s,\lambda) &\equiv \sum_{n=2}^{\sqrt{s}} n\sigma_n(s,\lambda) \\ &= \sum_{n=2}^{L} n\sigma_n(s,\lambda) + \sum_{n>L}^{\sqrt{s}} n\sigma_n(s,\lambda) \,.\end{aligned} \tag{2.26}$$

Again the second term above can be made $O(1/s^M)$, and in the first sum $n \leq L$; hence,

$$\langle n\rangle \sigma_{\text{tot}} \leq L \sum_{n=2}^{L} \sigma_n + O(1/s^M) \,, \tag{2.27}$$

and using Eqs. (2.25) and (2.24), we get

$$\langle n\rangle \leq L = C\ln s \,. \tag{2.28}$$

Similarly, one can easily show, following identical steps, that for fixed finite p

$$\langle n^p\rangle \leq L^p = C^p \ln^p s, \quad p = 2, 3, 4, \ldots . \tag{2.29}$$

This completes the proof.

We conclude this section with a series of remarks:

(1) Condition (b) can be weakened to give us a bound on $\langle n\rangle$ that depends on the maximum of $|\sigma_n(s,\lambda)|$ on the circle $|\lambda| = R$. For example, if instead of (b) we define

$$\underset{n,\phi}{\text{Max}} |\sigma_n(s, Re^{i\phi})| = M(s) \,, \tag{2.30}$$

and restrict ourselves to cases where $M(s) \geq s^\epsilon$ for any $\epsilon > 0$, at least for some sequence of s values, this gives us an effective cutoff at $n \cong L$, with

$$L = C\ln M(s) \,. \tag{2.31}$$

The resulting bounds for $\lambda < R$ are

$$\langle n^p\rangle \leq C^p[(\ln M(s))]^p, \quad p = 1, 2, 3, \ldots . \tag{2.32}$$

As an example, consider a case where $M(s) = \exp(\ln s)^\beta$, $\beta > 1$. Such a value of $M(s)$ is definitely larger than any polynomial in s. From Eq. (2.32) we get in this case the bounds

$$\langle n\rangle \leq C(\ln s)^\beta$$

and

$$\langle n^p\rangle \leq C^p(\ln s)^{\beta p}, \quad p = 1, 2, 3, \ldots . \tag{2.33}$$

This example shows that one still can get strong bounds even if $|\sigma_n(s,\lambda)|$ for some complex λ grows faster than any polynomial in s but not as fast as $e^{(s)^\epsilon}$. The case where $M(s) \leq (\ln s)^\beta$ is discussed in Ref. 11.

(2) Conversely, given a power behavior for $\langle n\rangle$, i.e.,

$$\langle n\rangle \cong s^a, \quad \tfrac{1}{2} > a > 0 \tag{2.34}$$

for large s, then there must exist an $\epsilon > 0$ and a sequence of points $\{s_j\}$, $s_j \to \infty$ as $j \to \infty$, such that

$$M(s_j) \geq \exp[\epsilon(s_j)^a] \,, \tag{2.35}$$

for if no such sequence exists, then for any $\epsilon > 0$,

$$\ln M(s) < \epsilon s^a \ . \tag{2.36}$$

But from our previous remark, we obtain

$$\langle n\rangle \leq C \ln M \leq C\epsilon s^a \ . \tag{2.37}$$

This contradicts Eq. (2.34) for ϵ small enough and we see that the power behavior for $\langle n\rangle$ necessarily implies that the maximum of σ_n for some complex λ must have an exponential behavior $e^{\delta s^a}$. The power of s in the exponent is the same as the power in $\langle n\rangle$. We are, of course, assuming that condition (a) holds true and perturbation expansions converge.

This is indeed the situation in the Cheng-Wu model,[5] which leads to $\langle n\rangle \sim s^a$ as we shall discuss in Sec. III.

(3) The validity of the Jin-Martin lower bound is essential for the theorem. Without it one can easily construct a counterexample. For example, take for $\sigma_n(s,\lambda)$ the ansatz

$$\sigma_n(s,\lambda) \equiv \frac{(\lambda s^{\lambda^2})^n}{n!} e^{-s^{1/2}} \ . \tag{2.38}$$

This form certainly satisfies conditions (a) and (b) for $|\lambda| < 1/\sqrt{2}$. However, the value of $\langle n\rangle$ is given by $\langle n\rangle \cong \lambda s^{\lambda^2}$ and increases like a power. The total cross section resulting from the ansatz of Eq. (2.38) is

$$\sigma_{\text{tot}}(s,\lambda) \cong \exp(\lambda s^{\lambda^2}) e^{-s^{1/2}} \ , \tag{2.39}$$

and decreases exponentially with s for large s and $|\lambda| < 1/\sqrt{2}$. It thus violates the Jin-Martin lower bound which follows from analyticity and positivity.

One can use the expression given in (2.38) only for certain values of n which are near $n = O(\lambda s^{\lambda^2})$ and assume that contributions from other values of n, which decrease like a power of s, lead to a $\sigma_{\text{tot}}(s) \geq s^{-6}$ satisfying the lower bound. But in that case $\langle n\rangle$ will not be $O(\lambda s^{\lambda^2})$, but will lie in the latter interval where the σ_n's are much larger. To be more specific, suppose we have σ_n given by Eq. (2.38) for all n except for an interval, $n_1 \leq n \leq n_2$, where the σ_n's are such that $\sigma_n(s) \geq s^{-6.5}$ and $n_2 < (\ln s)^p$. Then clearly the contributions to $\langle n\rangle$ from the terms of the form (2.38) are exponentially damped and $\langle n\rangle$ is bounded by $n_1 \leq \{\langle n\rangle\} \leq n_2$.

(4) Our result is essentially a weak coupling result. Even in the case of models we cannot say anything about the behavior of $\langle n\rangle$ for values of λ outside the radius of convergence.[8]

(5) In the final section of this paper we shall comment on the possibility of obtaining similar bounds on $\langle n^p\rangle$ with weaker assumptions than those in (a) and (b). But while the assumptions (a) and (b) are not demonstrated to be true in rigorous field theory and (a) is almost certainly not true for field theories, the assumptions are true in many models. They certainly hold in many multiperipheral type models. The theorem we have just proved shows that the fact that $\langle n\rangle$ does not grow faster than $\ln s$ in these models is not a dynamical detail of the models, but a general feature controlled by temperedness and convergence properties.

One model that does not satisfy our bounds is the nova model.[9] But that model has no power-series expansion in any effective coupling constant. So condition (a) cannot be even formulated for it.

Another model which gives $\langle n\rangle \sim s^a$, $a > 0$, is that of Cheng and Wu.[5] There, as we shall see in Sec. III, one effectively has an ansatz which violates condition (b) and for which $\sigma_n(s,\lambda)$ grows exponentially in s for complex values of λ.

III. REMARKS ON THE CHENG-WU MODEL

As mentioned previously, Cheng and Wu[5] have recently proposed an extension of their impact model that enables them to calculate the average multiplicity. The result in this model is $\langle n\rangle \sim s^a$, where a is a function of the coupling constant. As noted by the authors, this extension treats only a special set of diagrams and "the basis of the conclusions reached is perhaps not as general as that for the ones reached in previous papers." For a critical discussion of the results of this extension, the reader is referred to a recent paper by Mueller.[10]

In this section we shall consider an ansatz for $\sigma_n(s,\lambda)$ which, for the problems discussed in this paper, has all the main features of the Cheng and Wu result. We show that the crucial property that this ansatz does not satisfy is the assumption (b), polynomial boundedness.

We write for $\sigma_n(s,\lambda)$

$$\sigma_n(s,\lambda) = \frac{(\lambda s^{\lambda^2})^n}{n!} \exp(-\lambda s^{\lambda^2}) \ . \tag{3.1}$$

This simple ansatz is not the Cheng-Wu result, but it does exhibit all the features of their model that are relevant to the discussion of the questions raised in this paper. For example all the $\ln s$ factors that appear in the expression for σ_n in Ref. 5 are not significant for our discussion.

The average multiplicity given by (3.1) is

$$\langle n\rangle = O(s^{\lambda^2}) \ . \tag{3.2}$$

Furthermore, the convergence of the power-series expansion in λ is evident in this case. Also for real λ the Jin-Martin lower bound is respected,

$$\sigma_{\text{tot}}(s,\lambda) \geq \operatorname*{Max}_n \sigma_n(s,\lambda) \geq Cs^{-\lambda^2/2} \ . \tag{3.3}$$

For real λ, $\sigma_n(s,\lambda)$ as given in Eq. (3.1) is certainly polynomially bounded. The increasing factor $(s^{\lambda^2})^n$ is always damped by the exponential factor even for $n \cong \lambda s^{\lambda^2}$, the maximum point. However, as soon as λ becomes complex this is no longer necessarily true. For example, it is easy to choose a phase for λ such that $|\exp(-\lambda s^{\lambda^2})|$ is exponentially increasing for some large values of real s. Thus the ansatz (3.1) does not satisfy condition (b). Indeed the quantity $\mathrm{Max}_{n,\phi}|\sigma_n(s, Re^{i\phi})| = M(s)$ for this model behaves as predicted in remark (2) of Sec. II and has an exponential growth.

At this stage one might remark that condition (b) as stated is somewhat unphysical since it involves the behavior for complex λ. However, one can show, as we shall immediately proceed to do, that having $\langle n\rangle \sim s^a$ also implies restrictions on the asymptotic behavior of the Feynman diagrams contributing to $\sigma_n(s,\lambda)$ for real λ and for some high order and some large n.

Suppose for the moment we restrict ourselves to a subset of Feynman diagrams for which perturbation expansions converge. Then from Eq. (2.7) and Eq. (2.13b) we have the convergent expansions

$$\sigma_n(s,\lambda) = \sum_{k=n}^{\infty} \lambda^k \alpha_n^{(k)}(s)$$

and (3.4)

$$\bar{\sigma}_n(s,\lambda) = \sum_{j=0}^{\infty} \lambda^j \alpha_n^{(j+n)}(s)\ ,$$

where $\alpha_n^{(k)}(s)$ is the contribution from all the diagrams in our subset to order k for the production process $2\to n$. By construction $\alpha_n^{(k)}$ is independent of λ and is calculated from the diagrams by setting the vertices equal to unity. For fixed finite n and k, $\alpha_n^{(k)}(s)$ must be tempered for large s. The behavior of $\alpha_n^{(k)}(s)$ for large s, when k and n also grow, is not known. Nevertheless, if we are willing to assume a uniform tempered bound on $\alpha_n^{(k)}(s)$, then the same bound on $\langle n\rangle$ we obtained in Sec. II holds for weak coupling, $\lambda<1$.

Indeed, let us assume for large s

$$|\alpha_n^{(k)}(s)| \le C(n,k) s^{N'}\ , \tag{3.5}$$

where N' is independent of n and k. Namely, what we are assuming is that the limit for large s and k and n large is bounded in the same way as the case for large s and fixed k and n. Furthermore, since we are dealing with a convergent subset of diagrams, we shall assume that the $C(n,k)$ in Eq. (3.5) are such that $\sum_{k=n}^{\infty} C(n,k)\lambda^k$ converges absolutely for values of λ, $|\lambda|<R$, and is tempered in n, i.e., $\sum C(n,k)|\lambda|^k \le n^\alpha$, α finite, n large. This immediately leads us to condition (b) and the results of Sec. II, since

$$|\sigma_n(s,\lambda)| \le \left[\sum_{k=0}^{\infty} |\lambda|^k C(n,k)\right] s^{N'},\quad n \le \sqrt{s}$$

for any λ real or complex inside the radius of convergence defined by the $C(n,k)$. Thus we see that condition (b) can be stated in terms of the asymptotic behavior of Feynman diagrams with λ set equal to unity.

Any model with a convergent perturbation expansion and a Jin-Martin lower bound, but with $\langle n\rangle \sim s^a$, must either violate Eq. (3.5) for large values of n and/or k, or violate the convergence and temperedness assumption on $C(n,k)$.

We are certainly aware that the assumption (3.5) is strong. For example, the bound in Eq. (3.5) might hold, but only with coefficients $C(n,k)$ for which the series $\sum_k \lambda^k C(n,k)$ is divergent, even though one starts with a model in which the series in Eq. (3.4) have a finite radius of convergence. The discussion we have just gone through is only intended to show that assumption (b) can be restated in terms of the asymptotic properties of physical Feynman diagrams.

Finally we should stress again the importance in assumption (b) of not just the temperedness in s but the temperedness, in a sense, in n also. Since $n\le\sqrt{s}$ always, the temperedness in n can be absorbed into the temperedness in s. It is not enough for our results to just assume temperedness in s for fixed finite n.

IV. DISCUSSION AND POSSIBLE GENERALIZATIONS

The main question that comes up at this stage concerns the generalization of our results to axiomatic field theory. As they stand, assumptions (a) and (b) are not known to be true in any field theory. In fact assumption (a), which involves the convergence of perturbation theory, is most probably not true.[3] Thus if conditions (a) and (b) turn out to be both necessary and sufficient for our bounds, then the task for obtaining rigorous bounds will become almost hopeless. This, fortunately, is not the case.

The simplest way to see that assumption (a) is not necessary is by constructing an example. One can take a set $\sigma_n(s,\lambda)$, which is analytic only in a semicircle of radius R and $\mathrm{Re}\lambda>0$, with a finite number of poles in the semicircle with $\mathrm{Re}\lambda<0$. If $|\sigma_n(s,Re^{i\phi})| \le s^N$ on the circle, then our bounds will still hold for $0<\lambda<R$ even though perturbation expansions will diverge for $|\lambda|$ equal to the modulus of the nearest pole or larger. Thus, in this example (a) does not hold while (b) is still true (excluding the neighborhoods of the poles), and the results are the same as in Sec. II. One can construct more sophisticated examples with cuts instead of poles in the left half λ plane. The cru-

cial ingredient needed for our bounds is always some form of temperedness.

Results both on analyticity in λ and temperedness for complex values of λ have recently been obtained by Glimm, Jaffe, and Spencer.[4] They have shown that in $\lambda P(\phi)_2$ theories, the Euclidean Schwinger Green's functions are analytic in a small semicircle in the λ plane centered at the origin with $\text{Re}\lambda > 0$. Inside this domain the Schwinger functions remain tempered even for complex values of λ. This result has not yet been continued to the case of real time and the physical Wightman functions. After such a continuation the physical values of λ, i.e., λ real and positive, will most probably end up on the boundary of the resulting domain of analyticity, but it will still be interesting to see if temperedness survives inside the final domain. However, if the physical values of λ end up on the boundary of the domain of analyticity, then the information implied by temperedness for complex values of λ will be essentially useless for our purposes in this paper.

The situation at this stage does seem bleak. However, in the case of weak coupling, $\lambda < 1$, what we need to get the bound $\langle n\rangle \leq C \ln s$ is quite minimal. It does not involve either analyticity in λ, or polynomial boundedness for complex λ. We only need to know if the reduced cross-sections $\tilde{\sigma}_n(s,\lambda)$ as defined in Eq. (2.13a) are tempered or not for physical real λ, $\lambda < 1$. Indeed, one can easily prove that the inequality

$$\tilde{\sigma}_n(s,\lambda) \leq s^N, \quad \lambda < 1 \tag{4.1}$$

leads to the bound $\langle n\rangle \leq C \ln s$ for $\lambda < 1$. All that is needed is the definition $\sigma_n(s,\lambda) \equiv \lambda^n \tilde{\sigma}_n(s,\lambda)$ and the Jin-Martin lower bound. If we take $\lambda < 1$, and $\gamma = \ln(1/\lambda)$, then it is easy to check from Eq. (4.1) that (2.17) holds. The rest of the proof is identical to that in Sec. II.

Of course we have no proof for Eq. (4.1). But we have bounds on $\tilde{\sigma}_n(s,\lambda)$, both for $\lambda \geq 1$ and for $\lambda = 0$. In the first case, $\lambda \geq 1$, we know that $\sigma_n(s,\lambda) \leq \sigma_{\text{tot}}(s,\lambda) \leq S^N$. Hence from positivity and temperedness, without even using unitarity,[11] we have

$$\tilde{\sigma}_n(s,\lambda) \leq s^N, \quad \lambda \geq 1 \ . \tag{4.2}$$

This result is rigorous. Furthermore, in the limit $\lambda = 0$, the only diagrams that contribute to $\tilde{\sigma}_n(s,\lambda=0)$ are the tree diagrams[12] shown in Fig. 1. These are essentially the multiperipheral model with a coupling constant of unity. Hence, here again we have, at least when perturbation expansions converge,

$$\tilde{\sigma}_n(s,0) \leq s^{N'} \ . \tag{4.3}$$

The question is then simply this: Can $\tilde{\sigma}_n(s,\lambda)$ be tempered for $\lambda \geq 1$, and yet have exponential growth for λ in between, $0 < \lambda < 1$? For, in fact, if $\langle n\rangle \cong s^{a(\lambda)}$ for $\lambda < 1$, then following the argument of remark (2) in Sec. II, we can show that there must exist at least one value of $n = n_0$ such that $\tilde{\sigma}_{n_0}(s,\lambda)$ behaves at least like e^{cs^a} for large s, on some sequence of points $\{s_j\}$, $s_j \to \infty$ as $j \to \infty$.

The ansatz considered in Sec. III does indeed violate the bound (4.1). However, it is perhaps instructive to note that it satisfies the rigorous bound (4.2) for $\lambda \geq 1$, and all n.

Finally, we see that for weak coupling the general validity of the bounds (2.11) and (2.12) depends on the simple question of whether the trivial factor λ^n that must appear in $\sigma_n(s,\lambda)$ is an integral part of the temperedness property, or whether the expression for $\sigma_n(s,\lambda)$ is still tempered with the λ^n factor taken out.

For fixed finite n, and even for $\lambda < 1$, we know from the convergence of the dispersion relation that $\tilde{\sigma}_n(s,\lambda)$ is tempered. The integral $\int_c^\infty \sigma_{\text{tot}}(s,\lambda) s^{-2} ds$ must converge. Hence also the integral $\int^\infty \sigma_n(s,\lambda) s^{-2} ds$ is convergent. Thus for fixed n and λ we must have for large enough s, at least on a sequence of s values,

$$\tilde{\sigma}_n(s,\lambda) \leq C(n,\lambda) s \ , \tag{4.4}$$

where $C(n,\lambda)$ is a positive function of n and λ. The whole question hinges on how $C(n,\lambda)$ behaves for large n for fixed λ, $\lambda < 1$, with n always such that $n \leq \sqrt{s}$.

The alternative conditions we have just listed are deceptively simple. While they do not involve any analyticity assumptions, they are too close to the final result to be considered as necessarily weaker than (a) and (b).

ACKNOWLEDGMENTS

Much of this work was done while the author was visiting the National Accelerator Laboratory. He wishes to thank several members of the NAL theory group for many helpful discussions. He is also indebted to H. Cornille and A. Mueller for several helpful remarks.

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*Work supported in part by the U. S. Atomic Energy Comission under Contract No. AT(11-1)-2232.

[1]M. Froissart, Phys. Rev. 123, 1053 (1961). The Froissart bound also holds in Araki-Haag field theories; see H. Epstein, V. Glaser, and A. Martin, Commun. Math. Phys. 13, 257 (1969).

[2]A. A. Logunov and M. A. Mestvirishvily, CERN Report No. Th. 1707, 1973 (unpublished).

[3] A. Jaffe, Commun. Math. Phys. 1, 127 (1965).

[4] J. Glimm, A. Jaffe, and T. Spencer (unpublished).

[5] H. Cheng and T. T. Wu, Phys. Lett. 45B, 367 (1973).

[6] Y. S. Jin and A. Martin, Phys. Rev. 135, B1369 (1964). See also H. Cornille, Nuovo Cimento 4A, 549 (1971).

[7] See A. Martin, in *Strong Interactions and High Energy Physics, Scottish Universities Summer School Edinburgh, 1963*, edited by R. G. Moorhouse (Oliver and Boyd, Edinburgh, Scotland, 1964), pages 105–108.

[8] S.-J. Chang and T.-M. Yan, Phys. Rev. Lett. 25, 1586 (1970). This paper presents a ϕ^3-type model with $\langle n\rangle \sim \ln s$ for weak coupling and $\langle n\rangle \sim s^{\alpha}$ for strong coupling.

[9] M. Jacob and R. Slansky, Phys. Rev. D 5, 1847 (1972).

[10] A. Mueller, rapporteur's talk, in proceedings of the second Aix-en-Provence International Conference on Elementary Particles, September, 1973 (unpublished).

[11] The reader might guess at this stage that if we use the strong assumption $\tilde{\sigma}_n(s,\lambda) \le (\ln s)^{\beta}$, for $\lambda < 1$, we will automatically get $\langle n\rangle \le C \ln \ln s$. This is *not* necessarily true. The explanation of this fact also explains why in remark (2) of Sec. II we limited ourselves to $M(s)$ such that $M(s) \ge s^{\epsilon}$, $\epsilon > 0$. Indeed if $M(s)$, as defined in Eq. (2.30), is such that $M(s) \le (\ln s)^{\beta}$, then the sum in Eq. (2.4) will have an effective cutoff at $n = L = C \ln \ln s$. However, the terms of the sum for $n > L$ could sum up to $O((\ln s)^{-M})$. This will not be a negligible contribution to $\sigma_{\text{tot}}(s,\lambda)$ unless we are given a lower bound stronger than the Jin-Martin bound, namely something like $\sigma_{\text{tot}}(s,\lambda) \ge (\text{const}) \times (\ln s)^{-\alpha}$, $\alpha > 0$, and α finite. Only if such a bound is true, and $M(s) \le (\ln s)^{\beta}$, will $L = C \ln \ln s$ give a useful cutoff for large enough C.

[12] This obviously can only be guaranteed when perturbation expansions are convergent near $|\lambda| = 0$. If perturbation series are asymptotic, the contributions to $\tilde{\sigma}_n(s,\lambda)$ as $\lambda \to 0$ are not necessarily just the tree diagrams. The bound (4.3) is thus not rigorous in general.

PHYSICAL REVIEW D VOLUME 9, NUMBER 6 15 MARCH 1974

Sixth-order electron g factor: Mass-operator approach. I*

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(Received 15 October 1973)

A magnetic-moment contribution, due to the sixth-order process in which the lowest-order radiative correction to the internal-electron propagator is twice iterated, is calculated analytically, using an improved mass-operator method developed by Schwinger. Our result confirms that of Levine and Roskies.

The next generation of experiments[1] to measure the g factor of the electron or muon should be of sufficient accuracy to discern with some precision the order-α^3 electrodynamic correction[1] as well as strong-interaction[2] and perhaps weak-interaction effects.[3] There have been a number of calculations of sixth-order electrodynamic contributions[4–7]; all such processes have been evaluated numerically,[4,5] while a few have been computed analytically.[6,7] Insofar as there exist discrepancies between various computations of perhaps 20%,[4] there is room for improvement. Indeed, it would be desirable to have a complete analytic evaluation, although this would appear to be a very formidable undertaking.

It is in part a question of efficient organization. The usual diagrammatic expansion involves 40 distinct diagrams.[4] Actually, the number of contributing processes can be greatly reduced if instead one employs the mass-operator method,[5,8–10] in which one considers the propagation of an electron in a weak homogeneous magnetic field. Schwinger[10] has recently developed an improved version of this method in an efficient rederivation of the α^2 moment, and the extension of his methods to an analytic evaluation of the α^3 effects does not seem unreasonable.

In this paper we will first review certain aspects of his calculation, those in which the internal-electron propagation function undergoes radiative corrections. Then we apply this method to a particular sixth-order process in which the lowest-order radiative correction to this propagator is twice iterated.[11] Although this result is not new, we present this calculation in order to demonstrate how such an analytic evaluation can be carried out simply and quickly, and specifically, to check the result of Levine and Roskies.[7] Hopefully, related mass-operator methods should effect simplifications in other sixth-order processes.

Consider an electron moving in a weak, homogeneous, magnetic field, described by the action term

$$-\tfrac{1}{2}\int (dx)(dx')\psi(x)\gamma^0 M(x,x')\psi(x') . \tag{1}$$

Given M, we extract the magnetic moment by applying the mass-shell condition