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Minimal hierarchy of Faddeev-type equations for N-particle scattering*

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Operators are introduced, which make up the kernels of all N-body Faddeev-type scattering equations. The hierarchy of these operators, and the equations they satisfy, are shown to provide the minimal description of all Faddeev-type formalisms. By means of this hierarchy, the N-body formalisms proposed by Yakubovskii and by Alt, Grassberger, and Sandhas are shown to be equivalent. The four-body case is treated in some detail.

I. INTRODUCTION

After the pioneering work of Faddeev on the formulation and solution of the three-body problem,[1] several attempts have been made to obtain viable systems of equations for N particles. Among these, two approaches are particularly interesting, namely the ones presented by Yakubovskii[2] and by Alt, Grassberger, and Sandhas[3] (AGS).

The method of Yakubovskii relies on a powerful index notation to handle the channel structure arising from the separation of N particles into subgroups. It can be understood as a repeated application of the Faddeev procedure of removing from the kernel of the N-body Lippmann-Schwinger (LS) equations the pieces representing disconnected subprocesses.

The AGS approach is based on a scheme for writing down three-body relations as matrix versions of two-body relations; in particular the three-body Faddeev equations correspond to matrix Lippmann-Schwinger-type equations. For the four-body case, a matrix version of the Faddeev procedure is applied to such LS equations, and the resulting Faddeev-type matrix equations are again written in two-body-like form. In this way, an inductive prescription is established for the generation of matrices of operators for the N-body case; their equations are obtained by simply writing down the N-body matrix version of the corresponding two-body relations.

In this paper we show that these two approaches are equivalent, but that neither of them provides the most concise description of the hierarchy of N-body equations with Faddeev-type kernels. By generalizing to the N-body case an alternative formalism based on the three-body K operators,[4] we obtain a hierarchy of equations for precisely the operators of the kernels, and we identify these as the minimal hierarchy for the N-body problem.

A detailed description of the Faddeev-Yakubovskii (FY) procedure for $N=4$ is given in Sec. II, where, in addition to reproducing the Yakubovskii[2] results, we obtain symmetric four-body M operators that form a more natural generalization of the three-body Faddeev $M_{\beta\alpha}$'s than the operators obtained by Yakubovskii.

The AGS formalism for $N=4$ is outlined in Sec. III; M operators are also obtained within this scheme.

In Sec. IV, the three-body K operators and their equations are generalized to the four-body case. In order to see the relevance of the K operators, we show that all the four-body equations obtained in Secs. II to IV have the same kernel, namely maximal[5] subsystem K operators. In other words, just as the two-body t operators make up the three-body Faddeev kernel, the (3+1)- and (2+2)-subsystem K operators make up the four-body Faddeev-type kernel. Since they produce four-body equations with identical kernels, we conclude that the AGS and FY four-body formalisms are equivalent. We end Sec. IV with a detailed explanation of the minimal characteristics of the N-body K-operator hierarchy and its relation to other hierarchies.

The N-body scattering problem is treated in Sec. V. The K-operator hierarchy is constructed within both the AGS and FY formalisms, thereby proving that the N-body equations of both formalisms have identical kernels. The equivalence of these two formalisms is thus established for an arbitrary number of particles.

Finally, in Sec. VI, the wave-function formulation of the N-body problem is discussed, and it is shown that the N-body K operators yield the Faddeev-type components of the full wave function out of the initial-state components. Their Faddeev-type equations have as a driving term the components of the initial-state wave function, and have, as do all N-body equations, the maximal subsystem K operators as elements of the kernel.

II. THE FADDEEV FORMALISM

In two-body scattering theory, the basic equation for the transition operator t in terms of the potential v and the resolvent operator $g_0=(h_0-z)^{-1}$ is the Lippmann-Schwinger equation

$$t=v-vg_0t=v-tg_0v\ . \tag{2.1}$$

As is well known, by removing the two-body disconnected piece of the kernel in the three-body version of the first of Eqs. (2.1), Faddeev obtains operators ${}^{\beta}T$ with equations

$${}^{\beta}T=t_\beta-t_\beta G_0\sum_\gamma \bar\delta_{\beta\gamma}\,{}^{\gamma}T \tag{2.2}$$

and

$${}^{\beta}T=v_\beta-{}^{\beta}TG_0V\ , \tag{2.3}$$

where $\bar\delta_{\beta\gamma}=1-\delta_{\beta\gamma}$. Equation (2.2) is a mathematically satisfactory three-body equation for ${}^{\beta}T$, but (2.3) is not. However, by applying the same procedure again to (2.3), Faddeev obtains equations for operators[1] $M_{\beta\alpha}$ that are symmetric in the sense that the counterparts of both (2.2) and (2.3) have connected kernels:

$$\begin{aligned} M_{\beta\alpha}&=\delta_{\beta\alpha}t_\beta-t_\beta G_0\sum_\gamma \bar\delta_{\beta\gamma}M_{\gamma\alpha}\ ,\\ M_{\beta\alpha}&=\delta_{\beta\alpha}t_\beta-\sum_\gamma \bar\delta_{\gamma\alpha}M_{\beta\gamma}G_0t_\alpha\ . \end{aligned} \tag{2.4}$$

In (2.4) the three-body indices α,β,γ label pairs of particles and run over the values 12,13,23.

If we now turn to the four-body problem—still with pairwise potentials—we can interpret (2.4) as being equations for a four-body operator $M^{(4)}_{\beta\alpha}$, where now β and α run over all possible values 12, 13, 23, 14, 24, 34. So interpreted, (2.4) does not form a satisfactory set of four-body equations, since the kernel contains three-body disconnected pieces. This problem is solved by generalizing to the four-body level the basic Faddeev procedure of removing disconnected parts from the kernels.

To do so, we introduce four-body indices σ,τ,ρ, which label the seven different ways in which four particles can be separated in two groups,

$$\begin{aligned}&(123)(4),\ (421)(3),\ (341)(2),\ (432)(1),\\ &(12)(34),\ (13)(24),\ (14)(23)\ .\end{aligned} \tag{2.5}$$

They define channels in the four-body system when it is understood that particles within the same group can interact, but no interaction exists between particles belonging to different groups. The indices α, β, and γ label the six ways in which four particles can be split into three groups $(\cdot\cdot)(\cdot)(\cdot)$, i.e., they label interacting pairs as before. Pair indices will usually appear as subordinate indices, in the sense that they label interacting pairs within a certain channel of the type σ. In such a case we write $\beta\subset\sigma$.

When applying the Faddeev procedure at the four-body level, it is convenient to exclude the two-body disconnected pieces in (2.4) from the definition of four-body operators; we thus define

$$Y^{\sigma\cdot}_{\beta\alpha}=-t_\beta G_0\sum_{\gamma\subset\sigma}\bar\delta_{\beta\gamma}M^{(4)}_{\gamma\alpha}\ , \tag{2.6}$$

where γ in the sum is now restricted to σ, and it is understood that $\beta\subset\sigma$. The superscript dot emphasizes the asymmetric character of $Y^{\sigma\cdot}_{\beta\alpha}$, as will be seen below. From (2.4) we see that

$$\sum_{\sigma\supset\beta}Y^{\sigma\cdot}_{\beta\alpha}=M^{(4)}_{\beta\alpha}-\delta_{\beta\alpha}t_\beta\ , \tag{2.7}$$

where the sum in (2.7) runs over all values of σ in (2.5) such that $\beta\subset\sigma$.

Continuing the generalization of the Faddeev procedure, we now shift the $Y^{\sigma\cdot}_{\gamma\alpha}$ piece of $M^{(4)}_{\gamma\alpha}$ in (2.6) to the left-hand side, and multiply by the operator

$$\left(\delta_{\beta\beta'}-\sum_{\gamma\subset\sigma}\bar\delta_{\gamma\beta'}M^{\sigma}_{\beta\gamma}G_0\right)\ ,$$

where $M^{\sigma}_{\beta\gamma}$ satisfies (2.4) with α,β,γ all restricted to σ. For σ of the type $(\cdot\cdot\cdot)(\cdot)$, $M^{\sigma}_{\beta\alpha}$ is the familiar three-body Faddeev operator. For σ of the type $(\cdot\cdot)(\cdot\cdot)$, $M^{\sigma}_{\beta\alpha}$ is a "2+2" operator, discussed for instance in Ref. 6. We thus obtain the equation

$$\begin{aligned}Y^{\sigma\cdot}_{\beta\alpha}&=W^{\sigma}_{\beta\alpha}\,\delta(\alpha\subset\sigma)\\ &\quad-\sum_{\gamma\subset\sigma}\left(\sum_{\gamma'\subset\sigma}\bar\delta_{\gamma'\gamma}M^{\sigma}_{\beta\gamma'}\right)G_0\sum_{\rho\supset\gamma}\bar\delta^{\sigma\rho}\,Y^{\rho\cdot}_{\gamma\alpha}\ ,\end{aligned} \tag{2.8}$$

where $W^{\sigma}_{\beta\alpha}=M^{\sigma}_{\beta\alpha}-\delta_{\beta\alpha}t_\beta$ is the connected part of $M^{\sigma}_{\beta\alpha}$, and $\delta(\alpha\subset\sigma)$ indicates that the driving term in (2.8) is nonzero only when $\alpha\subset\sigma$. $Y^{\sigma\cdot}_{\beta\alpha}$ also satisfies the relation

$$Y^{\sigma\cdot}_{\beta\alpha}=-t_\beta G_0t_\alpha\bar\delta_{\beta\alpha}\,\delta(\alpha\subset\sigma)-\sum_{\gamma'}\bar\delta_{\gamma'\alpha}\,Y^{\sigma\cdot}_{\beta\gamma'}\,G_0t_\alpha\ . \tag{2.9}$$

In addition to $Y^{\sigma\cdot}_{\beta\alpha}$, it is sometimes useful to define four-body operators that do include two-body disconnected pieces:

$$Y^{\sigma\tau}_{\beta\alpha}=Y^{\sigma\cdot}_{\beta\alpha}+\delta^{\sigma\tau}\,\delta_{\beta\alpha}t_\beta\;. \qquad (2.10)$$

The operators $Y^{\sigma\tau}_{\beta\alpha}$ are the ones obtained by Yakubovskii.[2] They satisfy an equation similar to (2.8) but with $\delta^{\sigma\tau}M^{\sigma}_{\beta\alpha}$ as driving term.

From (2.8) and (2.9) we see that $Y^{\sigma\cdot}_{\beta\alpha}$—and also $Y^{\sigma\tau}_{\beta\alpha}$—has an asymmetric character, as was the case for the ${}^{\beta}T$ in (2.2) and (2.3). In other words, only (2.8) is satisfactory as a four-body equation.

As in the three-body case, we proceed to construct symmetric four-body operators $M^{\sigma\tau}_{\beta\alpha}$ out of $Y^{\sigma\cdot}_{\beta\alpha}$ by an additional splitting from the opposite side:

$$M^{\sigma\tau}_{\beta\alpha}=-\,\delta^{\sigma\tau}\,\bar{\delta}_{\beta\alpha}t_\beta G_0 t_\alpha +t_\beta G_0\left(\sum_{\gamma\subset\sigma}\;\sum_{\gamma'\subset\tau}\bar{\delta}_{\beta\gamma}M^{(4)}_{\gamma\gamma'}\,\bar{\delta}_{\gamma'\alpha}\right)G_0 t_\alpha\;, \qquad (2.11)$$

with

$$\sum_{\tau\supset\alpha}M^{\sigma\tau}_{\beta\alpha}=Y^{\sigma\cdot}_{\beta\alpha}\;. \qquad (2.12)$$

Equation (2.11) clearly displays the symmetric character of $M^{\sigma\tau}_{\beta\alpha}$. These operators satisfy the equations

$$M^{\sigma\tau}_{\beta\alpha}=\delta^{\sigma\tau}\,W^{\sigma}_{\beta\alpha}-\sum_{\gamma\subset\sigma}\left(\sum_{\gamma'\subset\sigma}\bar{\delta}_{\gamma'\gamma}\,M^{\sigma}_{\beta\gamma'}\right)G_0 \times\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}\,M^{\rho\tau}_{\gamma\alpha}\;,$$

$$M^{\sigma\tau}_{\beta\alpha}=\delta^{\sigma\tau}\,W^{\sigma}_{\beta\alpha}-\sum_{\gamma\subset\sigma}\;\sum_{\rho\supset\gamma}\bar{\delta}^{\rho\tau}M^{\sigma\rho}_{\beta\gamma}\,G_0 \times\left(\sum_{\gamma'\subset\tau}\bar{\delta}_{\gamma\gamma'}M^{\tau}_{\gamma'\alpha}\right). \qquad (2.13)$$

We thus see that from a one-sided Faddeev procedure Yakubovskii obtains operators with an asymmetric character, and that it is easy to proceed in a way that yields symmetric four-body operators more in analogy with the three-body case. We finally note that

$$W^{\sigma\tau}_{\beta\alpha}=M^{\sigma\tau}_{\beta\alpha}-\delta^{\sigma\tau}\,W^{\sigma}_{\beta\alpha}$$

is the four-body connected part of $M^{\sigma\tau}_{\beta\alpha}$, and that, if T stands for the full four-body transition operator,

$$\sum_{\sigma,\tau}\;\sum_{\substack{\beta\subset\sigma\\ \alpha\subset\tau}}M^{\sigma\tau}_{\beta\alpha}=T-\sum_{\gamma}t_\gamma\;, \qquad (2.14)$$

i.e., the six transition amplitudes corresponding to two-body disconnected processes are not present in $M^{\sigma\tau}_{\beta\alpha}$.

III. THE AGS FORMALISM

Consider now the Faddeev-type equations for the AGS three-body transition operators $U_{\beta\alpha}$,

$$U_{\beta\alpha}=-\,\bar{\delta}_{\beta\alpha}\,G_0^{-1}-\sum_{\gamma}\bar{\delta}_{\beta\gamma}\,t_\gamma G_0 U_{\gamma\alpha}\;,$$
$$U_{\beta\alpha}=-\,\bar{\delta}_{\beta\alpha}\,G_0^{-1}-\sum_{\gamma}\bar{\delta}_{\gamma\alpha}\,U_{\beta\gamma}\,G_0 t_\gamma\;. \qquad (3.1)$$

If matrices of operators T, V, and G_0 are defined according to[3]

$$\mathsf{T}=\{U_{\beta\alpha}\}\;,$$
$$\mathsf{V}=\{-\,\bar{\delta}_{\beta\alpha}\,G_0^{-1}\}\;, \qquad (3.2)$$
$$\mathsf{G}_0=\{-\,\delta_{\beta\alpha}\,G_0 t_\beta G_0\}\;,$$

(3.1) can be recast into a form similar to (2.1),

$$\mathsf{T}=\mathsf{V}-\mathsf{V}\,\mathsf{G}_0\,\mathsf{T}=\mathsf{V}-\mathsf{T}\,\mathsf{G}_0\,\mathsf{V}\;. \qquad (3.3)$$

Other three-body operator relations can also be written as matrix versions of familiar two-body relations. For instance, Eqs. (2.4) take the form

$$\mathsf{G}=\mathsf{G}_0-\mathsf{G}_0\,\mathsf{V}\,\mathsf{G}=\mathsf{G}_0-\mathsf{G}\,\mathsf{V}\,\mathsf{G}_0\;, \qquad (3.4)$$

where

$$\mathsf{G}=\{-\,G_0 M_{\beta\alpha}\,G_0\}\;. \qquad (3.5)$$

Moreover, $M_{\beta\alpha}$ is related to $U_{\beta\alpha}$ through

$$\mathsf{G}\,\mathsf{V}=\mathsf{G}_0\,\mathsf{T}\;,$$
$$\mathsf{V}\,\mathsf{G}=\mathsf{T}\,\mathsf{G}_0\;, \qquad (3.6)$$
$$\mathsf{T}=\mathsf{V}-\mathsf{V}\,\mathsf{G}\,\mathsf{V}\;.$$

The AGS matrix notation clearly displays the structural similarity of the two- and three-body scattering equations.

Using a matrix version of the Faddeev procedure, one can now construct matrix Faddeev equations for the four-body case. However, the resulting equations can also be obtained by direct analogy,

$$\mathsf{T}^{(4)}=\mathsf{V}^{(4)}-\mathsf{V}^{(4)}\,\mathsf{G}_0^{(4)}\,\mathsf{T}^{(4)}=\mathsf{V}^{(4)}-\mathsf{T}^{(4)}\,\mathsf{G}_0^{(4)}\,\mathsf{V}^{(4)}\;, \qquad (3.7)$$

$$\mathsf{G}^{(4)}=\mathsf{G}_0^{(4)}-\mathsf{G}_0^{(4)}\,\mathsf{V}^{(4)}\,\mathsf{G}^{(4)}=\mathsf{G}_0^{(4)}-\mathsf{G}^{(4)}\,\mathsf{V}^{(4)}\,\mathsf{G}_0^{(4)}\;,$$

using (3.2) to guide the definition of appropriate four-body matrices of operators $\mathsf{V}^{(4)}$ and $\mathsf{G}_0^{(4)}$,

$$\mathsf{V}^{(4)}=\{-\,\bar{\delta}^{\sigma\tau}\,\mathsf{G}_0^{-1}\}$$
$$=\{\bar{\delta}^{\sigma\tau}\,\delta_{\beta\alpha}(G_0 t_\beta G_0)^{-1}\}\;,$$
$$\mathsf{G}_0^{(4)}=\{-\,\delta^{\sigma\tau}\,\mathsf{G}_0\,\mathsf{T}^{\sigma}\,\mathsf{G}_0\} \qquad (3.8)$$
$$=\{-\,\delta^{\sigma\tau}\,G_0 t_\beta G_0 U^{\sigma}_{\beta\alpha}\,G_0 t_\alpha\,G_0\}$$
$$=\{-\,\delta^{\sigma\tau}\,G_0\,W^{\sigma}_{\beta\alpha}\,G_0\}\;.$$

In fact, it can be shown that all four-body equations within the AGS formalism can be obtained by analogy. We shall always do so in the present work. The superscript (4) in (3.7), etc. indicates that the matrices so labeled involve four-body operators. [A similar superscript (3) in (3.2), etc., has been suppressed, so it is understood that all matrices without superscript in this section and the next involve three-body operators.]

With the notation

$$\mathsf{T}^{(4)}=\{\mathsf{T}^{\sigma\tau}\}=\{U^{\sigma\tau}_{\beta\alpha}\}\ ,\qquad \mathsf{G}^{(4)}=\{\mathsf{G}^{\sigma\tau}\}=\{-G_0M^{\sigma\tau}_{\beta\alpha}G_0\}\ ,\tag{3.9}$$

we obtain from (3.7) more explicit four-body equations,

$$\mathsf{T}^{\sigma\tau}=-\bar\delta^{\sigma\tau}\mathsf{G}_0^{-1}-\sum_\rho\bar\delta^{\sigma\rho}\,\mathsf{T}^\sigma\mathsf{G}_0\mathsf{T}^{\rho\tau}\ ,\qquad \mathsf{G}^{\sigma\tau}=-\delta^{\sigma\tau}\mathsf{G}_0\mathsf{T}^\sigma\mathsf{G}_0-\mathsf{G}_0\mathsf{T}^\sigma\sum_\rho\bar\delta^{\sigma\rho}\mathsf{G}^{\rho\tau}\ ,\tag{3.10}$$

or

$$U^{\sigma\tau}_{\beta\alpha}=\bar\delta^{\sigma\tau}\,\delta_{\beta\alpha}(G_0t_\beta G_0)^{-1}+\sum_{\rho\supset\beta}\sum_{\gamma\subset\rho}\bar\delta^{\sigma\rho}U^\rho_{\beta\gamma}G_0t_\gamma G_0U^{\rho\tau}_{\gamma\alpha}\ ,$$
$$M^{\sigma\tau}_{\beta\alpha}=\delta^{\sigma\tau}W^\sigma_{\beta\alpha}+t_\beta G_0\sum_{\gamma\subset\sigma}U^\sigma_{\beta\gamma}G_0\sum_{\rho\supset\gamma}\bar\delta^{\sigma\rho}M^{\rho\tau}_{\gamma\alpha},\tag{3.11}$$

and similarly for the remaining equations in (3.7). The operators $U^\rho_{\beta\gamma}$ in (3.11) can be obtained from (3.1) with α,β,γ all restricted to ρ. For ρ of the type $(\cdots)(\cdot)$, $U^\rho_{\beta\gamma}$ is the familiar three-body AGS operator. As in the previous section, the compatibility conditions between four- and three-body indices must be taken into account when carrying out the double sums in (3.11).

The interpretation of the first of Eqs. (3.11) as an operator relation in the four-body Hilbert space is made dubious by the presence of the term $(G_0t_\beta G_0)^{-1}$ arising from the prescription (3.8) for $\mathsf{V}^{(4)}$. However, if we formally rewrite (3.11) as an equation for the difference

$$U^{\sigma\tau}_{\beta\alpha}-\bar\delta^{\sigma\tau}\,\delta_{\beta\alpha}(G_0t_\alpha G_0)^{-1}\ ,$$

this problem disappears. Furthermore, $U^{\sigma\tau}_{\beta\alpha}$ is eventually to be evaluated between Faddeev-type components $\phi^{(\tau)}_\alpha$ of the initial and final states (τ), and in that case the meaning of $(G_0t_\beta G_0)^{-1}$ is clear: Indeed, since $\phi^{(\tau)}_\alpha=-G_0v_\alpha\phi^{(\tau)}$ and $\phi^{(\tau)}=\sum_{\alpha\subset\tau}\phi^{(\tau)}_\alpha$, we have the on-energy-shell relation

$$(G_0t_\alpha G_0)^{-1}|\phi^{(\tau)}_\alpha\rangle=-G_\alpha^{-1}|\phi^{(\tau)}\rangle.$$

The four-body M operators in (3.9) and the second equation in (3.11) differ from the corresponding ones introduced by AGS in Ref. 3. The reasons for the present choice will be explained in the next section.

IV. THE K-OPERATOR FORMALISM[7]

Consider now the three-body operators $K_{\beta\alpha}$ of Ref. 4; they are intermediate between the operators $M_{\beta\alpha}$ and $U_{\beta\alpha}$, in the sense that

$$K_{\beta\alpha}=\sum_\gamma\bar\delta_{\gamma\alpha}M_{\beta\gamma}\ ,\qquad U_{\beta\alpha}=-\bar\delta_{\beta\alpha}G_0^{-1}+\sum_\gamma\bar\delta_{\beta\gamma}K_{\gamma\alpha}\ ,\tag{4.1}$$

and they satisfy Faddeev equations[4]

$$K_{\beta\alpha}=\bar\delta_{\beta\alpha}t_\beta-t_\beta G_0\sum_\gamma\bar\delta_{\beta\gamma}K_{\gamma\alpha}\ ,\qquad K_{\beta\alpha}=\bar\delta_{\beta\alpha}t_\beta-\sum_\gamma\bar\delta_{\gamma\alpha}K_{\beta\gamma}G_0t_\gamma\ .\tag{4.2}$$

We now incorporate these operators into the AGS scheme. Defining a matrix of operators $\mathsf{N}=\{N_{\beta\alpha}\}=\{G_0K_{\beta\alpha}\}$ and using (3.2), we can write (4.1) and (4.2) in matrix form as

$$\mathsf{N}=\mathsf{G}\,\mathsf{V}=\mathsf{G}_0\mathsf{T}\ ,\qquad \mathsf{T}=\mathsf{V}\,(1-\mathsf{N})\ ,\qquad \mathsf{N}=\mathsf{G}_0\mathsf{V}\,(1-\mathsf{N})=(1-\mathsf{N})\,\mathsf{G}_0\mathsf{V}\ .\tag{4.3}$$

From (4.3) it can be seen that N is the matrix of three-body operators that corresponds to the two-body operator $n=g_0t=gv$.

Proceeding as in the previous section, we can obtain the matrix of four-body operators $\mathsf{N}^{(4)}$ by generalizing (4.3). We define

$$\mathsf{N}^{(4)}=\{\mathsf{N}^{\sigma\tau}\}=\{G_0K^{\sigma\tau}_{\beta\alpha}\}\ ,\tag{4.4}$$

where a factor G_0 has been made explicit, just as was done for $M^{\sigma\tau}_{\beta\alpha}$ in (3.3). Then the first of the four-body equations,

$$\mathsf{N}^{(4)}=\mathsf{G}_0^{(4)}\mathsf{V}^{(4)}(1-\mathsf{N})^{(4)}=(1-\mathsf{N})^{(4)}\mathsf{G}_0^{(4)}\mathsf{V}^{(4)}\ ,\tag{4.5}$$

becomes

$$\mathsf{N}^{\sigma\tau}=\bar\delta^{\sigma\tau}\mathsf{N}^\sigma-\mathsf{N}^\sigma\sum_\rho\bar\delta^{\sigma\rho}\mathsf{N}^{\rho\tau}\ ,\tag{4.6}$$

or

$$K^{\sigma\tau}_{\beta\alpha}=\bar\delta^{\sigma\tau}K^\sigma_{\beta\alpha}-\sum_\gamma K^\sigma_{\beta\gamma}G_0\sum_\rho\bar\delta^{\sigma\rho}K^{\rho\tau}_{\gamma\alpha}\ .\tag{4.7}$$

With the above procedure we obtain four-body operators that correspond to the three-body operators $K_{\beta\alpha}$. Alternatively, by directly defining

$$K^{\sigma\tau}_{\beta\alpha}=\sum_{\rho\supset\alpha}\bar{\delta}^{\rho\tau}\sum_{\gamma\subset\rho}\bar{\delta}_{\gamma\alpha}Y^{\sigma\rho}_{\beta\gamma}\ , \tag{4.8}$$

the same operators $K^{\sigma\tau}_{\beta\alpha}$ can be obtained within the FY formalism. In this case we recover Eq. (4.7) from the Faddeev-Yakubovskii equations for $Y^{\sigma\tau}_{\beta\alpha}$ in Sec. II.

With the aid of (4.1) and the relation $K_{\beta\alpha} = -t_\beta G_0 U_{\beta\alpha}$, we now realize that the kernels of the four-body equations for the M operators of Secs. II and III are nothing but $K^{\sigma}_{\beta\alpha}$ operators (or their adjoints). For example, the first of Eqs. (2.13) and the second of Eqs. (3.11) both become

$$M^{\sigma\tau}_{\beta\alpha}=\delta^{\sigma\tau}W^{\sigma}_{\beta\alpha}-\sum_{\gamma\subset\sigma}K^{\sigma}_{\beta\gamma}G_0\sum_{\rho\supset\gamma}\bar{\delta}^{\sigma\rho}M^{\rho\tau}_{\gamma\alpha}\ . \tag{4.9}$$

We thus see that by defining a symmetric four-body operator within the Faddeev formalism, and by a straightforward definition of the elements of the matrix $\mathsf{G}^{(4)}$ within the AGS scheme, we obtain identical M operators in both formalisms. In addition, we note that Eq. (4.9) for $M^{\sigma\tau}_{\beta\alpha}$ and Eq. (4.7) for $K^{\sigma\tau}_{\beta\alpha}$ have identical kernels. This is in fact the case for all four-body equations considered here, as is particularly evident from the matrix formulation, where $\mathsf{G}^{(4)}_0\mathsf{V}^{(4)}$, or possibly $\mathsf{V}^{(4)}\mathsf{G}^{(4)}_0$, is the only kernel that occurs. That is, just as $\mathsf{G}_0\mathsf{V}=\{G_0 t_\beta\bar{\delta}_{\beta\alpha}\}$ is the Faddeev kernel in the three-body case, $\mathsf{G}^{(4)}_0\mathsf{V}^{(4)}=\{G_0K^{\sigma}_{\beta\alpha}\bar{\delta}^{\sigma\tau}\}$ is the kernel of all four-body Faddeev-type equations.

The above considerations establish the equivalence between the Faddeev-Yakubovskii and the Alt-Grassberger-Sandhas formalisms for $N=4$.

It should be clear that both the FY and the AGS formalisms can be extended to the N-body case by successive applications of the procedures described in the previous sections. A closer look at the kernels for all the N-body matrix equations so obtained,

$$\begin{aligned}&\mathsf{G}_0\mathsf{V}=\{G_0t_\beta\bar{\delta}_{\beta\alpha}\}=\{n_\beta\bar{\delta}_{\beta\alpha}\}\ ,\\&\mathsf{G}^{(4)}_0\mathsf{V}^{(4)}=\{\mathsf{G}_0\mathsf{T}^{\sigma}\bar{\delta}^{\sigma\tau}\}=\{\mathsf{N}^{\sigma}\bar{\delta}^{\sigma\tau}\}\ ,\\&\quad\cdot\qquad\cdot\qquad\cdot\\&\quad\cdot\qquad\cdot\qquad\cdot\\&\quad\cdot\qquad\cdot\qquad\cdot\end{aligned} \tag{4.10}$$

shows that their elements are simply maximal subsystem K operators, that is, operators describing the two-cluster subsystems $(N-1)+(1)$, $(N-2)+(2)$, etc. We can thus identify the hierarchy of K operators as being the hierarchy of N-body Faddeev-type kernel operators. Therefore, they will be central to any Faddeev-type N-body theory.

We also note that the N-body K operators are directly obtainable from the maximal subsystem K operators, with no additional input. This feature follows from the fact that the driving term of the K-operator equations is identical to the kernel, and should be compared with the situation for other N-body operators, such as the U operators. There, in order to construct the kernel, it is necessary to evaluate products of U operators belonging to all subsystems. For example, the kernel in the five-body U-operator equations is simply the adjoint of (4+1)- and (3+2)-subsystem K operators; however, when rewritten in terms of subsystem U operators, it becomes

$$\sum_{\gamma}U^{\sigma\tau}_{\beta\gamma}G_0t_\gamma G_0U^{\tau}_{\gamma\alpha}G_0t_\alpha G_0\ .$$

Schematically,

$$\begin{aligned}&\left.\begin{matrix}t\rightarrow\mathsf{T}\\ t\end{matrix}\right\}\rightarrow\left.\begin{matrix}\mathsf{T}^{(4)}\\ \mathsf{T}\\ t\end{matrix}\right\}\rightarrow\cdots\ ,\\&\left.\begin{matrix}g\rightarrow\mathsf{G}\\ g\end{matrix}\right\}\rightarrow\left.\begin{matrix}\mathsf{G}^{(4)}\\ \mathsf{G}\\ g\end{matrix}\right\}\rightarrow\cdots\ ,\\&n\rightarrow\mathsf{N}\ \rightarrow\mathsf{N}^{(4)}\ \rightarrow\cdots\ ,\end{aligned} \tag{4.11}$$

where the first line corresponds to the AGS hierarchy for the U operators as presented in Sec. III, the second line to the equivalent hierarchy for the M operators, and the last line to the hierarchy of kernel operators.

The above considerations single out the K operators and their equations as forming the minimal hierarchy for the N-body problem. This can also be seen directly from the fact that they can be obtained from a simplified version of the AGS formalism. Indeed, (4.5) indicates that it is not necessary to consider separate hierarchies for G_0 and V, as was done in Sec. II; instead it is sufficient to use the simple prescription for the product G_0V suggested by (4.10),

$$\begin{aligned}&\mathsf{G}_0\mathsf{V}=\{n_\beta\bar{\delta}_{\beta\alpha}\}\ ,\\&\mathsf{G}^{(4)}_0\mathsf{V}^{(4)}=\{\mathsf{N}^{\sigma}\bar{\delta}^{\sigma\tau}\}\ ,\end{aligned} \tag{4.12}$$

and so on, in order to generate the complete hierarchy of K operators and their equations.

We thus see that the hierarchy of kernel operators in itself contains the basic structure of all Faddeev-type N-body formalisms. On the other hand, the more elaborate AGS scheme has the advantage of supplying the hierarchies for all operators needed to construct physical transition amplitudes.[8]

Finally, we point out that with the matrix formalism one can easily define a variety of operators other than the ones considered so far, which will also satisfy Faddeev-type equations. In fact, the number of different N-body operators that can be defined from the basic G and T hierarchies increases rapidly with increasing number of particles. As an example, instead of taking

$$\mathsf{G}^{(4)}=\{-G_0 M^{\sigma\tau}_{\beta\alpha} G_0\},\ldots,$$
$$\mathsf{N}^{(4)}=\{G_0 K^{\sigma\tau}_{\beta\alpha}\},\ldots, \tag{4.13}$$

as in (3.9) and (4.4), we can write

$$\mathsf{G}^{(4)}=\{-\mathsf{G}_0\,\overline{\mathsf{M}}^{\sigma\tau}\mathsf{G}_0\}=\{-G_0 t_\beta G_0 \overline{M}^{\sigma\tau}_{\beta\alpha} G_0 t_\alpha G_0\},\ldots,$$
$$\mathsf{N}^{(4)}=\{\mathsf{G}_0\,\overline{\mathsf{K}}^{\sigma\tau}\}=\{-G_0 t_\beta G_0 \overline{K}^{\sigma\tau}_{\beta\alpha}\},\ldots, \tag{4.14}$$

where $\overline{M}^{\sigma\tau}_{\beta\alpha}$ forms a different set of four-body operators, in fact the ones defined by AGS. The $\overline{K}$ operators in (4.14) will also differ from the K operators of this section. We have preferred the choice of Eq. (4.13) since in that case the M, K, and U operators are related to the G, N, and T hierarchies in a most straightforward manner, and the hierarchy of kernel operators is obtained directly.

V. THE N-BODY CASE

For the detailed discussion of the N-body case, we must introduce a more general subsystem labeling than that we have used so far. Following Yakubovskii, we will use the concept of a partition a_i of N particles into i different groups, such that only particles within the same group are interacting. For instance, $a_2, b_2, \ldots$ denote different possible partitions of the N particles into two groups that do not interact with each other; $a_{N-2}, b_{N-2}, \ldots$ denote configurations of the type $(\cdot)\cdots(\cdot)(\cdots)$ or $(\cdot)\cdots(\cdot)(\cdot\cdot)(\cdot\cdot)$—such as $\sigma, \tau, \ldots$—and a_{N-1}, $b_{N-1}, \ldots$ denote interacting pairs—such as $\alpha, \beta, \ldots$—in the preceding sections. Partitions can be performed one after another to form a sequence. In such a case they will be denoted by the same letter and symbolized by $a_i \subset a_k$, $i>k$.[9] Only sequences whose last partition is of the type a_{N-1} will be considered; they will be denoted by a Greek letter, and their subscript will be that of the first partition in the sequence. In this way,

$$\alpha_2=(a_2, a_3, \ldots, a_{N-1})=(a_2\supset a_3\supset\cdots\supset a_{N-1})$$

determines one way in which the N-body system is ultimately separated into N-1 groups that do not interact with each other. At times, we will make part of a sequence explicit, as in $\alpha_2=(a_2, \alpha_3)$. Finally, each possible partition a_i defines a particular, disconnected N-body scattering problem, in which specific interactions have been set equal to zero.

The dynamical equations for N-body scattering are now written in terms of matrices of operators which are labeled by full sequences β_2, α_2,

$$\mathsf{A}=\{A_{\beta_2\alpha_2}\}=\{\mathsf{A}^{b_2 a_2}\}\ . \tag{5.1}$$

In (5.1), the first bracket contains the simple elements of the matrix of operators A, and the second contains the submatrices obtained when the first partition label is made explicit. Matrices of operators related to disconnected (partitioned) N-body systems will be denoted

$$\mathsf{B}^{b_k}=\{B^{b_k}_{\beta_{k+1}\alpha_{k+1}}\}=\{\mathsf{B}^{b_k}_{b_{k+1}a_{k+1}}\}\ , \tag{5.2}$$

where it is understood that α_{k+1} is included in b_k.

The notation just introduced will be recognized as the general form of the notation used in previous sections for $N=4$.

We can now write down the N-body generalization of (4.6),[10]

$$\mathsf{N}^{b_2a_2}=\overline{\delta}^{\,b_2a_2}\,\mathsf{N}^{b_2}-\mathsf{N}^{b_2}\sum_{d_2}\overline{\delta}^{\,b_2d_2}\,\mathsf{N}^{d_2a_2}\ . \tag{5.3}$$

For individual matrix elements, (5.3) takes the form

$$K^{b_2a_2}_{\beta_3\alpha_3}=\overline{\delta}^{\,b_2a_2}K^{b_2}_{\beta_3\alpha_3}-\sum_{\delta_3}K^{b_2}_{\beta_3\delta_3}G_0\sum_{d_2}\overline{\delta}^{\,b_2d_2}K^{d_2a_2}_{\delta_3\alpha_3}\ , \tag{5.4}$$

where δ_2 denotes the sequence $\delta_2=(d_2,\ldots,d_{N-1})$, and $\sum_{\delta_3}$ stands for

$$\sum_{d_3}\sum_{d_4}\cdots\sum_{d_{N-1}}\ .$$

For $N=4$, by writing $b_2=\sigma$, $\beta_3=b_3=b_{N-1}=\beta$, etc., in (5.4), we simply recover (4.7). N^{b_2} in (5.3) is the K-operator matrix for an N-body system which has been split into two groups according to the partition b_2. This matrix satisfies an equation similar to (5.3), but with a matrix N^{b_3} in the driving term and the kernel. In general we have for any k, $3\le k\le N-1$,

$$\mathsf{N}^{b_{k-1}}_{b_ka_k}=\overline{\delta}^{\,b_ka_k}\,\mathsf{N}^{b_k}-\mathsf{N}^{b_k}\sum_{d_k\subset b_{k-1}}\overline{\delta}^{\,b_kd_k}\,\mathsf{N}^{b_{k-1}}_{d_ka_k}\ , \tag{5.5}$$

or, in component form,

$$K^{b_{k-1}}_{\beta_k\alpha_k}=\overline{\delta}^{\,b_ka_k}K^{b_k}_{\beta_{k+1},\alpha_{k+1}}-\sum_{\delta_{k+1}\subset b_k}K^{b_k}_{\beta_{k+1},\delta_{k+1}}G_0\sum_{d_k\subset b_{k-1}}\overline{\delta}^{\,b_kd_k}K^{b_{k-1}}_{\delta_k\alpha_k}\ . \tag{5.6}$$

Since for $k=N-1$ $K^{b_{N-2}}_{b_{N-1},a_{N-1}}$ is just $K^{\sigma}_{\beta\alpha}$, and (5.7) is the Faddeev equation (4.2) with $\alpha,\beta,\gamma\subset\sigma$, we identify $K^{b_{N-1}}_{\beta_N\alpha_N}$ as simply t_β.

With $K^{b_1}_{\beta_2\alpha_2}\equiv K_{\beta_2\alpha_2}=K^{b_2a_2}_{\beta_3\alpha_3}$, we can include (5.4) in (5.6) by including $k=2$ among the allowed values for k. In this way (5.6) becomes the complete hierarchy of K-operator equations, as obtained within the AGS scheme.

Let us now turn to the FY formalism, and consider Eq. (4.1) of Ref. 2 for $i=k-1$, and with $k=2,3,\ldots,N-1$[11]:

$$Y^{b_k -1}_{\beta_k \epsilon_k} = \delta^{b_k e_k} Y^{b_k}_{\beta_{k+1},\epsilon_{k+1}} - \sum_{\delta_{k+1}\subset b_k} \sum_{\gamma_{k+1}\subset b_k}^{\delta_{k+1}} Y^{b_k}_{\beta_{k+1},\gamma_{k+1}} G_0 \times \sum_{d_k \subset b_{k-1}} \bar{\delta}^{b_k d_k} Y^{b_k -1}_{\delta_k \epsilon_k} . \tag{5.7}$$

Here, $\gamma_k = (c_k, \ldots, c_{N-1})$, $\epsilon_k = (e_k, \ldots, e_{N-1})$, and

$$\sum_{\gamma_{k+1}\subset b_k}^{\delta_{k+1}} \equiv \sum_{\substack{c_{k+1}\supset d_{k+2}\\ c_{k+1}\neq d_{k+1}\\ c_{k+1}\subset b_k}} \sum_{\substack{c_{k+2}\supset d_{k+3}\\ c_{k+2}\neq d_{k+2}}} \cdots \times \sum_{\substack{c_{N-2}\supset d_{N-1}\\ c_{N-2}\neq d_{N-2}}} \sum_{c_{N-1}\neq d_{N-1}} .$$

The two sums in (5.7) correspond to the sums in the concatenation of Ref. 2, but their order has been inverted. For $k=N$, $Y^{b_N -1}_{\beta_N \alpha_N}$ equals t_β by definition. For $N=4$ and $k=2$, $Y^{b_1}_{\beta_2 \epsilon_2}$ and $Y^{b_2}_{\beta_3 \epsilon_3}$ are simply the operators $Y^{\sigma\tau}_{\beta\alpha}$ and $M^{\sigma}_{\beta\alpha}$ of Sec. II, and (5.7) is just the FY equation for $Y^{\sigma\tau}_{\beta\alpha}$.

Consider now the following operators:

$$\hat{K}^{b_k -1}_{\beta_k \alpha_k} = \sum_{\epsilon_k \subset b_{k-1}}^{\alpha_k} Y^{b_k -1}_{\beta_k \epsilon_k} . \tag{5.8}$$

Summing over the indices ϵ_k in (5.7) as indicated in (5.8), we get

$$\hat{K}^{b_k -1}_{\beta_k \alpha_k} = \bar{\delta}^{b_k a_k} \hat{K}^{b_k}_{\beta_{k+1}\alpha_{k+1}} - \sum_{\delta_{k+1}\subset b_k} \hat{K}^{b_k}_{\beta_{k+1},\delta_{k+1}} G_0 \sum_{d_k \subset b_{k-1}} \bar{\delta}^{b_k d_k} \hat{K}^{b_k -1}_{\delta_k \alpha_k} . \tag{5.9}$$

This equation is clearly equivalent to the original FY equation (5.7), since the summation over the parameter index ϵ_k leaves the kernel of the equation unchanged.

A comparison of (5.9) and (5.6) now shows that

$$K^{b_k -1}_{\beta_k \alpha_k} = \hat{K}^{b_k -1}_{\beta_k \alpha_k}, \quad \text{all } k \tag{5.10}$$

provided the operators at the bottom of the two hierarchies, $K^{b_N -1}_{\beta_N \alpha_N}$ and $\hat{K}^{b_N -1}_{\beta_N \alpha_N}$, coincide. However, this is the case since both are identical to t_β by definition.

We therefore conclude that the N-body equations generated by the AGS formalism are equivalent to the N-body FY equations, in the sense that both have the same kernel.

VI. THE WAVE-FUNCTION FORMALISM

The K operators of Secs. IV and V are very closely related to the Faddeev-type components of the N-body wave function. At the three-body level we have

$$\psi_{\beta(\alpha)} = [\delta_{\beta\alpha} - G_0(E_\alpha + i0) K_{\beta\alpha}(E_\alpha + i0)] \phi_{(\alpha)} , \tag{6.1}$$

where the initial state $\phi_{(\alpha)}$ corresponds to a bound state in the α channel and a third particle free. As indicated in (6.1), all operators of this section are to be taken at an energy corresponding to that of the initial state. From (6.1) we also see that the operator $\delta_{\beta\alpha} - G_0 K_{\beta\alpha}$ directly yields the Faddeev components of the scattered wave function out of the initial-state wave function.

We now introduce column matrices [again suppressing the superscript (3) in denoting three-body operators] $\Psi_{(\alpha)} = \{\psi_{\beta(\alpha)}\}$ and $\Phi_{(\alpha)} = \{\delta_{\beta\alpha}\phi_{(\alpha)}\}$, so that the relation (6.1) and the Faddeev equations for the wave-function components take the form

$$\Psi_{(\alpha)} = (1 - \mathsf{N})\Phi_{(\alpha)} , \qquad \Psi_{(\alpha)} = \Phi_{(\alpha)} - \mathsf{G}_0 \mathsf{V} \Psi_{(\alpha)} . \tag{6.2}$$

As in previous sections, these results can be generalized to the N-body case. For the four-body Faddeev-type components we get

$$^{(4)}\Psi^{(\tau)} = (1 - \mathsf{N}^{(4)})\, ^{(4)}\Phi^{(\tau)} , \qquad ^{(4)}\Psi^{(\tau)} = {}^{(4)}\Phi^{(\tau)} - \mathsf{G}_0^{(4)} \mathsf{V}^{(4)}\, ^{(4)}\Psi^{(\tau)} . \tag{6.3}$$

The initial state in (6.3), labeled by (τ), is considered to be either a three-body bound state and a free particle, or a pair of two-body bound states. With

$$^{(4)}\Psi^{(\tau)} = \{\Psi^{\sigma(\tau)}\} = \{\psi^{\sigma(\tau)}_\beta\} , \qquad ^{(4)}\Phi^{(\tau)} = \{\delta^{\sigma\tau}\Phi^{(\tau)}\} = \{\delta^{\sigma\tau}\phi^{(\tau)}_\beta\} , \tag{6.4}$$

we can write (6.3) in explicit form:

$$\psi^{\sigma(\tau)}_\beta = \delta^{\sigma\tau}\phi^{(\tau)}_\beta - G_0 \sum_{\gamma\subset\tau} K^{\sigma\tau}_{\beta\gamma}\phi^{(\tau)}_\gamma , \qquad \psi^{\sigma(\tau)}_\beta = \delta^{\sigma\tau}\phi^{(\tau)}_\beta - G_0 \sum_{\gamma\subset\sigma} K^{\sigma}_{\beta\gamma} \sum_{\rho\supset\gamma} \bar{\delta}^{\sigma\rho}\psi^{\rho(\tau)}_\gamma , \tag{6.5}$$

where $\phi^{(\tau)}_\beta$ is a Faddeev component of the initial-state wave function. It is not difficult to verify that the second equation in (6.5) is identical to the corresponding Faddeev-Yakubovskii equation, e.g., as given by Kharchenko and Kuzmichev.[6]

The analogous relations for the N-body case are[10]

$$\Psi^{b_2(a_2)} = \delta^{b_2 a_2} \Phi^{(a_2)} - \mathsf{N}^{b_2 a_2} \Phi^{(a_2)}, \qquad (6.6)$$

$$\Psi^{b_2 a_2} = \delta^{b_2 a_2} \Phi^{(a_2)} - \mathsf{N}^{b_2} \sum_{d_2} \bar{\delta}^{b_2 d_2} \Psi^{d_2(a_2)}.$$

Here we again see that the operator $1-\mathsf{N} \equiv \{\delta^{b_2 a_2} - \mathsf{N}^{b_2 a_2}\}$ yields the Faddeev-type components of the N-body scattered wave function out of the Faddeev-type components of the initial-state wave functions, and that the K operators, $\mathsf{N}^{b_2} = \{G_0 K^{b_2}_{\beta_3 \alpha_3}\}$, form the kernel of the N-body equations.

VII. CONCLUSIONS

The different formalisms of Alt, Grassberger, and Sandhas and of Faddeev and Yakubovskii have been shown to be equivalent generalizations of the Faddeev treatment of the three-body system to the N-body case. In particular, the partition notation of FY and the matrix index notation of AGS describe the channel structure of the N-particle system in equivalent ways. Consequently, the advantages of the intuitively appealing AGS notation can now be fully exploited when handling N-body operator relations.

The structure of the Faddeev-type N-body theories is best described in terms of the hierarchy of K operators obtained by generalizing to the N-body level the three-body K operators discussed in Ref. 4. In fact, this hierarchy is precisely the hierarchy of Faddeev-type kernels, and the K operators will therefore play a central role in any Faddeev-type treatment of the N-body problem.

By exploiting the close connection between wave functions and this minimal hierarchy of K operators, the Faddeev-type components of the N-body full wave function and their equations have been obtained in a most straightforward manner. In a subsequent paper, it will be shown that the transition amplitudes for elastic, rearrangement, and breakup scattering—obtained in terms of Faddeev-type N-body operators and wave-function components—are algebraically equivalent to the well-known expressions for these amplitudes in terms of potentials and full wave functions.

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[3] E. O. Alt, P. Grassberger, and W. Sandhas, JINR Report No. E4-6688, 1972 (unpublished); and in *Few Particle Problems in the Nuclear Interaction*, edited by I. Slaus *et al.* (North-Holland, Amsterdam, 1972), p. 299. See also P. Grassberger and W. Sandhas, Nucl. Phys. B2, 181 (1967).

[4] B. R. Karlsson and E. M. Zeiger, in *Few Particle Problems in the Nuclear Interaction*, edited by I. Slaus *et al.* (Ref. 3), p. 330; SLAC Report No. SLAC-PUB-1139, 1972 (unpublished).

[5] Maximal subsystems are splittings of N particles into two groups. For instance, $(N-1)+(1)$ is a maximal subsystem, but $(N-3)+(2)+(1)$ is not.

[6] V. F. Kharchenko and V. E. Kuzmichev, Nucl. Phys. A183, 606 (1972).

[7] The notation originates from Ref. 1, where kernel functions $\mathcal{K}_{\beta\alpha}$ closely related to the K operators are considered. The K operators should not be confused with the so-called Heisenberg K matrix.

[8] B. R. Karlsson and E. M. Zeiger (unpublished).

[9] This is equivalent to the "tree" concept, presented by Faddeev for $N=4$; L. D. Faddeev, in *Three Body Problem in Nuclear and Particle Physics*, edited by J. S. C. McKee and P. M. Rolph (North-Holland, Amsterdam, 1970).

[10] With the more general notation, superscripts (4), etc., as used previously, are now unnecessary.

[11] The $Y^{b_k-1}_{\beta_k \epsilon_k}$ operators defined here correspond to $M^{\beta_k \epsilon_k}_{b_k-1}$ of Ref. 2.