Eikonal expansion as the high-energy limit of the Born series*

Arthur R. Swift
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002
(Received 29 August 1973)

The Glauber version of the eikonal expansion in potential theory is shown to be the unique result of calculating the high-energy, fixed-momentum-transfer limit of the Born series. Correction terms are systematically calculated and agree with the form proposed by Wallace. These results are used to understand the fact that the Glauber eikonal is an accurate approximation to the exact scattering amplitude at all energies.

I. INTRODUCTION

The eikonal representation for potential-theory scattering amplitudes has been used innumerable times in atomic and nuclear physics problems. It has even been a subject of recent interest in approximations to high-energy amplitudes in particle physics. Among the various versions of the eikonal representation that have been proposed, the original form derived by Glauber,[1]

$$f(\vec{p},\vec{p}\,')=\frac{p}{2\pi i}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left(e^{i[\chi_0(b)/p]}-1\right), \qquad (1)$$

where $\vec{q}=\vec{p}-\vec{p}\,'$ and the eikonal function $\chi_0(b)$ is related to the potential $V(r)$ by

$$\chi_0(b)=-m\int_{-\infty}^{\infty} dz\, V(\vec{b},z), \qquad (2)$$

has proved to be the simplest and most accurate in actual applications. The z direction in (2) is chosen parallel to the average momentum $\vec{P}=(\vec{p}+\vec{p}\,')/2$, and the impact-parameter vector $\vec{b}$ lies in a plane perpendicular to $\vec{P}$. All derivations of (2) have involved one or more heuristic steps which have made it difficult to calculate correction terms.[2,3] Moreover, previous derivations lead to the conclusion that the Glauber eikonal should be accurate only at small angles, yet it does a surprisingly good job of approximating the exact amplitude at large angles for many potentials.[3] There exist a number of rigorous derivations of impact-parameter representations of the scattering amplitude which agree with the Glauber version at small angles.[4-6] Invariably, these other forms are worse than Glauber's at large angles or are too complicated to evaluate.

Clearly what is missing in all attempts to derive the eikonal representation is an understanding of the appropriate expansion parameter. In other words, in what limit does the Glauber eikonal become exact? Most derivations contain one or two approximations whose justification is one of convenience. The method used by Abarbanel and Itzykson[5], on one hand, that of Lévy and Sucher,[6] on the other, are examples of this approach. The eikonal representations derived by them turn out to be either too complicated or less accurate than the Glauber eikonal at large angles. Wallace[7] has systematically calculated correction terms to the Glauber eikonal. In his early work he used the approach developed by Abarbanel and Itzykson[5] and noted that if a certain function of the scattering angle is dropped, the Glauber form together with correction terms is obtained. He then showed that the correction terms are calculable in practice and improve the agreement with the exact amplitude at all angles. In his most recent paper Wallace[7] has shown that the eikonal expansion together with corrections can be obtained from the high-energy limit of the analytic continuation in angular momentum of the partial-wave expansion.

In this paper we show that if the high-energy limit at fixed momentum transfer of each term of the Born series is calculated and the resulting series is summed, the Glauber eikonal is the unique result. Moreover, it is possible to calculate and sum nonleading terms in the Born series and obtain the conjectured correction terms proposed by Wallace. Thus, the Glauber eikonal amplitude is shown to be the high-energy limit of the Born series at fixed momentum transfer, independent of the strength of the energy-independent potential.[8] The range of the potential R sets the energy scale in the sense that high energy implies that $Rp \gg 1$, where p is the incident momentum. Using p^{-1} as the appropriate expansion parameter, we systematically calculate corrections to the Glauber amplitude. Although conceptually simple, calculation of these terms quickly becomes algebraically very complicated. However, the first few correction terms can be used to understand the empirically observed fact that the eikonal is accurate at all angles. When the momentum transfer squared q^2 becomes large, the Glauber eikonal goes to zero like a power of q^{-1} or faster. Since the ratio of the correction term to the leading term is of order p^{-1}, the condition for validity of the eikonal at large angles is that the correction terms must vanish as $q\to\infty$ at least as rapidly as does the leading term. Moreover, the large-q behavior of the eikonal amplitude is controlled by the $b\to 0$ limit of the eikonal function $\chi_0(b)$. Hence, it is possible to relate the criteria of large-angle validity to $\chi_0(b)$, and thus to the potential itself. The conclusion is that the Glauber eikonal expansion should be useful at all angles for all smooth potentials except those for which $\chi_0(b)$ is analytic at $b=0$. A basic assumption in this discussion is that the fixed-angle limit of the Born series is obtained when first $p\to\infty$, then $q\to\infty$. This point is discussed in detail in the Appendix, where it is shown that the $p, q\to\infty$ limit is independent of the order in which the limits are taken. Moreover, for the second Born term, the limits agree with the $p, q\to\infty$, q/p-fixed limit.

In Sec. II a brief outline is presented to show how the standard tools of high-energy perturbation theory are combined with the summation procedure of Lévy and Sucher[6] to calculate the high-energy limit of each term in the Born series. The results are then written down and used to discuss the properties of the eikonal representation. Examples are discussed. Section III contains the details of the derivation for the more dedicated reader.

II. RESULTS AND DISCUSSION

A. Outline of derivation

The starting point in the systematic derivation of the eikonal expansion is the Born series in momentum space

$$f(\vec{p},\vec{p}\,')=\sum_{n=0}^{\infty} f_{n+1}(\vec{p},\vec{p}\,')\ , \tag{3}$$

where

$$\begin{aligned} f_{n+1}(\vec{p},\vec{p}\,') = -\frac{m}{2\pi}\int \frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_1}{(2\pi)^3} &\times V(\vec{k}_1-\vec{p})\,G(\vec{k}_1)\,V(\vec{k}_2-\vec{k}_1)\cdots G(\vec{k}_n)\\ &\times V(\vec{p}\,'-\vec{k}_n)\ , \end{aligned}\tag{4}$$

$$V(\vec{k})=\int d^3r\, V(r)\, e^{i\vec{k}\cdot\vec{r}}\ . \tag{5}$$

The Green's function $G(\vec{k})$ is given by

$$G(\vec{k})=\left(\frac{p^2}{2m}-\frac{k^2}{2m}+i\epsilon\right)^{-1}\ . \tag{6}$$

In the limit $p=|\vec{p}|=|\vec{p}\,'|\to\infty$ and $q=|\vec{p}-\vec{p}\,'|$ is fixed, each term of the Born series has the expansion

$$f_{n+1}(\vec{p},\vec{p}\,')=\frac{f^{(0)}_{n+1}(q)}{p^n}+\frac{f^{(1)}_{n+1}(q)}{p^{n+1}}+\frac{f^{(2)}_{n+1}(q)}{p^{n+2}}+\cdots\ . \tag{7}$$

When $f^{(0)}_{n+1}/p^n$ is summed over n, the Glauber eikonal, Eq. (1), is obtained. Similar summations of the second, third, and higher, terms in (7) lead to the correction terms whose form was conjectured by Wallace.[7] The details of the derivation are given in Sec. III. A brief outline is presented here.

The first step is to use techniques developed for calculating the high-energy limits of field-theory diagrams.[9] To do this we introduce a set of Feynman parameters by the identity

$$\frac{1}{a-i\epsilon}=i\int_0^{\infty} dy\, e^{-i(a-i\epsilon)y} \tag{8}$$

and the ansatz

$$V(k)=\int_0^{\infty} dx\,\rho(x)\, e^{-ixk^2}\ . \tag{9}$$

Most potentials have such a representation, and, in any case, the weight function $\rho(x)$ is never needed explicitly. When (8) and (9) are used in (4), the momentum integrations can be easily performed to provide a representation for $f_{n+1}(\vec{p},\vec{p}\,')$ as an integral over Feynman parameters. The similarity to a ladder diagram amplitude in field theory becomes manifest at this point. Calculation of $f^{(i)}_{n+1}(q)$ is straightforward, but the answers so obtained are complicated, multidimensional integrals which are impossible to evaluate directly. The eikonal form emerges from the fact that

$$f^{(i)}_{n+1}(q)=\lim_{\sigma\to 0}\left[\frac{1}{i!}\frac{\partial^i}{\partial\sigma^i}\int_0^{\infty} w^{n+i-1}\,dw\, I_{n+1}(q,\sigma,w)\right]\ , \tag{10}$$

where $I_{n+1}(q,\sigma,w)$ is exactly the function that is obtained if $f_{n+1}(\vec{p},\vec{p}\,')$ is calculated with

$$\begin{aligned}\vec{p}&=\frac{w}{\sigma}(\sin\theta, 0, \cos\theta)\ ,\\ \vec{p}\,'&=\frac{w}{\sigma}(-\sin\theta, 0,\cos\theta)\ ,\end{aligned}\tag{11}$$

where $\sin\theta=q\sigma/2w$. In addition the parameters y_i conjugate to the propagators $G(k_i)$ are restricted by $y_i=\sigma\bar{y}_i$, $\sum_1^n \bar{y}_i=1$. Thus, the momentum integrations can be reintroduced to write

$$\begin{aligned} f^{(i)}_{n+1}(q)=-\frac{m}{2\pi}\left(\frac{-2mi}{(2\pi)^3}\right)^n \frac{1}{i!}\frac{\partial^i}{\partial\sigma^i}\Bigg[\int_0^{\infty} w^{n+i-1}\,dw\int_0^1 \delta\Big(1-\sum_1^n \bar{y}_j\Big)\,d\bar{y}_j\int d^3k_1'\cdots d^3k_n'\ V(k_1')\cdots V(k_n')&\\ \times V\Big(q-\sum_1^n k_j'\Big)e^{-i\psi(k_j',\sigma,w,\bar{y}_j)}\Bigg]&\ . \end{aligned}\tag{12}$$

The integration momenta in (12) and (4) are related by $\vec{k}_i' = \vec{k}_i - \vec{k}_{i-1}$. The explicit form of the exponential factor in (12) is

$$\psi(k', \sigma, w, \bar{y}) = \sigma \left\{ \sum_{i=1}^{l-1} \bar{y}_i \left[(\vec{K}_i - \vec{p})^2 - \frac{w^2}{\sigma^2} \right] + \sum_{i=l}^{n} \bar{y}_i \left[(\vec{K}_i' + \vec{p}')^2 - \frac{w^2}{\sigma^2} \right] \right\}, \quad (13)$$

where $\vec{p}$ and $\vec{p}'$ are defined in (11). The lth rung in the ladder diagram corresponding to (13) carries momentum $\vec{q} - \sum_1^n \vec{k}_i'$; moreover, $\vec{K}_i = \sum_1^i \vec{k}_j'$, $\vec{K}_i' = \sum_i^n \vec{k}_j'$. Equation (12) has exactly the form studied by Lévy and Sucher.[6] Their analysis, with technical complications for the correction terms, converts (12) to an expression which sums to give the eikonal amplitude. In particular, the $(n!)^{-1}$ necessary to generate the exponential function comes from averaging over the equivalent ways of maintaining momentum conservation and labeling the momenta k_i in (12).

B. Results

The discussion here is restricted to spherically symmetric potentials $V(r)$, $r = (\vec{b}^2 + z^2)^{1/2}$. The impact-parameter vector $\vec{b}$ is perpendicular to the z direction, the direction of $\vec{p} + \vec{p}'$. The procedure outlined above leads to the following expressions:

$$f_{n+1}^{(0)}(q) = \frac{-i}{2\pi} \int d^2b\, e^{i\vec{q}\cdot\vec{b}} \frac{(i\chi_0)^{n+1}}{(n+1)!}, \quad (14a)$$

$$f_{n+1}^{(1)}(q) = \frac{-i}{2\pi} \int d^2b\, e^{i\vec{q}\cdot\vec{b}} \frac{(i\chi_0)^{n-1}}{(n-1)!} (i\chi_1), \quad (14b)$$

$$f_{n+1}^{(2)}(q) = \frac{-i}{2\pi} \int d^2b\, e^{i\vec{q}\cdot\vec{b}} \times \left[\frac{(i\chi_0)^{n-3}}{(n-3)!} \frac{(i\chi_1)^2}{2!} + \frac{(i\chi_0)^{n-2}}{(n-2)!} (i\chi_2) - \frac{(i\chi_0)^{n-1}}{(n-1)!} (\omega_1) \right], \quad (14c)$$

where

$$\chi_0(b) = -m \int_{-\infty}^{\infty} dz\, V(r), \quad (15a)$$

$$\chi_1(b) = -m^2 \int_{-\infty}^{\infty} dz \frac{1}{2r} \frac{d}{dr} [rV(r)]^2, \quad (15b)$$

$$\chi_2(b) = \frac{-m^3}{24b^2} \left[4 \int_{-\infty}^{\infty} dz \frac{1}{r} \frac{d}{dr} \left(r^2 \frac{d}{dr} [rV(r)]^3 \right) - \left(\int_{-\infty}^{\infty} dz \frac{d}{dr} [rV(r)] \right)^3 \right], \quad (15c)$$

$$\omega_1 = \frac{m^2}{8} \left(\int_{-\infty}^{\infty} dz \frac{d}{dr} [rV(r)] \right) \times \left\{ \int_{-\infty}^{\infty} dz \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{dV(r)}{dr} \right] \right\}. \quad (15d)$$

Although written in a different form, these expressions are those calculated by Wallace.[7] The derivation here makes clear their significance as high-energy limits of the Born series. The presence of $[(n+1)!]^{-1}$ in (14a) compared with $[(n-1)!]^{-1}$ in (14b) means that the inequality $f_{n+1}^{(0)}(q)/p^n > f_{n+1}^{(1)}(q)/p^{n+1}$ cannot be maintained for all n. In the discussion below on the large-p and large-q limits of the eikonal amplitude, we assum the complete Born series converges rapidly at high energy. In this case the contribution to the total amplitude of the terms violating this inequal ity is negligible. Admittedly, this procedure is not very rigorous. When (14) is summed over n, we find the amplitude

$$f(\vec{p}, \vec{p}') = \frac{-ip}{2\pi} \int d^2b\, e^{i\vec{q}\cdot\vec{b}} [e^{i\chi_0/p} G(b) - 1], \quad (16)$$

where

$$G(b) = 1 + \frac{i\chi_1}{p^3} + \frac{1}{2}\left(\frac{i\chi_1}{p^3}\right)^2 + \frac{i\chi_2}{p^5} - \frac{\omega_1}{p^4} = \exp\left[i\left(\frac{\chi_1}{p^3} + \frac{\chi_2}{p^5} \right) - \frac{\omega_1}{p^4} \right]. \quad (17)$$

Exponentiation of $G(b)$ is a conjecture based on th suggestive manner in which $\chi_1(b)$ appears.

Having obtained the Glauber eikonal expansion as the high-energy limit of the Born series, we are in a position to discuss the extent to which the eikonal representation is an accurate approximation to the exact amplitude. In the limit $p \to \infty$ and a weak potential, it is no better than the first few Born terms. However, the eikonal expansion should also be accurate when the energy is large and the potential is strong so that $\chi_0(b)/p$ is of order unity. As a measure of accuracy we requi that the first correction term be small compared to the leading term,

$$\frac{f_{n+1}^{(1)}(q)}{p f_{n+1}^{(0)}(q)} \ll 1.$$

Neglecting a factor of $n(n+1)$, we then find that a sufficient condition for this to happen is

$$\left| \frac{\chi_1(b)}{p[\chi_0(b)]^2} \right| \ll 1. \quad (18)$$

For a Yukawa potential $V(r)=ge^{-\mu r}/r$, this condition becomes

$$\frac{\mu}{p}\left|\frac{K_0(2\mu b)}{[K_0(\mu b)]^2}\right|\ll 1 ,$$

or the momentum p should be large compared to μ, the inverse range of the potential. More generally the Glauber eikonal will be an accurate representation of the scattering amplitude when

$$Rp\gg 1 , \tag{19}$$

where R is the range of the potential and p is the incident momentum. This condition is independent of the strength of the potential and is in agreement with what one expects on the basis of semiclassical, intuitive derivations of the eikonal expansion.[1-3] A better test of the accuracy of the Glauber eikonal would be a numerical comparison with the corrected form in (16). This approach would be particularly useful when exact solutions are not available.

C. The eikonal amplitude at large angles

The derivation of the Glauber eikonal was carried out in the $p\to\infty$, fixed-q limit. To investigate the large-angle behavior, we let q become large. This procedure is justified in detail in the Appendix. For the eikonal to be useful at large angles, correction terms should remain small compared to the leading terms in the $q\to\infty$ limit. As a general rule, the leading term goes to zero at least as fast as some power of q^{-1}. Since in the backward direction $q^2\approx p^2$ and the ratio of the leading term to the first correction term is of order p^{-1}, the correction terms themselves must vanish for large q at least as fast as the leading term. For potentials where this condition is satisfied, the eikonal expansion will remain roughly equally accurate at all angles contrary to expectations based on semiclassical derivations. Implicit in this discussion is the requirement that the Born series converges rapidly in the high-energy limit so that factors of $n(n+1)$ are unimportant. The large-angle validity of the eikonal representation has been discovered empirically by several workers.[3,7]

The $q\to\infty$ limit of (16) is determined by the $b\to 0$ limit of the integrand. In particular we must study integrals of the type

$$F(q)=\int d^2b\, e^{i\vec{q}\cdot\vec{b}}H(b)$$

$$=2\pi\int_0^\infty bdb\, J_0(qb)\, H(b) , \tag{20}$$

where $J_0(qb)$ is a zero-order Bessel function. Then if

$$\lim_{b\to 0} H(b)=c\,\frac{\ln^m b}{b^s} , \tag{21}$$

we find that

$$\lim_{q\to\infty} F(q)=c'(\ln q)^m q^{s-2},\ s\neq 0$$

$$=c'(\ln q)^{m-1}/q^2,\ s=0 . \tag{22}$$

If $H(b)$ is constant at $b=0$, then $F(q)=c'q^{-2-N}$, where $d^NH(b)/db^N$ is the first singular derivative of $H(b)$. If $H(b)$ has no singular derivatives, as for a Gaussian potential, $F(q)$ vanishes exponentially with q. Since for the leading term in the eikonal expansion $H(b)\sim[\chi_0(b)]^{n+1}$, $\chi_0(b)$ can have no more than a logarithmic divergence at $b=0$. A power divergence at $b=0$ in $\chi_0(b)$ would mean that there exists an N for which $f^{(0)}_{n+1}(q)$ is undefined for $n>N$. [There is an interesting question related to the validity of the eikonal expansion for situations where $f^{(0)}_{n+1}(q)$ may not be defined for $n>N$, yet the full integral in (16) exists.] Since a $\ln b$ behavior for $\chi_0(b)$ is generated by an r^{-1} singularity in the potential $V(r)$, the derivation of the eikonal expansion presented here is restricted to potentials for which $\lim_{r\to 0} r^{1+\epsilon}V(r)\to 0$, $\epsilon>0$.

Since $H(b)$ for the first correction term is given by

$$H(b)\sim[\chi_0(b)]^{n-1}\chi_1(b) ,$$

a necessary condition for the large-angle validity of the leading eikonal term is that

$$\lim_{b\to 0}\left|\frac{\chi_1(b)}{[\chi_0(b)]^2}\right|=c ,$$

where the constant c may be zero. If we test the Yukawa potential, we find from (15) that χ_0, χ_1, χ_2, and ω_1 are all proportional to $\ln b$ in the $b=0$ limit. Hence, the leading term and all correction terms fall off like $(\ln q)^m/q^2$. The leading eikonal amplitude should be accurate in the backward direction. Numerical calculations support this conclusion.[3,7]

If the potential is less singular than r^{-1}, $\chi_0(b)$ is a constant in the $b=0$ limit and, as indicated above, the large-q^2 behavior depends on a more detailed knowledge of the potential. Hence, for a Gaussian potential $V(r)=ge^{-\lambda r^2}$,

$$\chi_0(b)\sim e^{-\lambda b^2},\quad \chi_1(b)\sim(\alpha+\beta b^2)\,e^{-2\lambda b^2} .$$

Both χ's are analytic at $b=0$ so that $f_{n+1}^{(1)}(q)$ vanishes faster than any power of q:

$$f_{n+1}^{(0)}(q) \sim e^{-q^2/4\lambda n} ,$$

$$f_{n+1}^{(1)}(q) \sim q^2 e^{-q^2/4\lambda n} .$$

in this case, the correction term dominates in the backward direction, although both go to zero rapidly. For a polarization potential of the form $V(r)=g(r^2+\alpha^2)^{-N}$, both leading and correction terms are proportional to $e^{-\alpha q}$. Again the correction term dominates at large angles. In fact, from the cases considered, it appears that the leading term dominates at all angles for potentials where $\chi_0(b)$ has a singularity at $b=0$, so that $f_{n+1}^{(0)}(q)$ vanishes only as an inverse power of q. If $\chi_0(b)$ is analytic at $b=0$, $f_{n+1}^{(0)}(q)$ vanishes exponentially with $q\to\infty$, and the correction terms dominate at large q. It is interesting to note that in all cases the absolute accuracy of the Glauber eikonal is approximately independent of angle. In other words, $|f_{\text{exact}} - f_{\text{eikonal}}|$ is small at all angles. However, for some potentials the relative accuracy $|f_{\text{exact}}/f_{\text{eikonal}}|-1$ is strongly angle-dependent. The Gaussian potential is an example of this latter situation. The relative error is large in the backward direction, but the amplitudes are very small in this region. Most comparisons[3,7] of the amplitudes are presented on logarithmic plots which are sensitive to the relative accuracy.

III. DERIVATION OF THE EIKONAL EXPANSION

The $n+1$ term in the Born series is given by (4). After parametrization by (8) and (9), the momentum integrations are carried out to yield

$$\begin{aligned} f_{n+1}(p,q) &= f_{n+1}(\vec{p},\vec{p}\,') \\ &= -\frac{m}{2\pi}\left(\frac{2m}{(2\pi)^3}\right)^n (-i)^n \left(\frac{\pi}{i}\right)^{3n/2} \\ &\quad \times \int_0^\infty dx_1\cdots dx_{n+1}\,\rho(x_1)\cdots\rho(x_{n+1}) \\ &\quad \times \int_0^\infty \frac{dy_1\cdots dy_n}{\Delta_n^{3/2}}\, e^{-i\psi_n} . \end{aligned} \tag{23}$$

The function ψ_n is given by

$$\psi_n = -p^2\frac{f_n}{\Delta_n} + q^2\frac{g_n}{\Delta_n} - i\epsilon\sum_1^n y_i . \tag{24}$$

The only property of the functions f_n, g_n, and Δ_n which is needed here is that f_n is a quadratic function of the parameters y_i, which are conjugate to the propagators $G(k_i)$.[10] By quadratic is meant the fact that if a change of variables is made

$$y_i = \sigma\,\bar{y}_i, \quad \sum \bar{y}_i = 1 , \tag{25}$$

then $f_n(y_i)=\sigma^2\bar{f}_n(\bar{y}_i,\sigma)$ and $\bar{f}_n(\bar{y}_i,0)\neq 0$. The $p\to\infty$ limit of (23) is calculated by means of standard Mellin transform techniques.[9] First, one defines

$$f_{n+1}(\alpha,q) = \int_0^\infty dp\, p^{-\alpha-1} f_{n+1}(p,q) ,$$

and then finds the poles of $f_{n+1}(\alpha,q)$ in the α plane. If

$$f_{n+1}(\alpha,q) = \frac{F_{n+1}(q)}{\alpha-\alpha_0} ,$$

then

$$f_{n+1}(p,q) = p^{-\alpha_0} F_{n+1}(q) .$$

Thus, making the change of variables in (25), we have

$$\begin{aligned} f_{n+1}(\alpha,q) = c_n \int_0^\infty dx_1\cdots dx_{n+1}\,\rho_1\cdots\rho_{n+1} \int_0^\infty \sigma^{n-1}\,d\sigma \int_0^1 d\bar{y}_1\cdots d\bar{y}_n\,\delta\Big(1-\sum_1^n \bar{y}_i\Big) \\ \times\Gamma(-\tfrac{1}{2}\alpha)(-i)^{\alpha/2}\sigma^\alpha \frac{(\bar{f}_n)^{\alpha/2}}{\Delta_n^{3/2+\alpha/2}}\, e^{-iq^2(\epsilon_n/\Delta_n)} . \end{aligned} \tag{26}$$

Poles at $\alpha=-n, -n-1, -n-2, \ldots$ are generated by divergences of the integration over σ at the end point $\sigma=0$. The residues are calculated by doing the integration by parts for $\alpha>-n$ and analytically continuing the resulting expression. For example, to calculate the residue at $\alpha=-n-2$, one integrates by parts three times:

$$\int_0^\infty \sigma^{n+\alpha-1}F(\alpha,\sigma)\,d\sigma = \frac{(-1)^3}{(n+\alpha)(n+\alpha+1)(n+\alpha+2)} \times\int_0^\infty d\sigma\,\sigma^{n+\alpha+2}\frac{\partial^3}{\partial\sigma^3}F(\alpha,\sigma) .$$

In the neighborhood of $\alpha=-n-2$, this becomes

$$\begin{aligned} \int_0^\infty \sigma^{n+\alpha-1}F(\alpha,\sigma)\,d\sigma &\approx \frac{1}{n+\alpha+2}\,\frac{(-1)^3}{2!} \times\int_0^\infty d\sigma\,\frac{\partial^3}{\partial\sigma^3}F(-n-2,\sigma) \\ &= \frac{1}{n+\alpha+2}\,\frac{1}{2!}\,\frac{\partial^2}{\partial\sigma^2}F(-n-2,\sigma)\Big|_{\sigma=0} . \end{aligned} \tag{27}$$

The residues at the other poles are calculated in the same way. Hence, from (26) we obtain

$$f_{n+1}^{(i)}(q) = -\frac{m}{2\pi}\left(\frac{2m}{(2\pi)^3}\right)^n (-i)^n \left(\frac{\pi}{i}\right)^{3n/2} \times \int_0^\infty dx_1 \cdots dx_{n+1}\, \rho_1 \cdots \rho_{n+1} \times \Gamma\left(\frac{n+i}{2}\right)(-i)^{-(n+i)/2}\frac{1}{i!}\frac{\partial^i}{\partial\sigma^i} \times \int_0^1 \frac{\delta(1-\sum \bar{y}_i)\,d\bar{y}_i}{\Delta_n^{3/2}}\left(\frac{\bar{f}_n}{\Delta_n}\right)^{-(n+i)/2} e^{-i(q^2 g_n/\Delta_n)}. \tag{28}$$

This equation can be rewritten in the form of (10), where

$$I_{n+1}(q,\sigma,w) = -\frac{m}{2\pi}\left(\frac{2m}{(2\pi)^3}\right)^n (-i)^n\left(\frac{\pi}{i}\right)^{3n/2} \times \int_0^\infty dx_i\, \rho(x_i) \int_0^1 \frac{\delta(1-\sum\bar{y}_i)\,d\bar{y}_i}{\Delta_n^{3/2}} \times e^{-i(q^2 g_n/\Delta_n)}\, e^{+i(w^2/\sigma^2)(\sigma^2\bar{f}_n/\Delta_n)} . \tag{29}$$

As discussed in Sec. II, (29) is just (23) with the momentum $|\vec{p}| = w/\sigma$, q fixed at its physical value, all the $y_i = \sigma\bar{y}_i$, and the σ integral removed. This observation permits us to reintroduce the momentum integrals in (28) and obtain (12).

The method of Lévy and Sucher[6] is now used. The momenta are chosen so that momentum conservation is maintained by having one of the $n+1$ rungs carry the momentum $\vec{q} - \sum \vec{k}_i$. The amplitude is averaged over the $n+1$ ways of doing this. Consider $f_{n+1}^{(0)}$ as the simplest case:

$$f_{n+1}^{(0)}(q) = -\frac{m}{2\pi}\left(\frac{-2mi}{(2\pi)^3}\right)^n \frac{1}{n+1} \times \sum_{l=1}^{n+1}\int d^3k_1 \cdots d^3k_n\, V(k_1)\cdots V(k_n)\, V(q-\sum k_i) \times \int_0^\infty w^{n-1}\,dw \int_0^1 d\bar{y}_1\cdots d\bar{y}_n \times \delta(1-\sum\bar{y}_i)\, e^{-i\psi_l} , \tag{30}$$

with

$$\psi_l = \lim_{\sigma\to 0}\left\{\sigma\sum_{i=1}^{l-1}\bar{y}_i\left[(\vec{K}_i - \vec{p})^2 - \frac{w^2}{\sigma^2}\right] + \sigma\sum_{i=l}^{n}\bar{y}_i\left[(\vec{K}_i' + \vec{p}')^2 - \frac{w^2}{\sigma^2}\right]\right\} . \tag{31}$$

The momenta $\vec{p}$ and $\vec{p}'$ are given by (11). Hence, the limit $\sigma\to 0$ in (31) yields

$$\psi_l = \sum_{i=1}^{l-1} w\,\bar{y}_i(-2\vec{K}_i\cdot\vec{n}) + \sum_{i=l}^{n} w\,\bar{y}_i(2\vec{K}_i'\cdot\vec{n}) , \tag{32}$$

where $\vec{n}$ is a unit vector in the z direction, the direction of $\vec{p}+\vec{p}'$. The vectors $\vec{K}_i$ and $\vec{K}_i'$ are exactly those of Lévy and Sucher[6] (Sec. II of their paper), $\vec{K}_i = \sum_1^i \vec{k}_j$ and $\vec{K}_i' = \sum_i^n \vec{k}_j$. The integrals over w and y_i are combined to give

$$f_{n+1}^{(0)}(q) = -\frac{m}{(2\pi)}\left(\frac{-2mi}{(2\pi)^3}\right)^n \frac{1}{n+1}\sum_{l=1}^{n+1}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r) \times \prod_{i=1}^{n}\int d^3k_i\, V(k_i)\, e^{-i\vec{k}_i\cdot\vec{r}} \int_0^\infty dy_1\cdots dy_n \exp\left(-2i\sum_1^{l-1} y_j\vec{K}_j\cdot\vec{n} + 2i\sum_l^n y_j\vec{K}_j'\cdot\vec{n}\right) . \tag{33}$$

The Fourier transform of $V(q-\sum k_i)$, Eq. (8), has been used in (33). The next step is to change variables from y_i to z_i and z_i' where

$$z_i = \sum_{j=i}^{l-1} y_j,\quad z_i' = \sum_{j=l}^{i} y_j .$$

The integral over the y parameters becomes

$$\left[\int_0^\infty dz_1\int_0^{z_1} dz_2\cdots\int_0^{z_{l-2}} dz_{l-1}\exp\left(-2i\sum_{j=1}^{l-1} z_j\vec{K}_j\cdot\vec{n}\right)\right]\left[\int_0^\infty dz_n'\int_0^{z_n'} dz_{n-1}\cdots\int_0^{z_{l+1}'} dz_l'\exp\left(2i\sum_{j=l}^{n} z_j'\vec{K}_j'\cdot\vec{n}\right)\right] .$$

Next, as shown by Lévy and Sucher,[6] the first $l-1$ momenta can be relabeled in $(l-1)!$ equivalent ways, and the corresponding integrals added together. This has the effect of dividing (33) by $(l-1)!$ and changing all the z_i integrations to the interval 0 to ∞. The final $n+1-l$ momenta are treated similarly to give

$$f_{n+1}^{(0)}(q) = -\frac{m}{2\pi}\left(\frac{-2mi}{(2\pi)^3}\right)^n \frac{1}{n+1} \times \sum_{l=1}^{n+1}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)\frac{(F_+)^{l-1}}{(l-1)!}\frac{(F_-)^{n+1-l}}{(n+1-l)!} , \tag{34}$$

where

$$F_\pm = \int_0^\infty dz \int d^3k\, V(k)\, e^{-i\vec{k}\cdot\vec{r}\pm 2iz\vec{k}\cdot\vec{r}} .$$

When (34) is summed over l, it yields

$$f_{n+1}^{(0)}(q) = -\frac{m}{2\pi}\left(\frac{-2mi}{(2\pi)^3}\right)^n \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)\, \frac{F^n}{(n+1)!} , \quad (35)$$

with

$$F \equiv F_+ + F_-$$

$$= \tfrac{1}{2}(2\pi)^3 \int_{-\infty}^{\infty} dz\, V(r) ,$$

and $r = (b^2+z^2)^{1/2}$. Incorporating the factor of $2m/(2\pi)^3$ and using the definition (15a) for $\chi_0(b)$ together with the fact that $\vec{q}\cdot\vec{n}=0$, we find that (35) becomes

$$f_{n+1}^{(0)}(q) = \frac{-i}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\, \frac{[i\chi_0(b)]^{n+1}}{(n+1)!} .$$

The expression is just (14a) and completes the derivation of the Glauber eikonal.

The correction terms are calculated in exactly the same manner. We sketch the procedure for the first correction term. To obtain $f_{n+1}^{(1)}(q)$, we start from (30) with

$$e^{-i\psi_l}\Big|_{\sigma=0} \to \frac{\partial}{\partial\sigma}(e^{-i\psi_l})\Big|_{\sigma=0} = -i\,\frac{\partial\psi_l}{\partial\sigma}\, e^{-i\psi_l}\Big|_{\sigma=0} .$$

Since

$$\psi_l = \sum_{i=1}^{l} \bar{y}_i\,(\sigma K_i^2 - 2w\vec{K}_i\cdot\vec{n}\cos\theta - \sigma\vec{K}_i\cdot\vec{q})$$

$$+ \sum_{i=l}^{n} \bar{y}_i\,(\sigma K_i'^2 + 2w\vec{K}_i'\cdot\vec{n}\cos\theta - \sigma\vec{K}_i'\cdot\vec{q}) , \quad (36)$$

and $\cos\theta = (1+q^2\sigma^2/4w^2)^{1/2}$,

$$\frac{\partial\psi_l}{\partial\sigma}\Big|_{\sigma=0} = \sum_{i=1}^{l}\bar{y}_i\,(K_i^2 - \vec{K}_i\cdot\vec{q}) + \sum_{i=l}^{n}\bar{y}_i\,(K_i'^2 - \vec{K}_i\cdot\vec{q}) .$$

Thus,

$$f_{n+1}^{(1)}(q) = \frac{im}{(2\pi)}\left(\frac{-2mi}{(2\pi)^3}\right)^n \frac{1}{n+1}$$
$$\times \sum_{l=1}^{n+1}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r) \prod_{i=1}^{n}\int d^3k_i\, V(k_i)\, e^{-i\vec{k}_i\cdot\vec{r}} \int_0^\infty dy_1\cdots dy_n \exp\left(-2i\sum_1^{l-1} y_i\,\vec{K}_i\cdot\vec{n} + 2i\sum_l^n y_i\,\vec{K}_i'\cdot\vec{n}\right)$$
$$\times\left[\sum_{j=1}^{l} y_j(K_j^2 - \vec{K}_j\cdot\vec{q}) + \sum_{j=l}^{n} y_j(K_j'^2 - \vec{K}_j'\cdot\vec{q})\right] . \quad (37)$$

When the integration variables are changed from y_i to z_i as before, the first term in the square brackets becomes

$$\sum_{j=1}^{l} z_j(k_j^2 - \vec{q}\cdot\vec{k}_j) + 2\sum_{j=1}^{l}\sum_{m=j+1}^{l} \vec{k}_j\cdot\vec{k}_m\, z_m .$$

When (37) is averaged over the permutations of the $\vec{k}_i$, it has the form

$$f_{n+1}^{(1)}(q) = \frac{im}{(2\pi)}\left(\frac{-2mi}{(2\pi)^3}\right)^n \frac{1}{n+1}\sum_{l=1}^{n+1}\int \frac{d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)}{(l-1)!(n+1-l)!}$$
$$\times\left\{(F_-)^{n+1-l}\left[(l-1)(G_1^+ - G_2^+)(F_+)^{l-2} + \tfrac{1}{2}(l-1)(l-2)\,G_3^+(F_+)^{l-3}\right] + (+\longleftrightarrow -)\right\}$$
$$= \frac{im}{2\pi}\left(\frac{-2mi}{(2\pi)^3}\right)^n \frac{1}{n+1}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)\left[\frac{F^{n-1}}{(n-1)!}(G_1^+ + G_1^- - G_2^+ - G_2^-) + \frac{F^{n-2}}{2(n-2)!}(G_3^+ + G_3^-)\right] , \quad (38)$$

where

$$G_1^\pm = \int_0^\infty z\,dz \int d^3k\, V(k)\, k^2 e^{-i\vec{k}\cdot\vec{r}\pm 2iz\vec{k}\cdot\vec{n}} ,$$

$$G_2^\pm = \int_0^\infty z\,dz \int d^3k\, V(k)\,\vec{q}\cdot\vec{k}\, e^{-i\vec{k}\cdot\vec{r}\pm 2iz\vec{k}\cdot\vec{n}} ,$$

$$G_3^\pm = 2\int_0^\infty dz_1 \int_0^{z_1} z_2\,dz_2$$
$$\times \int d^3k_1\, d^3k_2\, V(k_1)\, V(k_2)\, 2\vec{k}_1\cdot\vec{k}_2$$
$$\times e^{-i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}}\, e^{\pm 2i(z_1\vec{k}_1+z_2\vec{k}_2)\cdot\vec{n}} . \quad (39)$$

There is cancellation between G_2 and the other two terms. To see this we write the G_i in the form

$$G_1=G_1^+ + G_1^-$$
$$=-\frac{(2\pi)^3}{4}\nabla_r^2\int_0^\infty z'dz'[V(\vec{r}+\vec{n}z')+V(\vec{r}-\vec{n}z')]$$
$$=-\nabla_r^2 G,$$
$$G_2=i\vec{q}\cdot\vec{\nabla}_r G$$
$$G_3=-\frac{(2\pi)^6}{2}\int_0^\infty dz_1'$$
$$\times\int_0^{z_1'} z_2'dz_2'[\vec{\nabla}_r V(\vec{r}+\vec{n}z_1')\cdot\vec{\nabla}_r V(\vec{r}+\vec{n}z_2') + \vec{\nabla}_r V(\vec{r}-\vec{n}z_1')\cdot\vec{\nabla}_r V(\vec{r}-\vec{n}z_2')]\ . \tag{40}$$

Next, integration by parts is used to rearrange the G_2 term in (38):

$$\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)(-i\vec{q}\cdot\vec{\nabla}_r G)F^{n-1}$$
$$=\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\,[\nabla_r^2 G\, V(r)F^{n-1} + \vec{\nabla}_r G\cdot\vec{\nabla}_r V(r)F^{n-1} + \vec{\nabla}_r G\cdot\vec{\nabla}_r F(n-1)V(r)F^{n-2}]\ . \tag{41}$$

The first term on the right in (41) cancels G_1. The second term is evaluated by use of the relation, valid for central potentials,

$$\int_{-\infty}^\infty dz\,\vec{\nabla}_r V(r)\cdot\vec{\nabla}_r\int_0^\infty w\,dw\,[V(\vec{r}+\vec{n}w)+V(\vec{r}-\vec{n}w)]$$
$$=-2\int_{-\infty}^\infty dz\left[V(r)^2+2b^2\frac{V(r)}{r}\frac{d}{dr}V(r)\right]$$
$$=-2\int_{-\infty}^\infty dz\,\frac{1}{r}\frac{d}{dr}[rV(r)]^2=-2H(b)\,.$$

The third term in (41) combines with the G_3 part of (38) according to

$$\int_{-\infty}^\infty dz\,V(r)\left\{\int_0^\infty dw_1[\vec{\nabla}V(\vec{r}+\vec{n}w_1)+\vec{\nabla}V(\vec{r}-\vec{n}w_1)]\cdot\int_0^\infty w_2dw_2[\vec{\nabla}V(\vec{r}+\vec{n}w_2)+\vec{\nabla}V(\vec{r}-\vec{n}w_2)]\right.$$
$$\left.-2\int_0^\infty dw_1\int_0^{w_1} w_2dw_2[\vec{\nabla}V(\vec{r}+\vec{n}w_1)\cdot\vec{\nabla}V(\vec{r}+\vec{n}w_2)+\vec{\nabla}V(\vec{r}-\vec{n}w_1)\cdot\vec{\nabla}V(\vec{r}-\vec{n}w_2)]\right\}=-\int_{-\infty}^\infty dz\,V(r)H(b)\,.$$

With these results $f^{(1)}_{n+1}(q)$ becomes

$$f^{(1)}_{n+1}(q)=\frac{im}{(2\pi)}\left(\frac{-2mi}{(2\pi)^3}\right)^n\frac{1}{n+1}\times\int d^2b\,e^{i\vec{q}\cdot\vec{b}}\times\frac{(2\pi)^3}{4}\left[-\frac{2F^{n-1}H(b)}{(n-1)!}-\frac{F^{n-1}H(b)}{(n-2)!}\right]$$

or

$$f^{(1)}_{n+1}(q)=\frac{-i}{2\pi}\int d^2b\,e^{i\vec{q}\cdot\vec{b}}\frac{(i\chi_0)^{n-1}}{(n-1)!}\times[i(-\tfrac{1}{2}m^2H(b))]\ . \tag{42}$$

This is just (14b) with (15b) for $\chi_1(b)$.

The derivation of the second correction term proceeds in much the same way although the algebraic details are far more complicated. Details are available.

ACKNOWLEDGMENTS

I wish to thank F. Byron for suggesting this investigation and B. Holstein for many useful conversations.

APPENDIX

The discussion of the large-angle validity of the eikonal representation is based on a calculation of the $p\to\infty$ followed by $q\to\infty$ limit of the Born series. Here we show that for a Yukawa potential this limit is independent of the order in which $q, p\to\infty$ and, for the second Born term, is identical to the $p, q\to\infty$, q/p-fixed limit. These results indicate that the Glauber eikonal with $q\to\infty$ does indeed generate the high energy, fixed-angle limit of the Born series.

We start with

$$f_{n+1}(p,q)=\frac{f^{(0)}_{n+1}(q)}{p^n}\ . \tag{A1}$$

The expression for $f^{(0)}_{n+1}(q)$ in (14a) can be written in the form

$$f^{(0)}_{n+1}(q)=\frac{1}{i}\int_0^\infty b\,db\,J_0(qb)\frac{[i\chi_0(b)]^{n+1}}{(n+1)!}\ . \tag{A2}$$

$J_0(qb)$ is a Bessel function. If $V(r)=ge^{-\mu r}/r$,

$$\chi_0(b)=-2mgK_0(\mu b)\ , \tag{A3}$$

where $K_0(x)$ is a modified Bessel function. Hence, we have

$$f^{(0)}_{n+1}(\alpha)=\int_0^\infty q^{-\alpha-1}f^{(0)}_{n+1}(q)dq$$

$$=\frac{1}{i}\frac{(-2mig)^{n+1}}{(n+1)!}\frac{\Gamma(-\frac{1}{2}\alpha)}{2^{\alpha+1}\Gamma(\frac{1}{2}\alpha+2)}$$

$$\times\int_0^\infty b^{\alpha+1}[K_0(\mu b)]^{n+1}db\,. \qquad (A4)$$

The integral in (A4) has an $(n+2)$-order pole at $\alpha=-2$ arising from a combination of the divergence at $b=0$ of $b^{\alpha+1}$ and the fact that $\lim_{b\to 0}K_0(\mu b)\sim \ln b+c$. Isolating this pole, we find

$$f^{(0)}_{n+1}(\alpha)=\frac{1}{i}\frac{(-2mig)^{n+1}}{(n+1)!}\frac{2\Gamma(1)}{\Gamma(\frac{1}{2}\alpha+2)}\frac{(n+1)!}{(\alpha+2)^{n+2}}$$

$$=\frac{1}{i}(-2mig)^{n+1}\frac{1}{(\alpha+2)^{n+1}}\,. \qquad (A5)$$

The large-q behavior of $f^{(0)}_{n+1}(q)$ is determined from (A5) to be proportional to $(\ln q)^n/q^2$. Thus, we have

$$\lim_{q\to\infty}\lim_{p\to\infty}f_{n+1}(p,q)\sim\frac{1}{i}\frac{(-2mig)^{n+1}}{n!}\frac{(\ln q)^n}{p^n q^2}\,. \qquad (A6)$$

The calculation of the $q\to\infty$ limit of $f_{n+1}(p,q)$ amounts to the standard perturbation theory calculation of a Regge trajectory.[9] The Mellin transform of (23) with respect to q^2 is

$$f_{n+1}(p,\alpha)=A_{n+1}\Gamma(-\alpha)(i)^\alpha$$

$$\times\int_0^\infty dx_1\cdots dx_{n+1}(x_1\cdots x_{n+1})^\alpha$$

$$\times\int_0^\infty\frac{dy_1\cdots dy_n}{\Delta_n^{3/2}}e^{-iQ_n}, \qquad (A7)$$

where

$$A_n=-\frac{m}{2\pi}\left(\frac{-2mig}{(2\pi)^3}\right)^n\left(\frac{\pi}{i}\right)^{3n/2}(4\pi ig)^{n+1},$$

$$Q_n=-p^2\frac{f_n}{\Delta_n}+\mu^2\sum_1^{n+1}x_i-i\epsilon\sum_1^n y_i-i\epsilon\sum_1^{n+1}x_i\,.$$

We have used the fact that

$$\rho(x)=4\pi ig\,e^{-i\mu^2x}$$

for a Yukawa potential. There is an $(n+1)$-order pole at $\alpha=1$ coming from a divergence at each $x_i=0$. Isolating the pole, we find

$$f_{n+1}(p,\alpha)=A_{n+1}\frac{\Gamma(1)}{i}\frac{[K(p)]^n}{(\alpha+1)^{n+1}}, \qquad (A8)$$

where

$$K(p)=\int_0^\infty\frac{dy}{\sqrt{y}}e^{-iy(-p^2-i\epsilon)}$$

$$=\frac{i}{p}\left(\frac{\pi}{i}\right)^{1/2}. \qquad (A9)$$

From (A8) and (A9) we find the large-q^2 limit of $f_{n+1}(p,q)$ to be

$$\lim_{q^2\to\infty}f_{n+1}(p,q)\sim\frac{-m}{2\pi}\left(\frac{-2mi}{(2\pi)^3}\right)^n\left(\frac{\pi}{i}\right)^{3n/2}(4\pi ig)^{n+1}$$

$$\times\frac{1}{i}\frac{(\ln q^2)^n}{n!q^2}\left[\frac{i}{p}\left(\frac{\pi}{i}\right)^{1/2}\right]^n$$

$$=\frac{1}{i}\frac{(-mig)^{n+1}}{n!}\frac{(\ln q^2)^n}{p^nq^2}\,. \qquad (A10)$$

The identity of (A10) and (A6) suggests that they also represent the $q, p\to\infty$, q/p-fixed limit.

Additional support for the above statement is provided by a calculation of $f_2(p,q)$ with $q^2=\lambda p^2$, $p\to\infty$, λ fixed. Starting again from (23) and transforming with respect to p^2, we have

$$f_2(\alpha,\lambda)=A_2\Gamma(-\alpha)(-i)^\alpha$$

$$\times\int_0^\infty dy\,dx_1dx_2\frac{(y^2-\lambda x_1x_2)^\alpha}{(y+x_1+x_2)^{\alpha+3/2}}$$

$$\times e^{-i\mu^2(x_1+x_2)}\,. \qquad (A11)$$

If $y=\rho(1-x)$, $x_2=\rho xw$, $x_1=\rho x(1-w)$, the ρ integration can be performed to yield

$$f_2(\alpha,\lambda)=\frac{A_2\Gamma(-\alpha)(-i)^\alpha\Gamma(\alpha+\frac{3}{2})}{(i\mu^2)^{\alpha+3/2}}$$

$$\times\int_0^1 x^{-\alpha-1/2}\int_0^1 dw\,[(1-x)^2-\lambda x^2w(1-w)]^\alpha\,. \qquad (A12)$$

The integral in (A12) has a pole at $\alpha=-\frac{3}{2}$ from divergences at $u=1-x=0$, $w=0$ or $w=1$. The residue is obtained from an evaluation of

$$f_2(\alpha,\lambda)=\frac{A_2\Gamma(\frac{3}{2})(-i)^{-3/2}}{(\alpha+\frac{3}{2})}\,2\int_0^\epsilon du\int_0^\epsilon dw(u^2-\lambda w)^\alpha\,. \qquad (A13)$$

Wherever possible, we used $x=1$ and $\alpha=-\frac{3}{2}$. From (A13) we find

$$f_2(\alpha,\lambda)=\frac{A_2\pi^{1/2}(i)^{3/2}}{(\alpha+\frac{3}{2})}\left(\frac{-1}{\lambda(\alpha+\frac{3}{2})}\right). \qquad (A14)$$

Thus

$$\lim_{p\to\infty}f(p,\lambda)\sim\frac{1}{i}\frac{(-2mig)^2}{2}\frac{\ln p^2}{p^3\lambda}\,. \qquad (A15)$$

Since $p^2\lambda=q^2$, (A15) is identical to (A6) with $n=1$.

*Work supported in part by the National Science Foundation.
[1]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Interscience, New York, 1959), Vol. I, p. 315.
[2]For example, there is a discussion in L. I. Schiff, *Quantum Mechanics*, 3rd edition (McGraw-Hill, New York, 1968), p. 339.
[3]F. W. Byron, C. J. Joachain, and E. H. Mund, Phys. Rev. D 8, 2622 (1973).
[4]R. Blankenbecler and M. L. Goldberger, Phys. Rev. 126, 766 (1962).
[5]H. Abarbanel and C. Itzykson, Phys. Rev. Lett. 23, 53 (1968).
[6]M. Lévy and J. Sucher, Phys. Rev. 186, 1656 (1969).
[7]S. J. Wallace, Phys. Rev. Lett. 27, 622 (1971); Ann. Phys. (N.Y.) 78, 190 (1973); Phys. Rev. D 8, 1846 (1973).
[8]The authors of Ref. 3 anticipate this result on the basis of a careful analysis of the first few terms in the Born and eikonal series. R. J. Moore [Phys. Rev. D 2, 313 (1970)] has a proof that the Glauber eikonal is the high-energy limit of the Born series. His proof is valid only for Yukawa potentials and for large, but finite, values of the momentum transfer. Moreover, since his proof starts from the Glauber eikonal, it in no way demonstrates uniqueness.
[9]R. Eden, P. Landshoff, D. Olive, and J. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, New York, 1966), Chap. 3; A. R. Swift, J. Math. Phys. 8, 2420 (1967).
[10]A detailed discussion of the properties of f_n, g_n, and Δ_n can be found in A. R. Swift, Phys. Rev. 176, 1848 (1968).

PHYSICAL REVIEW D VOLUME 9, NUMBER 6 15 MARCH 1974

Proton-neutron mass difference and the pion mass in a gauge model

Judy Lieberman*
Rockefeller University, New York, New York 10021
(Received 11 December 1973)

In a variation of the Weinberg SU(2)×SU(2)×U(1) gauge model of the weak and electromagnetic interactions, we study the proton-neutron mass difference, which is calculable, and investigate the appearance of pions as part of the Higgs system. We find that the proton-neutron mass difference is a function of the way in which the symmetry is broken. We exhibit a possible symmetry breaking which produces the correct sign for the mass difference. In the Higgs sector, we have a mass-degenerate pseudoscalar triplet which interacts with nucleons as pions do in the SU(2)×SU(2) σ model. Therefore we identify this triplet with pions. They are massive in zeroth order, but we can calculate the mass difference δm^2. We find that δm^2 is of order $\alpha\mu^2$ which is too large. If we impose a reflection symmetry on the Lagrangian, the symmetry group of the potential is enlarged and we find that the theory contains three pseudo-Goldstone bosons. These are the pion triplet, which are now massless in zeroth order. When we calculate the pion mass in the one-loop approximation, the Π^0 remains massless while the charged pions pick up mass of order $m^2 \sim \alpha\mu^2$. This may perhaps be damped numerically to give a suitable estimate of the pion mass, but the mass difference is still too large.

I. INTRODUCTION

One of the most promising features of gauge theories of the weak and electromagnetic interactions is the possibility of calculating masses and mass differences. Previously, in renormalizable field theories, if a bare mass or mass difference vanished, then either it remained zero to all orders because of an underlying symmetry of the Lagrangian or it was infinite in higher orders. In a spontaneously broken gauge theory, if a mass difference or mass is zero in zeroth order for all possible coupling constants even after the symmetry is broken, then that quantity is necessarily finite and calculable. Since there are no possible counterterms to cancel infinities if the zeroth-order relation holds in the presence of all coupling constants not subject to artificial constraints, all higher corrections must be finite because the theory is renormalizable.[1]

This paper studies particular questions in the domain of this new calculability in the framework of an SU(2)×SU(2)×U(1) model of the weak and electromagnetic interactions. The model is basically due to Weinberg, but our interpretation of it is quite different.[2] One of the aims of this paper is to study the proton-neutron mass difference which is calculable in this model. The second aim is to investigate mechanisms for incorporating pions into gauge theories.

If a gauge theory is to describe the weak and electromagnetic interactions, the gauge symmetry