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Theory of massive and massless Yang-Mills fields*

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Introducing the Lagrangian multiplier field $\vec{\chi}(x)$, a canonical formalism for the Yang-Mills fields $\vec{f}_\mu(x)$ with mass $M \geq 0$ is proposed within the framework of an indefinite-metric quantum field theory. The *formalism* for the massive $\vec{f}_\mu$ has a well-defined zero-mass limit, and the reduction of the physical components of $\vec{f}_\mu$ as $M \to 0$ is embodied in an elegant way. Using the field equation for $\vec{\chi}(x)$ and the path integral, we find that the "extra" factor in the amplitude due to the interaction of $\vec{\chi}(x)$ in the intermediate states is $[\det(1+(\Box+M^2)^{-1} g\vec{f}_\mu \times \partial^\mu)]^{-1/2} \equiv D_M{}^{-1/2}$ for the massive $\vec{f}_\mu$, and that the extra factor is $D_{M=0}{}^{-1}$ for the massless $\vec{f}_\mu$ because of their different degrees of observable freedom. Thus, the *resultant rules* for the Feynman diagrams for $M > 0$ and $M = 0$ are *not* smoothly connected. The theory is covariant, renormalizable, and unitary after the extra parts are removed from the amplitudes. The problems of unitarization and renormalizability are discussed.

I. INTRODUCTION

We propose a canonical formalism for the Yang-Mills fields $\vec{f}_\mu(x)$ with mass $M \geq 0$, where the mass M has nothing to do with spontaneously broken gauge symmetry. A Lagrange-multiplier field $\vec{\chi}$ is introduced in the Lagrangian of the formalism,[1] so that the Lagrangian describes a pure vector particle f and a scalar particle χ with a mass M and a negative norm. The formalism for the massive $\vec{f}_\mu$ has a well-defined limit $M \to 0$ and it is covariant and renormalizable. It is inevitable to introduce an indefinite-metric Hilbert space if one wishes to formulate quantum electrodynamics in a manifestly covariant way.[2] Although it is easy to construct a renormalizable theory of a vector field employing an indefinite metric, it is more difficult to have a unitary "physical" field-theoretic S matrix for the vector field.[3] Usually, the physical S matrix can be consistently defined in field theory if and only if the S matrix satisfies the "physical-state condition," namely, if the initial state is a physical state, then the final state is also a physical state and vice versa. This condition is satisfied in Abelian gauge field theories because the Lagrange multiplier obeys the free-field equation by virtue of the source current being conserved. Yet, it is extremely difficult to satisfy this condition in other field theories.

In the theory of the Yang-Mills field, the Lagrange multiplier field $\vec{\chi}(x)$ does not obey the free-field equation and, therefore, the physical-state condition is not satisfied. The field-theoretic definition of the S matrix $\bar{S}$ in the physical-state subspace is *not unitary* due to the coupling between $\vec{\chi}(x)$ and $\vec{f}_\mu(x)$. So, the burning question is: How can we unitarize the S matrix without violating causality and analyticity and thus define the physical S matrix S_{ph}? The present Lagrange-multiplier formalism is constructed to overcome this difficulty. The field equations, especially the equation for the Lagrange multiplier $\vec{\chi}(x)$, are extensively used. For instance, though $\vec{\chi}(x)$ might superficially appear to have several different couplings, the field equations are used to show cancellations between different couplings and to simplify the $\vec{\chi}$ coupling. Furthermore, the equation for $\vec{\chi}$ is used to identify the extra absorptive part in $\overline{S}$ due to the interaction of $\vec{\chi}$.

The Lagrangian for the massive $\vec{f}_\mu$ and $\vec{\chi}$ is

$$\mathcal{L}^M = \mathcal{L}_{YM} + \tfrac{1}{2}M^2 \vec{f}_\mu \cdot \vec{f}^\mu + \eta \partial_\mu \vec{f}^\mu \cdot \vec{\chi} + \tfrac{1}{2}\eta^2 \vec{\chi}^2 - \overline{\psi}\gamma^\mu(-i\partial_\mu - \tfrac{1}{2}g\vec{\tau}\cdot\vec{f}_\mu)\psi - m\overline{\psi}\psi \,, \tag{1}$$

where $\vec{\tau}$ are the Pauli matrices for isospin, $x^\mu = (t, x, y, z)$, $g_{00} = -g_{kk} = 1$, $\partial_\mu \equiv \partial/\partial x^\mu$, and

$$\mathcal{L}_{YM} = -\tfrac{1}{4}\vec{f}_{\mu\nu}\cdot\vec{f}^{\mu\nu}, \tag{2}$$

$$\vec{f}_{\mu\nu} = \partial_\nu \vec{f}_\mu - \partial_\mu \vec{f}_\nu - g\vec{f}_\mu \times \vec{f}_\nu \,. \tag{3}$$

The parameter η will be specified as $\eta = M$ eventually for the massive $\vec{f}_\mu$; it is arbitrary when $M = 0$. The Lagrange-multiplier field $\vec{\chi}$ has no free Lagrangian and no apparent interaction with the transverse component of $\vec{f}_\mu$. However, we can derive the field equation for $\vec{\chi}(x)$. The equation indicates that $\vec{\chi}$ has a mass M and an interaction $g\vec{\chi}\cdot(\vec{f}_\mu \times \partial^\mu \vec{\chi})$, which completely determines the coupling of $\vec{\chi}$. It is shown below that the equation of motion for the Lagrangian-multiplier field $\vec{\chi}(x)$ together with the Feynman path integral is a powerful tool to extract and isolate the extra amplitudes to all orders in $\overline{S}$ in a simple way.[4] We find that the modification of the amplitude to all orders in $\overline{S}$ due to the interaction of $\vec{\chi}$ is coming from a factor [cf. the expression (47) below]

$$D_M^{-1/2} = [\det(1 + (\Box + M^2)^{-1} g\vec{f}_\mu \times \partial^\mu)]^{-1/2} \,. \tag{4}$$

Once this extra amplitude is removed from $\overline{S}$, the Yang-Mills theory for the massive $\vec{f}_\mu$ leads to the physical S matrix S_{ph} which is unitary. This has been verified by explicit calculations up to and including two loops.

When $M = 0$, we have a gauge-invariant $\mathcal{L}_{YM}$ and we may replace $\tfrac{1}{2}\eta^2\vec{\chi}^2$ by $\tfrac{1}{2}\beta\eta^2\vec{\chi}^2$, where β is the gauge parameter. In this case, the χ particle also has zero mass for all β. The extra amplitude in $\overline{S}$ due to the coupling between the unphysical and the physical $\vec{f}_\mu$ is found to be $D_{M=0}{}^{-1}$ instead of $D_{M=0}{}^{-1/2}$. This is because now there are two unphysical components in the massless 4-vector field $\vec{f}_\mu$. The massless Yang-Mills theory is unitary after the extra amplitude $D_{M=0}{}^{-1}$ is removed from $\overline{S}|_{M=0}$ to obtain the physical S matrix S_{ph}.

After unitarization, the resultant physical S matrices for the massive and the massless Yang-Mills theories are not smoothly related in the sense that

$$D_{M\to 0}{}^{1/2} \neq D_{M=0} \,. \tag{5}$$

This reflects the fact that there are two unphysical components in the massless $\vec{f}_\mu$, while there is only one in the massive $\vec{f}_\mu$. However, it should be emphasized that the *formalisms* for $M > 0$ and $M = 0$ are smoothly related. The sudden reduction of the observable degree of freedom in the limit $M \to 0$ can be clearly seen in the formalism.

II. FIELD EQUATIONS AND QUANTIZATION

The Lagrangian (1) can be written as

$$\mathcal{L}^M_\zeta = \mathcal{L}_{YM} + \tfrac{1}{2}M^2 \vec{f}_\mu \cdot \vec{f}^\mu - \tfrac{1}{2}(\partial_\mu \vec{f}^\mu)^2 + \tfrac{1}{2}\vec{\zeta}^2(x) - \overline{\psi}\gamma^\mu(-i\partial_\mu - \tfrac{1}{2}g\vec{\tau}\cdot\vec{f}_\mu)\psi - m\overline{\psi}\psi \,, \tag{6}$$

where $\vec{\zeta}(x) = (\partial_\mu \vec{f}^\mu + \eta\vec{\chi})$ can be regarded as a new field. The equation for $\vec{\zeta}$ derived from (6) is $\vec{\zeta} = 0$, so that $\vec{\zeta}(x)$ in (6) can be ignored. The Lagrange-multiplier formalism for the massive Yang-Mills field is based on the Lagrangian (1), which leads to the following field equations ($\Box \equiv \partial_\mu \partial^\mu$):

$$(\Box + M^2)\vec{f}_\mu + \vec{J}_\mu = 0 \,, \tag{7}$$

$$\partial_\mu \vec{f}^\mu + \eta\vec{\chi} = 0 \,, \tag{8}$$

$$\gamma^\mu(-i\partial_\mu - \tfrac{1}{2}g\vec{\tau}\cdot\vec{f}_\mu)\psi + m\psi = 0 \,, \tag{9}$$

$$\vec{J}_\mu \equiv \tfrac{1}{2}g\overline{\psi}\gamma_\mu\vec{\tau}\psi + g\vec{f}^\nu \times \vec{f}_{\mu\nu} + g\partial^\nu(\vec{f}_\nu \times \vec{f}_\mu) \,. \tag{10}$$

The divergence of the field equation (7) together with the constraint (8) leads to the equation of motion for the Lagrange-multiplier field $\vec{\chi}$:

$$(\Box + M^2)\vec{\chi} = \eta^{-1}\partial_\mu \vec{J}_\mu \,, \tag{11}$$

where $\vec{J}_\mu$ is given by (10). Superficially, the χ particle has a rather complicated interaction as shown in (11). However, using Eqs. (7), (8), and (9), the field equation (11) becomes

$$(\Box + M^2)\vec{\chi} = -g\vec{f}_\mu \times \partial^\mu \vec{\chi} \,. \tag{12}$$

The canonical conjugate π^a_μ, where $a = 1, 2, 3$ of the field f^a_μ is defined by

$$\pi^a_k \equiv \frac{\delta\mathcal{L}^M}{\delta(\partial_0 f^{ak})} = \partial_0 f^a_k - \partial_k f^a_0 \,, \quad k = 1, 2, 3$$
$$\pi^a_0 \equiv \frac{\delta\mathcal{L}^M}{\delta(\partial_0 f^{a0})} = \eta\chi^a \,. \tag{13}$$

The equal-time commutators are given by

$$[f_\mu^a(x), \pi_\nu^b(y)] = i\delta_{\mu\nu}\delta_{ab}\delta^3(\vec{x}-\vec{y}), \quad x_0 = y_0$$
$$[f_\mu^a(x), f_\nu^b(y)] = [\pi_\mu^a(x), \pi_\nu^b(y)] = 0, \quad x_0 = y_0 . \tag{14}$$

It is convenient to rewrite (14) directly in terms of f_μ^a, $\partial_0 f_k^a$, and χ^a (for $x^0 = y^0$):

$$[f_\mu^a(x), \partial_0 f_k^b(y)] = i\delta_{ab}\delta_{\mu k}\delta^3(\vec{x}-\vec{y}),$$
$$[f_\mu^a(x), \eta\chi^b(y)] = +i\delta_{ab}\delta_{\mu 0}\delta^3(\vec{x}-\vec{y}), \tag{15}$$
$$[\partial_0 f_k^a(x), \eta\chi^b(y)] = +i\delta_{ab}\partial_k^x\delta^3(\vec{x}-\vec{y});$$

all others vanish. These commutation relations already imply an indefinite metric, but it is not very transparent in (15); see (31) below.

In the absnece of ψ, the free Hamiltonian $\mathcal{H}_f$ for $\vec{f}_\mu$ is defined by

$$\mathcal{H}_f = \vec{\pi}_k \cdot \partial_0 \vec{f}_k + \vec{\pi}_0 \cdot \partial_0 \vec{f}_0 - \mathcal{L}^M(\vec{f}_\mu, \psi=0, g=0)$$
$$= \tfrac{1}{4}(\partial_k f_l^a - \partial_l f_k^a)^2 + \tfrac{1}{2}[(\partial_0 f_k^a)^2 - (\partial_k f_0^a)^2]$$
$$+ \tfrac{1}{2}M^2(f_k^a f_k^a - f_0^a f_0^a) - \tfrac{1}{2}\eta^2\chi^a\chi^a + \eta\chi^a\partial_k f_k^a . \tag{16}$$

Here, we treat $\mathcal{L}^M(f_\mu^a, \psi, g=0)$ as the unperturbed Lagrangian. The Heisenberg equation

$$i\partial_0 Q = [Q, H_f], \quad Q = f_\mu^a, \pi_\mu^a, \quad H_f = \int \mathcal{H}_f d^3x \tag{17}$$

reproduces the free equation of motion for $f^a(x)$ and some identities. Thus, H_f is really the time displacement operator. In the absence of the fermion field $\psi(x)$, the interaction Hamiltonian is

$$\mathcal{H}_{\rm int} = \mathcal{H} - \mathcal{H}_f$$
$$= (\vec{\pi}_k \cdot \partial_0 \vec{f}_k + \vec{\pi}_0 \cdot \partial_0 \vec{f}_0 - \mathcal{L}^M) - \mathcal{H}_f$$
$$= -\tfrac{1}{2}g(\partial_\nu \vec{f}_\mu - \partial_\mu \vec{f}_\nu) \cdot (\vec{f}^\mu \times \vec{f}^\nu)$$
$$+ \tfrac{1}{4}g^2(\vec{f}_\mu \times \vec{f}_\nu) \cdot (\vec{f}^\mu \times \vec{f}^\nu), \tag{18}$$

which is an invariant under Lorentz transformation.

III. THE PROPAGATORS

For convenience in studying the limit $M \to 0$, let us consider the following modified *free* Lagrangian in this section:

$$\mathcal{L}_{\rm free}^\beta = -\tfrac{1}{4}(\partial_\nu \vec{f}_\mu - \partial_\mu \vec{f}_\nu) \cdot (\partial^\nu \vec{f}^\mu - \partial^\mu \vec{f}^\nu)$$
$$+ \tfrac{1}{2}M^2 \vec{f}_\mu \cdot \vec{f}^\mu + \eta\vec{\chi} \cdot \partial_\mu \vec{f}^\mu + \tfrac{1}{2}\beta\eta^2\vec{\chi}^2 . \tag{19}$$

This gives the following "free" field equations:

$$(\Box + M^2)\vec{f}_\mu - (1-\beta)\eta\partial_\mu\vec{\chi} = 0,$$
$$\partial_\mu \vec{f}^\mu + \beta\eta\vec{\chi} = 0, \tag{20}$$

$$(\Box + \beta M^2)\vec{\chi} = 0 . \tag{21}$$

It follows that

$$(\Box + \beta M^2)(\Box + M^2)\vec{f}_\mu = 0, \quad \beta \neq 1 \tag{22}$$

$$(\Box + M^2)\vec{f}_\mu = 0, \quad \beta = 1 . \tag{23}$$

From Lorentz covariance, local commutativity, (21), and (22), we have

$$[f_\mu^a(x), f_\nu^b(y)] = [(\bar{a}g_{\mu\nu} + \bar{b}\partial_\mu^x\partial_\nu^x)\Delta(x-y, M^2)$$
$$+ (\bar{c}g_{\mu\nu} + \bar{d}\partial_\mu^x\partial_\nu^x)\Delta(x-y, \beta M^2)]\delta_{ab} . \tag{24}$$

The coefficients $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ are completely determined by the equal-time commutators, and we obtain

$$[f_\mu^a(x), f_\nu^b(y)] = -i\delta_{ab}(g_{\mu\nu} + M^{-2}\partial_\mu^x\partial_\nu^x)\Delta(x-y, M^2)$$
$$+ iM^{-2}\partial_\mu^x\partial_\nu^x\delta_{ab}\Delta(x-y, \beta M^2), \tag{25}$$

$$\Delta(x, M^2) = -i(2\pi)^{-3}\int d^4p\, \epsilon(p_0)\delta(p^2 - M^2)e^{-ip\cdot x} . \tag{26}$$

One should be careful in taking the limit $M \to 0$ in (25). We find that (25) reduces to

$$[f_\mu^a(x), f_\nu^b(y)] = -i\delta_{ab}g_{\mu\nu}\Delta(x-y, 0)$$
$$+ i(1-\beta)\delta_{ab}\partial_\mu^x\partial_\nu^x E(x-y) \tag{27}$$

in the limit $M \to 0$, where

$$E(x) \equiv -\frac{\partial}{\partial M^2}\Delta(x, M^2)\bigg|_{M=0}$$
$$= -(8\pi)^{-1}\epsilon(x_0)\theta(x^2), \tag{28}$$

$$\Box E(x) = \Delta(x, 0). \tag{29}$$

From (20) and (25), we obtain

$$[f_\mu^a(x), \eta\chi^b(y)] = -i\delta_{ab}\partial_\mu^x\Delta(x-y, \beta M^2), \tag{30}$$

$$[\eta\chi^a(x), \eta\chi^b(y)] = -iM^2\delta_{ab}\Delta(x-y, \beta M^2). \tag{31}$$

The commutator (31) implies that $\vec{\chi}(x)$ is a negative-metric field if $M^2 > 0$ and that it is a zero-norm field if $M^2 = 0$. (A state having a zero or negative norm is called a ghost state.)

From (25), (30), and (31), we have

$$F_{\mu\nu}^{ab}(k) \equiv \int d^4x\, e^{ik\cdot x}\langle 0|T(f_\mu^a(x)f_\nu^b(0))|0\rangle$$
$$= \delta_{ab}\left[\frac{-i(g_{\mu\nu} - M^{-2}k_\mu k_\nu)}{k^2 - M^2 + i\epsilon} - \frac{ik_\mu k_\nu}{M^2(k^2 - \beta M^2 + i\epsilon)}\right], \tag{32}$$

$$\int d^4x\, e^{ik\cdot x}\langle 0|T(f_\mu^a(x)\chi^b(0))|0\rangle$$
$$= +\delta_{ab}\eta^{-1}k_\mu(k^2 - \beta M^2 + i\epsilon)^{-1}, \tag{33}$$

$$\int d^4x\, e^{ik\cdot x}\langle 0|T(\chi^a(x)\chi^b(0))|0\rangle$$
$$= -i\delta_{ab}\eta^{-2}M^2(k^2 - \beta M^2 + i\epsilon)^{-1} . \tag{34}$$

The propagator (32) can be written as

$$F_{\mu\nu}^{ab}(k) = \frac{\delta_{ab}[-ig_{\mu\nu} + i(1-\beta)k_\mu k_\nu(k^2 - \beta M^2)^{-1}]}{k^2 - M^2}, \tag{35}$$

which is consistent with (27) as $M \to 0$.

IV. THE MASSIVE YANG-MILLS FIELD

Within the framework of the indefinite-metric field theory, the physical state is defined by

$$\vec{\chi}^{(+)}(x)|\text{phys}\rangle=0\,, \tag{36}$$

where $\vec{\chi}^{(+)}$ is the positive-frequency part of the Heisenberg operator $\vec{\chi}(x)$. Let us consider the physical-state subspace at time $t=\pm\infty$. For definiteness, we only discuss the in field, since the out field can be discussed in the same manner. Suppose

$$Z(x)\to Z^{\text{in}}(x)\equiv \overline{Z}(x)\,,\qquad Z(x)=\{f^a_\mu(x),\chi^a(x),\psi(x)\}\,, \tag{37}$$

as time $t\to-\infty$. The in field $Z^{\text{in}}(x)$ satisfies the free-field equations (20)–(23), with $\beta=1$. The commutation relations for $\overline{f}^a_\mu$ and $\overline{\chi}^a$ are given by (25), (30), and (31), where f^a_μ and χ^b are replaced by $\overline{f}^a_\mu$ and $\overline{\chi}^b$, respectively. In particular, we have

$$[\overline{f}^a_\mu(x),\eta\overline{\chi}^b(y)]=-i\delta_{ab}\partial^x_\mu\Delta(x-y,\beta M^2)\,, \tag{38}$$

$$[\overline{\chi}^a(x),\overline{\chi}^b(y)]=-i\eta^{-2}M^2\delta_{ab}\Delta(x-y,\beta M^2)\,, \tag{39}$$

$$[\overline{\chi}^a(x),\psi^{\text{in}}(y)]=0\quad(\beta=1,\ \eta=M\neq0). \tag{40}$$

The constraint for the physical in state is

$$\overline{\chi}^{(+)}_a(x)|\text{phys}\rangle=0,\quad \chi_a\equiv\chi^a\,. \tag{41}$$

The commutator (39) implies that the χ bosons are unphysical for $M>0$. In this case, we may divide $\overline{f}^a_\mu(x)$ into a spin-one part and a spin-zero part:

$$\overline{f}^a_\mu=[\overline{f}^a_\mu-\eta(M^2)^{-1}\partial_\mu\overline{\chi}^a]+\eta(M^2)^{-1}\partial_\mu\overline{\chi}^a\quad(\eta=M). \tag{42}$$

From (20), (21), (38), and (39), one can easily verify that the spin-one part $\overline{F}^a_\mu\equiv\overline{f}^a_\mu-\eta(M^2)^{-1}\partial_\mu\overline{\chi}^a$ satisfies $\partial^\mu\overline{F}^a_\mu=0$ and

$$[\overline{F}^a_\mu(x),\overline{\chi}^b(y)]=0\,. \tag{43}$$

The expressions (39), (40), and (43) imply that the physical subspace is generated by the Hermitian conjugates of $[\overline{F}^a_\mu(x)]^{(+)}$ and $[\psi^{\text{in}}(x)]^{(+)}$ from the vacuum.

If $\vec{\chi}(x)$ obeys the free-field equation, then (36) can be consistently defined for all time and the physical-state condition is satisfied. There will be no problem of unitarity, but this is of no interest to us, since there is no interaction. However, the equation (12) for the Lagrange multiplier shows that $\vec{\chi}$ couples to the physical components of $\vec{f}_\mu$. Thus, the physical state (36) cannot be consistently defined for all time and the physical-state condition is violated. This means that the S matrix $\overline{S}$, defined in the physical-state subspace, is not "physical," because it contains an extra amplitude due to the interaction of $\vec{\chi}(x)$ in the intermediate states, and is, therefore, not unitary.

So, the problem is: How can we isolate and remove this extra amplitude to all orders (in g) contained in $\overline{S}$ and thus define the physical S matrix S_{ph}? This problem can be solved with the help of the equation (12) for the Lagrange-multiplier field $\vec{\chi}$ and the Feynman path integral. The matrix elements of $\overline{S}$ are given by the amplitude

$$\begin{aligned}A&=\int\exp\Big[i\int d^4x(\mathcal{L}^M+\mathcal{L}_S)\Big]d[\vec{f}_\lambda,\vec{\chi},\overline{\psi},\psi]\\&=\text{const}\int\exp\Big[i\int d^4x(\mathcal{L}^M_{\zeta=0}+\mathcal{L}_S)\Big]d[\vec{f}_\lambda,\overline{\psi},\psi]\,,\end{aligned} \tag{44}$$

$$\begin{aligned}\mathcal{L}_S&=\overline{\psi}\eta(x)+\overline{\eta}(x)\psi+\vec{\eta}_\mu(x)\cdot\vec{f}^\mu\,,\\ d[\vec{f}_\lambda,\vec{\chi},\psi,\ldots]&=d[\vec{f}_\lambda]d[\vec{\chi},\psi,\ldots]\,,\end{aligned} \tag{45}$$

with external *physical* particles. The Lagrangians $\mathcal{L}^M$ and $\mathcal{L}^M_\zeta$ are given by (1) and (6), respectively. The extra amplitude in (44) is completely determined by Eq. (12). We observe that the equation (12) for the negative-metric field $\vec{\chi}$ could be derived from the Lagrangian $\mathcal{L}(\vec{\chi})$:

$$\mathcal{L}(\vec{\chi})=-\tfrac{1}{2}[\partial_\mu\vec{\chi}\cdot\partial^\mu\vec{\chi}-M^2\vec{\chi}^2-g\vec{\chi}\cdot(\vec{f}_\mu\times\partial^\mu\vec{\chi})] \tag{46}$$

if Eq. (8) (i.e., $\partial_\mu\vec{f}_\mu=-\eta\vec{\chi}$) holds. Therefore, in (44) the extra amplitude due to the production of χ in the intermediate state can be expressed by

$$\exp\Big\{i\int d^4\chi\,\mathcal{L}(\vec{\chi})\Big\}d[\vec{\chi}]=[\det(\delta^{ac}+(\Box+M^2)^{-1}g\epsilon^{abc}f^b_\mu\partial^\mu)]^{-1/2}\equiv D_M^{-1/2}\,. \tag{47}$$

Thus, the extra amplitude in (44) is completely isolated in a determinant factor. Effectively, after removing $D_M^{-1/2}$ from (44), we are left with a functional form which gives unitary amplitude. Unfortunately, such a functional form cannot, in contrast with the Abelian gauge theories,[4] be expressed in a simple closed and local form. So, the unitarized amplitude can only be expressed by

$$U=\int D_M^{+1/2}\Big[\exp\ i\int d^4x(\mathcal{L}^M_{\zeta=0}+\mathcal{L}_S)\Big]d[\vec{f}_\lambda,\psi,\overline{\psi}]\,, \tag{48}$$

$$D_M^{+1/2}=\exp\{\tfrac{1}{2}\operatorname{Tr}\ln[1+(\Box+M^2)^{-1}g\vec{f}_\mu\times\partial^\mu]\}\,. \tag{49}$$

The new rules for the Feynman diagrams in the massive Yang-Mills field theory are as follows: The propagator for $\vec{f}_\mu$ and $\vec{\chi}$ are, respectively,

$$\delta_{ab}[-ig_{\mu\nu}/(k^2-M^2+i\epsilon)] \tag{50}$$

and

$$\delta_{ab}[-i/(k^2-M^2+i\epsilon)]\,, \tag{51}$$

which are obtained from (32) and (34) with $\beta=1$ and $\eta=M$. The 3-vertex $V^{abc}_{\mu\nu\lambda}$ [i.e., $f^a_\mu(p)f^b_\nu(q)f^c_\lambda(k)$, where $p_\mu+q_\mu+k_\mu=0$] and the 4-vertex $U^{abcd}_{\mu\nu\lambda\rho}$ [i.e., $f^a_\mu(p)f^b_\nu(q)f^c_\lambda(k)f^d_\rho(k')$, where $p_\mu+q_\mu+k_\mu+k'_\mu=0$] are given by

$$V^{abc}_{\mu\nu\lambda} = g\epsilon_{abc}[g_{\mu\nu}(p-q)_\lambda + g_{\nu\lambda}(q-k)_\mu + g_{\lambda\mu}(k-p)_\nu], \tag{52}$$

$$U^{abcd}_{\mu\nu\lambda\rho} = ig^2[\epsilon_{fab}\epsilon_{fcd}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) + \epsilon_{fac}\epsilon_{fbd}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\rho\lambda}) + \epsilon_{fad}\epsilon_{fcb}(g_{\mu\nu}g_{\rho\lambda} - g_{\mu\lambda}g_{\nu\rho})]. \tag{53}$$

The new vertex implied by the determinant factor in (48) or (49) is

$$W^{abc}_\mu = \epsilon^{abc}k_\mu, \tag{54}$$

which can also be seen from the source term in (12) or the interaction term in (46). Because of the factor $D_M^{1/2}$ in the *numerator* of the expression (48), any "fictitious" loop from this square-root determinant factor carries an additional factor $(-\frac{1}{2})$. This is to be contrasted with the determinant factor in the massless case [cf. expressions (64) and (65) below], where the fictitious loop carries a factor (–1). The indices $a, b, c = 1, 2, 3$, and the lines corresponding to the fields in the diagrams are all directed outward from the vertex. These rules indicate that the theory is renormalizable by standard power counting.

V. THE MASSLESS YANG-MILLS FIELD

For the gauge-invariant Yang-Mills theory, the Lagrangian specified by the gauge parameter β is obtained by taking the limit $M\to 0$ in (1) and replacing $\frac{1}{2}\eta^2\vec{\chi}^2$ by $\frac{1}{2}\beta\eta^2\vec{\chi}^2$:

$$\mathcal{L}^\beta_{\rm YM} = -\tfrac{1}{4}\vec{f}_{\mu\nu}\cdot\vec{f}^{\mu\nu} + \eta\vec{\chi}\cdot\partial_\mu\vec{f}^\mu + \tfrac{1}{2}\beta\eta^2\vec{\chi}^2 + \mathcal{L}_\psi, \quad \mathcal{L}_\psi \equiv -\bar{\psi}\gamma^\mu(-i\partial_\mu - \tfrac{1}{2}g\vec{\tau}\cdot\vec{f}_\mu)\psi - m\bar{\psi}\psi. \tag{55}$$

This Lagrangian leads to the field equations

$$\partial^\nu(\partial_\nu\vec{f}_\mu - \partial_\mu\vec{f}_\nu) + \partial_\mu(\partial_\nu\vec{f}^\nu)/\beta + \vec{J}_\mu = 0, \tag{56}$$

$$\partial_\mu\vec{f}^\mu + \beta\eta\vec{\chi} = 0, \tag{57}$$

where $\vec{J}_\mu$ is given by (10), and (56) with $\beta=1$ is the zero-mass limit of (7). The divergence of (56) together with (57) gives

$$\Box\vec{\chi} = \eta^{-1}\partial^\mu\vec{J}_\mu = -g\vec{f}_\mu\times\partial^\mu\vec{\chi}, \tag{58}$$

which shows that $\vec{\chi}(x)$ is a massless field for any value of gauge parameter β. As usual, we define the physical states $|\text{phys}\rangle$ by

$$\chi^{(+)}_a|\text{phys}\rangle = 0 \quad (a=1, 2, 3). \tag{59}$$

It follows that the expectation value of $\vec{\chi}(x)$ in any physical state vanishes. Thus, the *expectation values* of (56) and (57) in the physical states reproduce the classical equation and the conservation of current

$$\langle\partial^\nu\vec{f}_{\mu\nu}\rangle + \langle\vec{J}_\mu\rangle = 0, \qquad \langle\partial^\mu\vec{J}_\mu\rangle = 0.$$

Furthermore, the dynamical characteristics of the physical system (i.e., the energy-momentum tensor $T^{\mu\nu}$ and the angular momentum tensor $M^{\mu\nu}$) derived from $\mathcal{L}^\beta_{\rm YM}$ in (55) and those from $\mathcal{L}^\beta_{\rm YM}|_{\eta=0}$ are different. Nevertheless, these differences involve $\vec{\chi}(x)$, and their expectation values in the physical states vanish.

The supplementary condition (59) does not imply there are no ghost quanta. Since $\vec{\chi}(x)$ commutes with $\vec{\chi}(y)$ as seen from (31) for $M=0$, it is not possible to identify $\vec{\chi}^{(\pm)}$ as the destruction and creation operators for quanta with a definite metric. Rather, they correspond to equal-weight linear combinations of operators for longitudinal and timelike quanta. The supplementary condition (59) permits the excitation of the linearly independent combination (with opposite relative phase) of the operators for these two modes. Change of gauge corresponds to excitation (or extinction) of this combination of modes; see the discussion of (61) below. All these *excited* states, of course, have zero norm and zero scalar product with all the other states. The number of remaining degrees of freedom corresponds to the *two* transverse modes; this is to be contrasted with the massive case with *three* modes.[5] Correspondingly, we have the contrast between (46) and (63).

Because of different physical degrees of freedom for $\vec{f}_\mu(x)$ with $M=0$ and for $\vec{f}_\mu(x)$ with $M>0$, one must modify the steps (44)–(47) to isolate the extra absorptive amplitude due to the interactions of the bosons corresponding to the unphysical components in $\vec{f}_\mu(x)$. In fact, in the one-loop level, one can show that the extra amplitude is twice that of the amplitude given by $D_M^{-1/2}$ with $M=0$ (Refs. 6–8) [cf. Eq. (49)]. This double contribution does not correspond to an additional factor 2 in (48) because any constant factor in (48) does not affect the physics [or the rules for the Feynman diagrams that follow from (48)]. Since there are two unphysical components, say $\vec{\chi}(x)$ and $\vec{\chi}'(x)$, in the massless $\vec{f}_\mu(x)$, the amplitude (44) with $\mathcal{L}^M$ replaced by $\mathcal{L}^\beta_{\rm YM}$ can, in principle, be written as

$$A|_{M=0} = \int \exp\left\{i\int d^4x[\mathcal{L}'^\beta_{\rm YM}(\psi,\bar{\psi},\vec{f}^T_\mu) + \mathcal{L}(\vec{\chi},\vec{\chi}') + \mathcal{L}_S]\right\} \times d[\vec{f}^T_\lambda, \vec{\chi}, \vec{\chi}', \psi, \bar{\psi}], \tag{60}$$

where $\mathcal{L}(\vec{\chi},\vec{\chi}')$ is the Lagrangian for the two massless unphysical components of $\vec{f}_\mu$ and $\mathcal{L}'^\beta_{\rm YM}(\psi,\bar{\psi},\vec{f}^T_\mu)$ contains only the two physical (transverse) components of $\vec{f}_\mu$, i.e., $\vec{f}^T_\mu$. We know that $\vec{\chi}$ satisfies Eq. (58). Now we must find the equation for $\vec{\chi}'$. In analogy with quantum electrodynamics,[9] $\vec{\chi}'$ should be identified with the gauge excitation, which appears in the gauge transformation of $\vec{f}_\mu$:

$$\vec{f}_\mu \to \vec{f}_\mu + g\vec{\chi}'\times\vec{f}_\mu - \partial_\mu\vec{\chi}'. \tag{61}$$

The gauge excitation $\vec{\chi}'$ must obey the equation such that the pure transversal $\vec{f}_\mu$, i.e., $\partial^\mu \vec{f}_\mu = 0$, remains pure transversal after the gauge transformation (61). Therefore, $\vec{\chi}'$ must satisfy

$$\Box\vec{\chi}' + g\partial^\mu(\vec{f}_\mu \times \vec{\chi}') = 0$$

or (62)

$$\Box\vec{\chi}' + g\vec{f}_\mu \times \partial^\mu\vec{\chi}' = 0\,, \quad \text{if } \partial_\mu \vec{f}^\mu = 0\,.$$

Note that we could set $\partial_\mu \vec{f}^\mu = 0$ [i.e., $\beta = 0$ in (55)] because of gauge invariance. We observe that Eqs. (58) and (62), which completely determine the coupling of the unphysical components in $\vec{f}_\mu(x)$, could be derived from the Lagrangian

$$\mathcal{L}_A(\vec{\chi}, \vec{\chi}') = -\partial_\mu\vec{\chi}'\cdot\partial^\mu\vec{\chi} + g\vec{\chi}'\cdot(\vec{f}_\mu \times \partial^\mu\vec{\chi}) \tag{63}$$

or, if $\partial_\mu \vec{f}^\mu = 0$,

$$\mathcal{L}_B(\vec{\chi}_1, \vec{\chi}_2) = -\tfrac{1}{2}[\partial_\mu\vec{\chi}_1\cdot\partial^\mu\vec{\chi}_1 - g\vec{\chi}_1\cdot(\vec{f}_\mu\times\partial^\mu\vec{\chi}_1)] + \tfrac{1}{2}[\partial_\mu\vec{\chi}_2\cdot\partial^\mu\vec{\chi}_2 - g\vec{\chi}_2\cdot(\vec{f}_\mu\times\partial^\mu\vec{\chi}_2)]\,,$$

where

$$\vec{\chi} = (\vec{\chi}_1 - \vec{\chi}_2)/\sqrt{2}\,, \quad \vec{\chi}' = (\vec{\chi}_1 + \vec{\chi}_2)/\sqrt{2}\,.$$

From (60) and $\mathcal{L}(\vec{\chi}, \vec{\chi}') = \mathcal{L}_A(\vec{\chi}, \vec{\chi}')$ [or $\mathcal{L}_B(\vec{\chi}_1, \vec{\chi}_2)$], we find that

$$A|_{M=0} = \int D_{M=0}{}^{-1} \exp\left[i\int d^4x(\mathcal{L}'^{\beta}_{\rm YM} + \mathcal{L}_S)\right] d[\vec{f}^{\,T}_\lambda, \psi, \bar{\psi}]\,.$$

Again, after the extra amplitude is isolated, the remaining functional form, which gives the unitary amplitude, cannot be expressed in a simple closed and local form. The unitarized amplitude is

$$A_U|_{M=0} = \int D_{M=0} \exp\left[i\int d^4x(\mathcal{L}^{\beta}_{\rm YM} + \mathcal{L}_S)\right] d[\vec{f}_\lambda, \vec{\chi}, \bar{\psi}, \psi]\,,$$

$$= \int D_{M=0}\exp\left\{i\int d^4x[-\tfrac{1}{4}\vec{f}_{\mu\nu}\cdot\vec{f}^{\mu\nu} - (1/2\beta)(\partial_\mu\vec{f}^\mu)^2 + \mathcal{L}_\psi + \mathcal{L}_S]\right\} d[\vec{f}_\lambda, \bar{\psi}, \psi]\,, \tag{64}$$

$$D_{M=0} = \exp[\operatorname{Tr}\ln(1 + \Box^{-1}g\vec{f}_\mu \times \partial^\mu)]\,, \tag{65}$$

which agrees with the result obtained by other methods.[10] The dynamical reason for the determinant factor $D_{M=0}$ in (64) and (65) is transparent in our considerations.

Another simple method of isolating the extra amplitude is suggested by quantum electrodynamics (QED) with nonlinear gauge condition.[11] Since the method works perfectly in QED with nonlinear gauge, it should work here if such a method is of any value. First, we write (58) as

$$\Box(1 + \Box^{-1}g\vec{f}_\mu \times \partial^\mu)\vec{\chi} \equiv \Box\vec{\xi}' = 0\,, \tag{66}$$

then express $\mathcal{L}^{\beta}_{\rm YM}$ given by (55) in terms of $\vec{\xi}'$:

$$\mathcal{L}^{\beta}_{\rm YM} = \mathcal{L}''^{\beta}_{\rm YM} + \eta\vec{\xi}'\cdot\partial_\mu\vec{f}^\mu + \tfrac{1}{2}\beta\eta^2\vec{\xi}'^2\,, \tag{67}$$

where $\mathcal{L}''^{\beta}_{\rm YM}$ does not contain $\vec{\xi}'$ field and its structure must be such that (67) reproduced the "free" field equation $\Box\vec{\xi}' = 0$. This ensures that $\mathcal{L}''^{\beta}_{\rm YM}$ will give unitarity amplitudes. In the case of QED with nonlinear gauge condition, the Lagrangian $\mathcal{L}''^{\beta}_{\rm QED}$ can be expressed in a simple local form and does give unitarity amplitudes. Yet, here $\mathcal{L}''^{\beta}_{\rm YM}$ cannot be written in a local form. This really does not matter because we are only interested in extracting the extra unwanted amplitude from $\mathcal{L}^{\beta}_{\rm YM}$. The amplitude is

$$A|_{M=0} = \int \exp\left[i\int d^4x(\mathcal{L}^{\beta}_{\rm YM} + \mathcal{L}_S)\right] d[\vec{f}_\mu, \psi, \bar{\psi}]d[\vec{\chi}]$$

$$= \int \exp\left[i\int d^4x(\mathcal{L}''^{\beta}_{\rm YM} + \eta\vec{\xi}'\cdot\partial_\mu\vec{f}^\mu + \tfrac{1}{2}\beta\eta^2\vec{\xi}'^2 + \mathcal{L}_S)\right] \times d[\vec{f}_\mu, \psi, \bar{\psi}]d[\vec{\xi}'][\det(1 + \Box^{-1}g\vec{f}_\mu \times \partial^\mu)]^{-1}$$

$$= \int D_{M=0}{}^{-1}\exp\left\{i\int d^4x[\mathcal{L}''^{\beta}_{\rm YM} - (1/2\beta)(\partial_\mu\vec{f}^\mu)^2 + \mathcal{L}_S]\right\} \times d[\vec{f}_\mu, \psi, \bar{\psi}]\,, \tag{68}$$

where the extra amplitude is isolated in the factor $D_{M=0}{}^{-1}$. Therefore, the unitarized amplitude is the same as that of (64).

VI. RENORMALIZABILITY

We know the massless Yang-Mills field is renormalizable. And it has been shown that if the massive Yang-Mills field has a limit $M \to 0$ (to within logarithmic infrared singularities), it may be renormalizable.[12] From the above discussions, the formalism for massive Yang-Mills field has a smooth limit $M \to 0$. In this limit, we obtain massless Yang-Mills theory in the Feynman gauge. Furthermore, the present theory for massive $\vec{f}_\mu$ is renormalizable by standard power counting.

However, at the present time there are different opinions about the renormalizability of the massive Yang-Mills field (without spontaneous breaking of gauge symmetry). Some held the opinion that it is not renormalizable[13] and others held the opinion that the question requires further investigation.[7] This is not so surprising because these different opinions are based on different modifications of the original pure massive Yang-Mills field given by

$$\mathcal{L}_{\rm MYM} = -\tfrac{1}{4}\vec{f}_{\mu\nu}\cdot\vec{f}^{\mu\nu} + \tfrac{1}{2}M^2\vec{f}_\mu\cdot\vec{f}^\mu + \mathcal{L}_\psi\,. \tag{69}$$

Roughly speaking, what one does is as follows. One modifies (69) by adding an unphysical scalar particle, so that the asymptotic behavior of the propagator for massive $\vec{f}_\mu$ is proportional to k^{-2}. The coupling in (69) is renormalizable. Thus, if the new coupling due to the scalar particle is renormalizable (or unrenormalizable), then the theory is renormalizable (or unrenormalizable). For example, if the source term in the equation for $\vec{\chi}$ (the added scalar field) is $\partial_\mu \vec{J}^\mu = \partial_\mu(\frac{1}{2}g\bar{\psi}\gamma^\mu\vec{\tau}\psi + g\vec{f}^\nu \times \vec{f}_{\mu\nu})$ and *if* it *cannot* be further simplified, then the $\vec{\chi}$ coupling will have a term $\bar{\psi}\gamma^\mu\vec{\tau}\psi\partial_\mu\vec{\chi}$, which is known to be unrenormalizable. In our formalism, the complicated source $\eta^{-1}\partial_\mu\vec{J}^\mu$ can be reduced to $-g\vec{f}_\mu \times \partial^\mu\vec{\chi}$ by using field equations.

There are many ways of modifying (69) to include a scalar field.[6,7,13,14] Since the scalar particle is unphysical, different authors assume *different* masses for it. Also, the relations between the scalar field and $\vec{f}_\mu$ are different in different modifications of (69). One way of modifying (69) is to apply a finite gauge transformation to $\vec{f}_\mu$ in (69):

$$\vec{\tau}\cdot\vec{f}'_\mu = \Omega\vec{f}_\mu\cdot\vec{\tau}\Omega^{-1} + \frac{1}{g}(\partial_\mu\Omega)\Omega^{-1}, \quad \Omega = e^{igM^{-1}\vec{\chi}_A\cdot\vec{\tau}}. \tag{70}$$

One obtains a very complicated Lagrangian with a very complicated relation between $\vec{\chi}_A$ and $\vec{f}_\mu$. Only when $g=0$ does one have the relation $\vec{\chi}_A \propto \partial_\mu \vec{f}^\mu$. One finds that $\vec{\chi}_A$ has unrenormalizable coupling and concludes that the massive Yang-Mills field is unrenormalizable because there are no further Ward identities to be enforced.[13] This, of course, does not rule out the possibility that the Lagrangian for the scalar field could be modified so as to produce renormalizability. In our formalism, the Lagrangian for the scalar field $\vec{\chi}$ is simple, and we have the simple relation $\partial_\mu\vec{f}^\mu \propto \vec{\chi}$ even if $g\neq 0$. The field $\vec{\chi}$ does interact, through its current, with the transverse components of $\vec{f}_\mu$ as shown in its equation of motion. Since the propagator of this current ($\propto \vec{\chi}\times\partial_\mu\vec{\chi}$) is, at large momenta, no more singular than that of a free scalar boson current, the theory is renormalizable.

We may remark that one should be cautious in making a finite gauge transformation to a non-gauge-invariant Lagrangian such as (69), because one may end up with a different result. For example, if one expresses

$$B = \int \exp\left[-i\int d^4x(\mathcal{L}_{\rm YM} + \tfrac{1}{2}M^2\vec{f}_\mu\cdot\vec{f}^\mu - M\vec{f}_\mu\cdot\partial^\mu\vec{p} + \tfrac{1}{2}M^2\vec{p}^2)\right] \times d[\vec{f}_\lambda, \vec{p}] \tag{71}$$

as the action of a vector and a scalar particle (both acting nonlinearly and in interaction) by making a finite gauge transformation of $\vec{f}_\mu$ to $\vec{f}'_\mu$,

$$\Omega = e^{+igM^{-1}\vec{\tau}\cdot\vec{p}}. \tag{72}$$

After a number of integrations by part, one obtains[14]

$$B = \exp\left\{-i\int d^4x[\mathcal{L}_{\rm YM} + \tfrac{1}{2}M^2\vec{f}_\mu\cdot\vec{f}^\mu + \tfrac{1}{2}M^2\vec{p}^2 - (\partial_\mu\vec{p} + g\vec{f}_\mu\times\vec{p})\cdot(\tfrac{1}{2}\partial^\mu\vec{p} - (g/3M)\vec{p}\times\partial^\mu\vec{p} + \cdots)]\right\} d[\vec{f}_\lambda, \vec{p}]. \tag{73}$$

However, (71) leads to the following equation for $\vec{p}$:

$$(\Box + M^2)\vec{p} = -g\vec{f}_\mu\times\partial^\mu\vec{p}, \tag{74}$$

while (73) leads to

$$(\Box + M^2)\vec{p} = -g\vec{f}_\mu\times\partial^\mu\vec{p} + \frac{2g}{3M}\vec{p}\times\Box\vec{p} + \frac{g^2}{3M}(\partial^\mu\vec{p}\cdot\vec{p})\vec{f}_\mu + \frac{g^2}{3M}(\vec{p}\cdot\vec{f}_\mu)\partial^\mu\vec{p} - \frac{2g^2}{3M}(\partial_\mu\vec{p}\cdot f^\mu)\vec{p} + \cdots, \tag{75}$$

which is different from (74). Therefore, it is no longer clear that (71) and (73) are the same thing.

VII. DISCUSSIONS AND CONCLUSIONS

The formalism described in this paper tries to remain as close to conventional quantum field theory as possible. The primitive Lagrangian (44) contains the two fields $\vec{f}_\lambda$ and $\vec{\chi}$. After the $\vec{\chi}$ field has been integrated out, we get a theory involving the four-vector isovector field $\vec{f}_\lambda$ and an additional unitarization factor $D_M^{1/2}$. In the massless limit, the unitarization factor is $D_{M=0}$. This factor may be viewed as being generated by a scalar isovector fictitious field with Fermi statistics. (The quantization of a massless scalar isovector field with Fermi statistics has been considered by Rudolph and Dürr.[10]) Thus, we may view the unitarized amplitude (48) as arising from a system with five degrees of freedom, including three normal physical degrees of freedom for the massive vector meson, one unphysical degree of freedom for the vector meson, and a fictitious degree of freedom. The last two cancel each other in computing physical (on-mass-shell) scattering amplitudes. Thus, in unitarity sums only the three physical degrees of freedom contribute; this is guaranteed for the unitarized amplitude (48) but not for the primitive amplitude (44).

The remaining degrees of freedom are all positive-norm states with the usual physical interpretation. In this theory, although initial and final states can be described in terms of physical components of $\vec{f}_\lambda$, we cannot dispense with the "unphysical" degrees of freedom altogether. This is somewhat similar to the situation in quantum electrodynamics, where we cannot dispense with longitudinal and timelike photons altogether. On the other hand, the standard massive Yang-Mills theory based on the Lagrangian (69) involves only three physical components of $\vec{f}_\lambda$. However, such a theory does not have a zero-mass limit[8] and does not seem to be renormalizable. The present theory does have a zero-mass limit and is renormalizable. In this sense, the present theory is a different and more satisfactory theory. Furthermore, the present formalism for the massive Yang-Mills fields might be a useful way of regulating the infrared behavior of the massless Yang-Mills fields because the Yang-Mills quanta can be given a mass according to the present approach without destroying renormalizability.

The unitarization of the amplitude in (48) is achieved without destroying the analyticity of the amplitude. This comes about because the added term

$$\tfrac{1}{2}\,\mathrm{Tr}\ln[1+(\Box+M^2)^{-1}g\vec{f}_\lambda\times\partial^\lambda]$$

in the transition amplitude coincides, apart from signs, term by term with the amplitude for a normal scalar field with local coupling. The corresponding Feynman amplitudes exhibit the usual analyticity properties of perturbation-theoretic amplitudes. Consequently, no loss of analyticity results from the unitarization. We have verified this by direct calculation in the one-loop approximation.

In our formalism of the Yang-Mills field $\vec{f}_\mu$ with mass $M>0$, the zero-mass limit of $\vec{f}_\mu$ exists. Aside from the problem of infrared divergence in the massless theory, we can discuss both $M>0$ and $M=0$ in a unified way. Because of the coupling between the physical and the unphysical components of $\vec{f}_\mu$, the field-theoretic definition of the S matrix, i.e., $\overline{S}$, in the physical state subspace is not unitary. From the field equation for $\vec{\chi}$, we know the interaction of $\vec{\chi}$ (although this interaction cannot be seen directly from the given Lagrangian). So one can unitarize $\overline{S}$ order by order. However, the combination of the equation of motion for $\vec{\chi}$ and the Feynman path integral is very powerful because it enables us to isolate the extra amplitude in $\overline{S}$ for all orders by simple reasoning. The matrix $\overline{S}$ can be unitarized to all orders in a simple way for both $M>0$ and $M=0$ cases. A minus sign for any fictitious loop in the massive theory and the difference of a factor 2 between the "fictitious loop" for massive $\vec{f}_\mu$ and that for massless $\vec{f}_\mu$ appear puzzling in other approaches of quantizing the Yang-Mills field.[8,14,6] Now, it becomes clear for all orders of diagrams in our approach. We note that the rules for the Feynman diagrams in the massive case (cf. Sec. IV) are different from those given by other authors.[15]

On the other hand, the observable physical state can be connected by the total S matrix to a state containing ghosts. Here we encounter the difficulty of probability interpretation. Can we just look at S_{ph} and simply ignore the production of ghost in the final state? Can a field with such properties be realized in nature? The answer is yes in the following sense: If one considers QED with a nonlinear gauge, e.g., $\partial_\mu A^\mu+\beta' A_\mu A^\mu=0$, $\beta'\neq 0$, the longitudinal photon can be created and the straightforward field-theoretic definition yields a "physical" S matrix which is not unitary. However, if one just uses the unitarized physical S matrix and simply ignores the production of ghost, one can describe nature accurately. This is because the physical S matrix after unitarization (in the same way discussed above) is exactly the same as that of the usual QED with linear gauge.[4] Therefore, we believe that a theory with such a unitarization procedure is all right. We note that analyticity of the S matrix is restored after such unitarization. In the case of the Yang-Mills field with $M\geq 0$, we have verified the analyticity of the unitarized $\overline{S}$ to the one-loop level.

Note added in proof. In calculation of higher-order processes one must take into account the constraint (8), which is closely related to the effective Lagrangian (46). The application of the present method to gauge theories leads to the results which are exactly the same as those obtained by the usual method; therefore, the possible effect of the corresponding constraints should also be examined by explicit calculations of higher-order diagrams.

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Functional evaluation of the effective potential*

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By use of the path-integral formulation of quantum mechanics, a series expansion for the effective potential is derived. Each order of the series corresponds to an infinite set of conventional Feynman diagrams, with a fixed number of loops. As an application of the formalism, three calculations are performed. For a set of n self-interacting scalar fields, the effective potential is computed to the two-loop approximation. Also, all loops are summed in the leading-logarithmic approximation when n gets large. Finally, the effective potential for scalar, massless electrodynamics is derived in an arbitrary gauge. It is found that the potential is gauge-dependent, and a specific gauge is exhibited in which all one-loop effects disappear.

I. INTRODUCTION

The effective potential for a field theory (that is the generating functional for zero-momentum single-particle irreducible Green's functions[1]), introduced by Euler, Heisenberg, and Schwinger, is useful in studies of spontaneous symmetry breaking, as was first pointed out by Jona-Lasinio,[2] and more recently by several authors.[3,4] Calculation of this object has proceeded by summing infinite series of Feynman graphs at zero momentum.[3,4] Obviously this is an onerous task, especially when several interactions are present which complicate the combinatorial factors that multiply each graph. Moreover, the calculation has been only performed in the one-loop approximation, since higher-loop contributions appear extremely difficult to evaluate.

However, it is important to be able to study the higher-order multiloop graphs, if not explicitly, at least in general terms. Two circumstances can be envisioned where multiloop graphs are needed. The one-loop approximation is very simple; indeed it will be seen that it is not typical of the higher-order terms. Thus it may be that relevant effects do not set in until the two-loop level. More importantly, bound states which, as has been recently suggested, can provide a mechanism for spontaneous mass generation[5] can never be observed in a finite order of the loop expansion. Necessarily they require at least an infinite subset of all orders.

In this paper, I shall use the Feynman path-integral method to obtain a simple formula for the effective potential. The formula has the advantage of summing all the relevant Feynman graphs to a given order of the loop expansion. Furthermore, in a natural way it generates all orders of the loop expansion, representing each order by a *finite* number of graphs. Before stating the result, some notation must be introduced.

Consider a theory described by a Lagrangian $\mathcal{L}$ depending on a set of fields $\phi_a(x)$ and construct the classical action,