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Relativistic fermion gas interacting through a scalar field. I. Hartree approximation

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A high-density fermion gas interacting through a scalar field is studied in the relativistic Hartree approximation, as a model for superhigh-density astrophysical systems. The Hartree quasiparticle states are the superposition of positive- and negative-energy states and can be constructed by a Bogoliubov-type canonical transformation. The resulting effective mass is strongly field-dependent, and approaches 0 as the density increases into the ultrarelativistic domain. The ensuing equation of state exhibits a phase transition and a bound state in the intermediate-density range, but becomes perfect-gas-like at high densities. There is no collapse, and stronger coupling enhances the perfect-gas-like behavior. An alternate picture is provided by considering the fermion gas in the background of zero-momentum bosons. Numerical comparisons with existing scalar-meson coupling data are provided, with emphasis on regions of neutron-star densities.

I. INTRODUCTION

Relativistic many-particle systems have gained an increasing importance in models for exotic astrophysical situations of extreme density. The understanding of relativistic many-particle systems, however, both from the methodological point of view and insofar as the novel physical situations are concerned, is rather meager. The present paper is devoted to the description of, conceptually, probably the simplest system, where the interaction is mediated exclusively by a massive *scalar* meson. None of the known physical systems correspond to this simplified model (we will return to this point later) and therefore our motivation is not that we expect the conclusions which emerge from this study to be immediately applicable to concrete situations, although it is conceivable, indeed likely, that some features of the model are of more general character than their derivation would suggest. The primary concern of this paper, is, however, to provide a relativistically consistent description and to explore the consequences of peculiar relativistic effects related to the transformation properties of the field.

In the theory of nuclear matter the contribution by scalar mesons to the two-nucleon interaction describable in terms of a one-boson exchange potential (OBEP) has been given considerable attention. The calculations and results pertaining to the scattering matrix elements and the effective interaction potential (expanded to lowest order in p^2/m^2c^2) are discussed and summarized by Green and Sawada[1] and Moszkowski.[2] A detailed analysis has recently been given by Gross.[3] The necessity of including a scalar-meson contribution in the OBEP is now fairly well established. Contributions both from phenomenological OBE description of correlated two-pion exchange processes[3,4] and particle resonances[4,5] have been considered. Typical values for the meson masses (μ, in MeV), and coupling strengths ($\gamma = g^2/4\pi$) are: $\mu_\delta = 963$, $\gamma_\delta = 1.14$ (Ref. 5), 9.63 (Ref. 4); $\mu_\epsilon = 572$ (Ref. 5), 702 (Ref. 6); $\gamma_\epsilon = 9.92$ (Ref. 5), 6.8 (Ref. 6); $\mu_\alpha = 363$, $\gamma_\alpha = 2.41$ (Ref. 3); $\mu_\sigma = 570$, $\gamma_\sigma = 4.08$ (Ref. 4). Since no direct comparison between the results of this paper and the behavior of neutron matter can be made, the exact numerical values are not significant, but they give an indication of the orders of magnitude to be of interest. Some further comments on this connection will be made at the conclusion of the paper.

In a different context, the scalar interaction has recently gained popularity as the principal candidate for quark-quark interaction in the quark model.[7]

Further studies, motivated by the unusual properties of the scalar interaction and its relative simplicity, explored two- and many-particle systems interacting through a scalar field. The one-particle and two-particle bound-state problems have been studied both classically[8,9] and quantum mechanically[10] while the classical scattering problem has also been given some attention.[11,12] The classical many-particle system (a "scalar plasma") has been considered by Kalman[13,14] (Ref. 13 will be referred to as I hereafter) and Hakim.[15] The unfamiliar features of all these systems stem from the field dependence of the effective mass: This fact is responsible for the appearance of apparent repulsion in the one-particle case and for the saturation of the interaction for the many-particle system.

In the present paper we investigate a zero-temperature Fermi gas interacting through an (attrac-

tive) scalar field. The thermodynamic properties of the system are calculated in the Hartree approximation; a subsequent paper deals with the role of the exchange[16,17] and with the Hartree-Fock approximation. The three parameters that determine the behavior of the system are the density (or Fermi momentum), the coupling strength $\gamma = g^2/4\pi$, and the meson mass μ; in the Hartree approximation, however, the latter two do not enter independently. The characteristics of the system are not unlike to those of the classical gas: At high densities, a perfect gas behavior obtains; at intermediate densities if the coupling strength is high enough a phase transition occurs. A new feature is the nonmonotonic behavior of the energy and the resulting appearance of bound states. Contrary, however, to what has tacitly or openly been assumed in studies relating to similar systems, there is no *collapse:* At sufficiently high densities the bound state disappears and the system reverts to a perfect-gas-like behavior.

In contrast to its nonrelativistic counterpart, the Hartree approximation constitutes a nontrivial many-body problem. The Hartree quasiparticle operators are obtained by a Bogoliubov-type canonical transformation, which mixes particle and antiparticle states. This formalism and questions of consistency are discussed in Sec. II. The actual calculation of the equilibrium properties is carried out in Sec. III.

There is no real "boson background" in the model; the existence or nonexistence of such background is, however, a distinction more of semantic than physical nature. Indeed, as is suggested by the static-source model, a state which contains a coherent mixture of zero-momentum N-boson states is a better approximation of the ground state than the mere Fermi distribution; but the results for physical observables are not unlike in the two cases, only the ways these results are obtained differ. This point will be somewhat further elaborated in Sec. IV.

II. QUASIPARTICLES

The behavior of a Fermi gas coupled to a scalar field is determined by the equations of motion of the field operators ψ and Φ:

$$\begin{aligned} &\Lambda\psi(x) = g\Phi(x)\psi(x)\,, \\ &\Delta\Phi(x) = 4\pi g\,\overline{\psi}(x)\psi(x)\,, \end{aligned} \tag{1}$$

where Λ and Δ are the free Dirac and Klein-Gordon operators

$$\begin{aligned} &\Lambda = -\gamma_\mu \frac{\partial}{\partial x_\mu} - m\,, \\ &\Delta = \frac{\partial^2}{\partial x_\mu \partial x_\mu} - \mu^2\,; \end{aligned} \tag{2}$$

and by the standard commutation relations.

(1) and (2) are derivable from the Hamiltonian

$$\mathcal{H} = \int dx \left\{ \overline{\psi}(x)\Lambda\psi(x) - g\overline{\psi}(x)\Phi(x)\psi(x) + \frac{1}{8\pi}\left[\mu^2\Phi^2(x) + \partial_i\Phi(x)\,\partial_i\Phi(x) + \Pi^2(x)\right] \right\}. \tag{3}$$

The unambiguous approach to the study of the many-body system described by (3) is through the standard many-body Green's-function technique. The generalization of this technique to a coupled relativistic fermion-boson system is fairly straightforward and is discussed in the Appendix. The simplicity of the Hartree and Hartree-Fock approximations makes, however, the use of the Green's-function approach a somewhat useless luxury. Therefore we prefer to handle the problem by introducing effective one-particle and two-particle Hamiltonians from which the boson operator has been eliminated. However, in the Appendix we demonstrate that the Green's-function approach leads to identical results.

From (1) and (2) the two-particle Hamiltonian can be inferred to be

$$\mathcal{H}^{(2)} = \sum_{\vec{p}} \psi^\dagger_{\vec{p}}\,(\vec{\alpha}\cdot\vec{p} + \beta m)\,\psi_{\vec{p}} - \tfrac{1}{2}\sum_{\vec{p},\vec{q},\vec{k}} \phi(\vec{k};\vec{p},\vec{q})\,\psi^\dagger_{\vec{p}+\vec{k}}\,\beta\psi_{\vec{p}}\,\psi^\dagger_{\vec{q}-\vec{k}}\,\beta\psi_{\vec{q}}\,. \tag{4}$$

$\phi(\vec{k};\vec{p},\vec{q})$ is the appropriately chosen two-particle interaction, which in the Hartree approximation can be taken as the static potential

$$\phi(\vec{k};\vec{p},\vec{q}) \equiv \phi(k) = \frac{4\pi g^2}{k^2+\mu^2}\,. \tag{5}$$

The one-particle Hartree Hamiltonian now will be

$$\mathcal{H} = \sum_{\vec{p}} \psi^\dagger_{\vec{p}}\,(\vec{\alpha}\cdot\vec{p} + \beta m)\,\psi_{\vec{p}} - \sum_{\vec{p},\vec{q},\vec{k}} \phi(k)\langle\psi^\dagger_{\vec{q}-\vec{k}}\,\beta\psi_{\vec{q}}\rangle\,\psi^\dagger_{\vec{p}+\vec{k}}\,\beta\psi_{\vec{p}}\,. \tag{6}$$

This Hamiltonian will be diagonalized by the same states $|\vec{p}r\rangle$ (r stands for both the helicity and the signature of energy) that diagonalize the free-particle Hamiltonian, provided in the state vector the substitution $m \to M$ is made and M is determined self-consistently. To see this we note that

$$\langle[\vec{p}]s + |\beta|[\vec{p}]s'+\rangle = \frac{M}{\epsilon_{\vec{p}}}\,, \tag{7}$$

$$\epsilon_{\vec{p}} = (M^2 + p^2)^{1/2}\,. \tag{8}$$

Thus (6) can be written as

$$\mathcal{H}=\sum_{\vec{p}}\psi^{\dagger}_{\vec{p}}\left[\vec{\alpha}\cdot\vec{p}+\beta\left(m-\phi\,\frac{1}{V}\sum_{\vec{p}}n_{\vec{p}}\,\frac{M}{\epsilon_{\vec{p}}}\right)\right]\psi_{\vec{p}}\,,$$

$$\phi\equiv\phi(0) \quad (9)$$

which provides the definition of M by

$$M=m-\phi\,\frac{1}{V}\sum_{\vec{p}}n_{\vec{p}}\,\frac{M}{\epsilon_{\vec{p}}}\,. \quad (10)$$

The one-particle energy is given by (8): Since M is momentum-independent, a Fermi momentum p_0 and a Fermi energy ϵ_0 can be defined in the usual way. Also, the identification between the chemical potential μ and ϵ_0 can be made,

$$\mu=[M^2(p_0)+p_0^2]^{1/2}, \quad (11)$$

although the implicit density dependence of M makes it less transparent that μ defined by (11) satisfies the required thermodynamic identities. This is, however, proven below.

We observe that M can be regarded as the effective (field-dependent) mass, replacing m in the single-particle energy. Also the average Hartree potential $\langle\Phi\rangle$ can be identified as

$$\langle\Phi\rangle=\psi\,\frac{1}{V}\sum_{\vec{p}}n_{\vec{p}}\,\frac{M}{\epsilon_{\vec{p}}}$$

$$=\phi\langle\nu\rangle, \quad (12)$$

$$\nu=\sum_{\vec{p}}\psi^{\dagger}_{\vec{p}}\beta\psi_{\vec{p}}\,, \quad (13)$$

the latter being the operator of proper density. The interpretation of $\langle\Phi\rangle$ being the average potential should also be clear from regarding (12) as the appropriate expectation value of (1). The salient feature of the scalar interaction is reflected through Eqs. (10), (12), and (13). M can be written as

$$M=m-g\langle\Phi\rangle\,. \quad (14)$$

Furthermore, the factor $M/\epsilon_{\vec{p}}$ renders the interaction momentum-dependent. Both these features are essentially identical to the characteristics of the corresponding classical system (cf. I and Ref. 15).

We also see that the Hartree potential plays a crucial role: Once it is known, the determination of the equilibrium properties is a routine matter. In contrast to the customary nonrelativistic Hartree approximation, however, the finding of the equilibrium potential is not trivial: This is a consequence of the implicit momentum dependence of the interaction which, ultimately, leads to a nonlinear self-consistency condition.

Although the states that diagonalize the one-particle Hamiltonian are the simple generalizations of free particle states, they are, in effect, fairly complex coherent superpositions of positive and negative energy or particle and antiparticle states. Equivalently, a canonical transformation can be introduced to define creation and annihilative operators, $c^{\dagger}$, $d^{\dagger}$ and c, d for the new quasiparticles:

$$c_{\vec{p}}=u_{\vec{p}}a_{\vec{p}}+v_{\vec{p}}b^{\dagger}_{-\vec{p}}\,,$$
$$d_{\vec{p}}=-v_{\vec{p}}a^{\dagger}_{\vec{p}}+u_{\vec{p}}b_{-\vec{p}}\,, \quad (15)$$

where $a^{\dagger}_{\vec{p}}$ and $b^{\dagger}_{\vec{p}}$ are the usual creation operators for particles and antiparticles. Spin indices, whose role is trivial, have been suppressed in (15). The one-particle Hamiltonian (6) written in terms of a and b has the form

$$\mathcal{H}=\sum_{\vec{p},\,\mathrm{spin}}\left[\epsilon_{\vec{p}}(m)\left(a^{\dagger}_{\vec{p}}a_{\vec{p}}+b^{\dagger}_{\vec{p}}b_{\vec{p}}\right)\right]$$
$$-g\langle\Phi\rangle\left[\frac{m}{\epsilon_{\vec{p}}(m)}\left(a^{\dagger}_{\vec{p}}a_{\vec{p}}+b^{\dagger}_{\vec{p}}b_{\vec{p}}\right)\right.$$
$$\left.+\frac{p}{\epsilon_{\vec{p}}(m)}\left(a^{\dagger}_{\vec{p}}b^{\dagger}_{-\vec{p}}+b_{-\vec{p}}a_{\vec{p}}\right)\right]. \quad (16)$$

A proper choice of $u_{\vec{p}}$, $v_{\vec{p}}$ now diagonalizes (16):

$$\mathcal{H}=\sum_{\vec{p},\,\mathrm{spin}}\epsilon_{\vec{p}}(M)\left(c^{\dagger}_{\vec{p}}c_{\vec{p}}+d^{\dagger}_{\vec{p}}d_{\vec{p}}\right)\,. \quad (17)$$

The similarity between the present problem and the familiar Bogoliubov transformation in the theory of superconductivity is rather evident. The values of $u_{\vec{p}}$ and $v_{\vec{p}}$ can be taken from there, which, when adapted to the present situation, yield

$$u_{\vec{p}}=\frac{1}{\sqrt{2}}\left(1+\frac{\epsilon_{\vec{p}}(m)-g\langle\Phi\rangle m/\epsilon_{\vec{p}}(m)}{\epsilon_{\vec{p}}(M)}\right)^{1/2},$$

$$v_{\vec{p}}=\frac{1}{\sqrt{2}}\left(1-\frac{\epsilon_{\vec{p}}(m)-g\langle\phi\rangle m/\epsilon_{\vec{p}}(m)}{\epsilon_{\vec{p}}(M)}\right)^{1/2}. \quad (18)$$

A little algebra shows that $u_{\vec{p}}$ and $v_{\vec{p}}$ can be rewritten as

$$u_{\vec{p}}=\frac{1}{2}\left(\frac{[\epsilon_{\vec{p}}(m)+m][\epsilon_{\vec{p}}(M)+M]}{\epsilon_{\vec{p}}(m)\,\epsilon_{\vec{p}}(M)}\right)^{1/2}\left(1+\frac{\epsilon_{\vec{p}}(m)-m}{\epsilon_{\vec{p}}(M)+M}\right),$$

$$v_{\vec{p}}=\frac{1}{2}\left(\frac{[\epsilon_{\vec{p}}(m)-m][\epsilon_{\vec{p}}(M)+M]}{\epsilon_{\vec{p}}(m)\,\epsilon_{\vec{p}}(M)}\right)^{1/2}\left(1-\frac{\epsilon_{\vec{p}}(m)+m}{\epsilon_{\vec{p}}(M)+M}\right), \quad (19)$$

which is indeed equivalent to the statement that $c_{\vec{p}}$, $d_{\vec{p}}$ have the structure of free-particle operators

with the $m \to M$ replacement. The similarity between our self-consistency criterion (10) and the BCS gap equation resulting from the Bogoliubov transformation can also be observed.

The Fermi sphere now is filled with *positive*-energy c particles. (To be consistent, the c vacuum has also to be redefined since it satisfied $d|0_c\rangle = 0$, which is obviously different from the a vacuum for which $b|0_a\rangle = 0$.) Such a coherent mixing of positive- and negative-energy (or particle and antiparticle) states is a general feature of consistent relativistic many-body calculations and will be shown to play an especially important role in the calculation of the exchange energy.

Turning to the calculation of the ground-state energy we note that the total energy (per particle) is the average of (4), and can be expressed as

$$\frac{1}{n}\langle H^{(2)}\rangle = \frac{1}{N}\sum_{\vec{p}} n_{\vec{p}}\epsilon_{\vec{p}} - \frac{1}{2N}\sum_{\vec{p}\vec{q}} \lambda_{\vec{p}\vec{q}} n_{\vec{p}} n_{\vec{q}} \,. \tag{20}$$

It is the usual feature of the Hartree approximation that this is not the average of the one-particle energies. The expression for $\lambda_{\vec{p}\vec{q}}$ is evaluated from (4):

$$\lambda_{\vec{p}\vec{q}} = -\phi \frac{M^2}{\epsilon_{\vec{p}}\epsilon_{\vec{q}}} \,. \tag{21}$$

This can be compared with the analogous expression for $V^{(i)}$ in I.

It should be noted that the second term in (20) is positive and is certainly *not* the "potential energy" of the system.

The consistency of the present scheme requires that the single-particle energy be derivable from the expression for the total energy as the quasiparticle energy:

$$\epsilon_{\vec{p}} = \frac{\delta U}{\delta n_{\vec{p}}} \,, \tag{22}$$

$$U = NE \tag{23}$$

$$= \sum_{\vec{p}} \epsilon_{\vec{p}}(\{n_{\vec{k}}\}) - \sum_{\vec{p}\vec{q}} \lambda_{\vec{p}\vec{q}}(\{n_{\vec{k}}\}) n_{\vec{p}} n_{\vec{q}} \,. \tag{24}$$

It is the implicit $\{n_k\}$ dependence of $\epsilon_{\vec{p}}$ which makes this requirement nontrivial. A further consequence of this implicit dependence is that the average potential, when disturbed by a variation $\delta n_{\vec{k}}$ of the momentum distribution, responds in a peculiar way, exhibiting a "screening" effect which can be described in terms of a quasipolarizability a and quasiresponse $e = 1 + a$ such that

$$\frac{\delta}{\delta n_{\vec{k}}} g\langle\Phi\rangle = \frac{1}{e}\frac{\partial}{\partial n_{\vec{k}}}\bigg|_{M} g\langle\Phi\rangle \,. \tag{25}$$

This quasipolarizability should not be confused with the actual static polarizability $\alpha(\vec{k})$ and response function $\epsilon(\vec{k}) = 1 + \alpha(\vec{k})$ which can be regarded as defined by the relation

$$\frac{\delta}{\delta\rho_{\vec{k}}} g\langle\Phi_{\vec{k}}\rangle = \frac{1}{\epsilon(\vec{k})}\frac{\partial}{\partial\rho_{\vec{k}}} g\langle\Phi_{\vec{k}}\rangle \,,$$

$$\rho_{\vec{k}} = \sum_{\vec{p}} \langle\psi^{\dagger}_{\vec{p}+\vec{k}}\psi_{\vec{p}}\rangle \,.$$

There is, however, an intimate connection between $\epsilon(\vec{k})$ and e which will be discussed in a later paper.

To see the validity of (22) we first note that

$$\frac{\delta\epsilon}{\delta n_{\vec{k}}} = \frac{\partial\epsilon_{\vec{p}}}{\partial M}\frac{\delta M}{\delta n_{\vec{k}}} \tag{26}$$

and

$$\frac{\delta\lambda_{\vec{p}\vec{q}}}{\delta n_{\vec{k}}} = \frac{\partial\lambda_{\vec{p}\vec{q}}}{\partial M}\frac{\delta M}{\delta n_{\vec{k}}} \,. \tag{27}$$

The two derivatives are

$$\frac{\partial\epsilon_{\vec{p}}}{\partial M} = \frac{M}{\epsilon_{\vec{p}}} \,,$$

$$\frac{\partial\lambda_{\vec{p}\vec{q}}}{\partial M} = -\frac{\phi}{V}\frac{M}{\epsilon_{\vec{p}}\epsilon_{\vec{q}}}\left(\frac{p^2}{\epsilon_{\vec{p}}^{\,2}} + \frac{q^2}{\epsilon_{\vec{p}}^{\,2}}\right) \,, \tag{28}$$

while $\delta M/\delta n_{\vec{k}}$ can be evaluated from (10) to lead to

$$\frac{\delta M}{\delta n_{\vec{k}}} = \frac{\phi/V}{1 + (\phi/V)\sum_{\vec{q}}(q^2/\epsilon_{\vec{q}}^{\,3}) n_{\vec{q}}}\frac{M}{\epsilon_{\vec{k}}} \,. \tag{29}$$

Since $\delta g\langle\Phi\rangle/\delta n_{\vec{k}} = -\delta M/\delta n_{\vec{k}}$, (29) serves to identify

$$e = 1 + a \,,$$

$$a = \frac{\phi}{V}\sum_{\vec{q}}\frac{q^2}{\epsilon_{\vec{q}}^{\,3}} n_{\vec{q}} \tag{30}$$

as the quasiresponse and the quasipolarizability of the medium. Now

$$\frac{\delta M}{\delta n_{\vec{k}}} = \epsilon_{\vec{k}} - \sum_{\vec{q}} \lambda_{\vec{k}\vec{q}} n_{\vec{q}} + \frac{\delta\epsilon_{\vec{q}}}{\delta n_{\vec{k}}} n_{\vec{q}} - \frac{1}{2}\sum_{\vec{q}\vec{p}} \frac{\delta\lambda_{\vec{q}\vec{p}}}{\delta n_{\vec{k}}} n_{\vec{q}} n_{\vec{p}} \,. \tag{31}$$

Introducing then the relations exhibited in Eqs. (26) through (29), one easily convinces oneself that all the terms but the first in the right-hand side of (31) cancel each other, thus verifying (22).

A further, somewhat similar, but distinct consistency requirement is that

$$\mu = \frac{\partial U}{\partial N} \,. \tag{32}$$

In other words

$$\epsilon_{p_0} = \frac{\partial}{\partial n}\left(\langle\epsilon_{\vec{p}}\rangle - \tfrac{1}{2}\langle\lambda_{\vec{p}\vec{q}}\rangle\right) \tag{33}$$

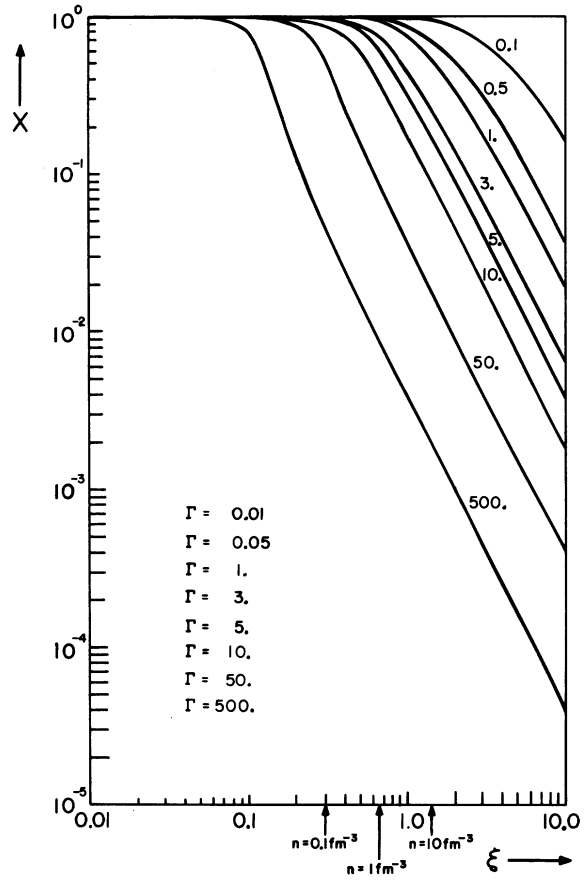

FIG. 1. Effective mass $x = M/m$ vs Fermi momentum $\xi = p_0/m$. The parameter is the effective coupling strength $\Gamma = (4/\pi)(m/\mu)^2 g^2$. The density scale is also indicated.

is to be satisfied. It is sufficient to combine

$$\frac{\partial}{\partial n}\langle \epsilon_{\vec{p}} \rangle = \epsilon_{p_0} + \left\langle \frac{\partial \epsilon_{\vec{p}}}{\partial n} \right\rangle \tag{34}$$

and

$$\frac{\partial}{\partial n}\langle \lambda_{\vec{p}\vec{q}} \rangle = -\phi \frac{M}{\epsilon_{\vec{p}}} \left\langle \frac{M}{\epsilon_{\vec{p}}} \right\rangle - \frac{1}{2} \left\langle \frac{\partial \lambda_{\vec{p}\vec{q}}}{\partial n} \right\rangle \tag{35}$$

with (28), and relations analogous to (27), in addition to using

$$\frac{dM}{dn} = -\frac{1}{V}\frac{\phi}{e}\frac{\partial}{\partial n}\bigg|_M \left\langle \frac{M}{\epsilon_{\vec{p}}} \right\rangle$$

$$= -\frac{1}{V}\frac{\phi}{e}\frac{M}{\epsilon_{p_0}} \tag{36}$$

to show the validity of (32).

III. EQUATION OF STATE

The equation of state is obtained by determining the density dependence of the energy, chemical potential, and pressure, and their critical values as functions of the density and of the coupling strength. The prerequisite for this is the self-consistent determination of the Hartree potential and of the effective mass.

A convenient form of the self-consistency conditions (10) through (13) will be given by employing the variables

$$x = \frac{M}{m}, \quad \eta = \frac{p_0}{M},$$
$$\xi = \frac{p_0}{m}, \quad \Gamma = \frac{4}{\pi}\gamma\left(\frac{m}{\mu}\right)^2. \tag{37}$$

The latter will be referred to as the (effective) coupling strength. By eliminating $\langle\phi\rangle$ in favor of M one can write

$$1 - x = \Gamma x^3 \int_1^{(1+\xi^2/x^2)^{1/2}} (\epsilon^2 - 1)^{1/2}\, d\epsilon$$
$$= \Gamma \tfrac{1}{2} x^3 G\left(\frac{\xi}{x}\right), \tag{38}$$

$$G(\eta) \equiv \eta(1+\eta^2)^{1/2} - \sinh^{-1}\eta .$$

Numerical solution of this equation in the form $x = x(\xi)$ has been obtained and is represented in Fig. 1. We note the following points:

(i) The effective mass is rapidly decreasing as the Fermi energy rises, but it remains finite for any density (cf. I);

(ii) in the low-density (nonrelativistic) limit ($\xi \to 0$) the expansion of $G(\eta)$ leads to

$$x = 1 - (\tfrac{1}{3}\Gamma)\xi^3, \tag{39}$$

(iii) while in the high-density (ultrarelativistic) limit ($\xi \to \infty$),

$$x = \frac{2}{\Gamma \xi^2}. \tag{40}$$

For the chemical potential μ one obtains, in view of (11),

$$\mu = [x^2 + \xi^2(x)]^{1/2}. \tag{41}$$

μ is plotted against the density in Fig. 2 for various values of the coupling strength Γ. The nonmonotonic behavior for a certain range of Γ is striking; we shall comment on this at a later point.

The total energy per particle, E, can be evaluated from (20) and (21) and with the aid of the self-consistency condition (38). Performing the averaging in (20) one finds that

$$E = \frac{2}{3} m\left[\frac{x}{4} H\left(\frac{\xi}{x}\right) + \frac{1}{\Gamma \xi^3}(1-x)^2\right],$$
$$H(\eta) = \frac{1}{\eta^3}\left[\eta(2\eta^2+1)(1+\eta^2)^{1/2} - \sinh^{-1}\eta\right]. \tag{42}$$

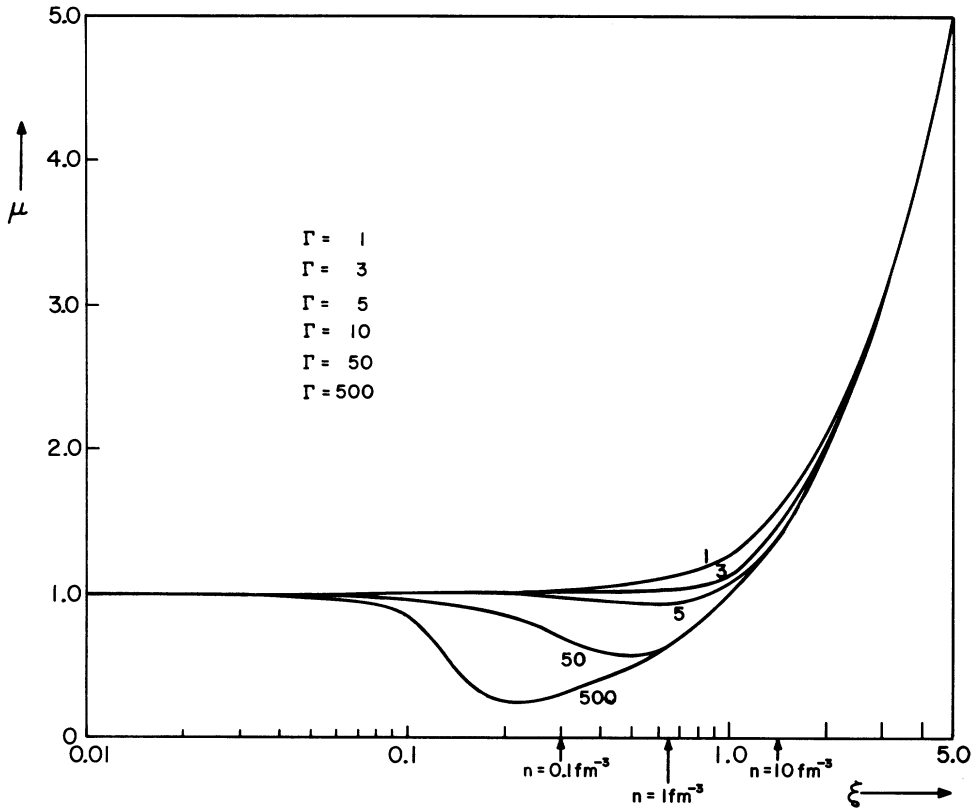

FIG. 2. Chemical potential μ relative to m in units of m vs the Fermi momentum $\xi = p_0/m$. The parameter is the effective coupling strength $\Gamma = (4/\pi)(m/\mu)^2 g^2$. The density scale is also indicated.

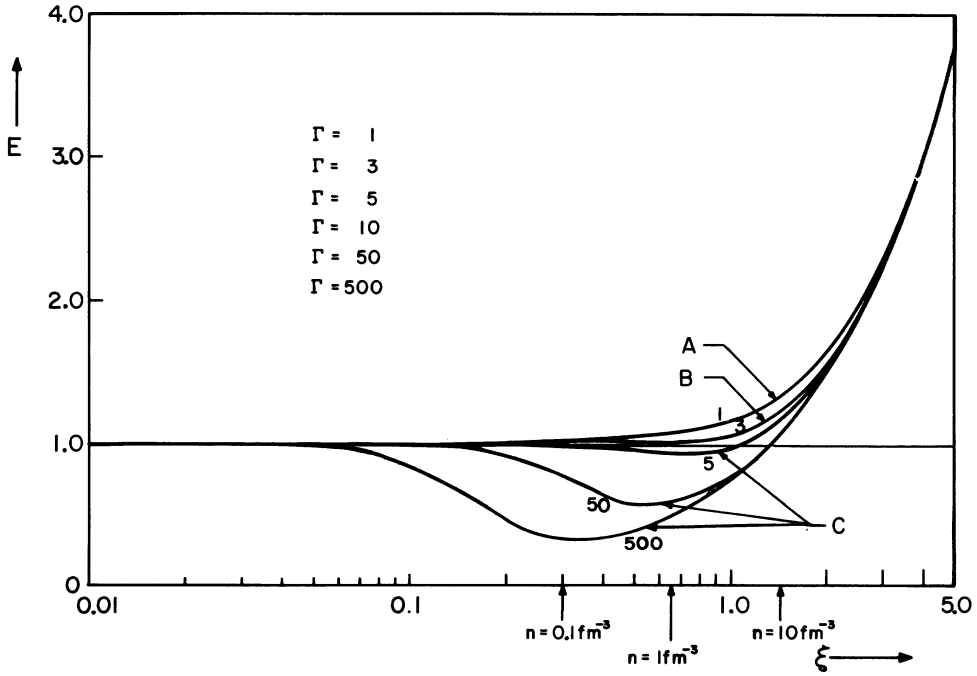

FIG. 3. Total energy per particle E in units of m vs the Fermi momentum $\xi = p_0/m$. The parameter is the effective coupling strength $\Gamma = (4/\pi)(m/\mu)^2 g^2$. The density scale is also indicated. Curves A, B, and C represent no-bound-state, local minima, and bound-state situations.

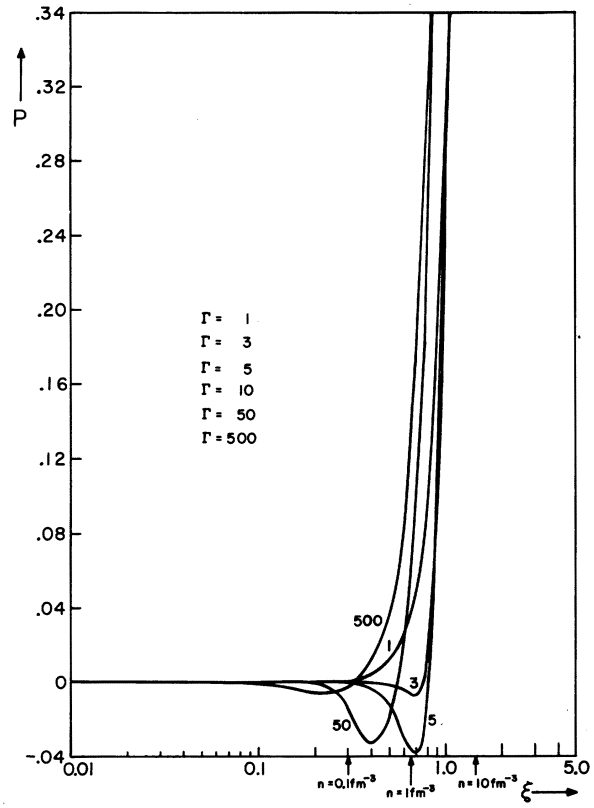

FIG. 4. Equation of state: pressure P in units of $m^4/9\pi^2$ vs Fermi momentum $\xi = p_0/m$. The pressure of an ideal ultrarelativistic Fermi gas is $\frac{3}{4}\xi^4$ in these units. The parameter is the effective coupling strength $\Gamma = (4/\pi)(m/\mu)^2 g^2$. The density scale is also indicated.

The knowledge of $x(\xi)$ enables one to explicitly determine $E = E(\xi)$. The result is plotted in Fig. 3. Again, we have a nonmonotonic dependence on the density, which will be discussed later.

The next quantity of interest is the pressure P. In a zero-temperature system

$$P = -\frac{\partial E}{\partial V} nV , \tag{43}$$

and a useful consequence of (43) is

$$P = n(\mu - E) . \tag{44}$$

The calculation of P from (44) by employing (38) is straightforward:

$$P = \frac{m^4}{12\pi^2}\left\{\xi^3 (x^2 + \xi^2)^{1/2} - (6/\Gamma)(1 - x)(1 - \tfrac{1}{2}x)\right\} . \tag{45}$$

Equation (45) in conjunction with Eq. (38) constitute the equation of state. Figure 4 represents the pressure-density relation for different values of the coupling strength.

Now we are prepared to construct a general picture of the equilibrium properties of the system. The behavior is determined by two parameters: the density (Fermi momentum, ξ) and coupling strength (Γ). Consider first the *low-density* case; x for this limit has been given by (39). We now consider the chemical potential, energy per particle, and pressure. On introducing (39) into the expansion of (41), (42), and (43), respectively, some algebra yields the result

$$\begin{aligned} \mu &= m(1 + \tfrac{1}{2}\xi^2 - \tfrac{1}{3}\Gamma\xi^3) , \\ E &= m(1 + \tfrac{3}{10}\xi^2 - \tfrac{1}{6}\Gamma\xi^3) , \\ P &= \frac{1}{3\pi^2} m^4 (\tfrac{1}{5}\xi^5 - \tfrac{1}{6}\Gamma\xi^6) . \end{aligned} \tag{46}$$

These are the expansions familiar from the nonrelativistic Hartree approximation with the only difference that μ and E now include the rest energy. The nonrelativistic Hartree potential is $-\frac{1}{6}\Gamma m\xi^3 \equiv -4\pi(g^2 n/\mu^2)$.

A less trivial result ensues from the evaluation of the *high-density* limit. The corresponding limiting value of x is provided by (40); the asymptotic expansions of μ, E, and P in (41), (42), and (43) now lead to

$$\begin{aligned} \mu &= m\left(\xi + \frac{2}{\Gamma^2\xi^3}\right), \\ E &= m\left(\frac{3}{4}\xi + \frac{3}{2}\frac{1}{\Gamma\xi^3}\right), \\ P &= m^4\left(\frac{1}{4}\xi - \frac{3}{2\Gamma}\frac{1}{\xi^3}\right) . \end{aligned} \tag{47}$$

The noteworthy feature of (47) is that the system at high density approaches a perfect-gas-like behavior: The higher the coupling constant and the density are, the more closely it resembles a perfect gas. Indeed, the leading terms in (47) are expressions one would find for a noninteracting gas of massless fermions. In addition, the expansion appears in *inverse* powers of the coupling strength. This weakening of the interaction at high densities is analogous to the similar effect occurring both at high densities and at high temperatures in a classical system. What happens can be described by recalling that the momentum dependence of the interaction is such that high-energy particles are ineffective in creating a potential. (For a more detailed physical picture the reader is referred to the corresponding discussion in I.) The competition of the two aspects of the interaction which are manifested by the low-density and high-density limits explains the van der Waals-type equation of state shown in Fig. 4. This is again similar to the character of the equation of state of the classical gas. In contrast, however, to the classical case, it is not only the pressure that exhibits a nonmono-

tonic behavior: Both the chemical potential (Fig. 2) and the total energy (Fig. 3) are nonmonotonic functions of the density, while the energy of a classical gas decreases towards an asymptotic limit as the density increases. The difference can easily be understood: In a Fermi gas the average kinetic energy varies as $\eta^{\lambda}(\frac{1}{3}<\lambda<\frac{2}{3})$; this, together with the discussed high-density behavior of the interaction and with the concomitant boundedness of the interaction energy, results in the resumed increase of the total energy at higher densities (were it not for the scalar character of the interaction, this would not be the case; cf. with the high-density behavior of a Fermi gas with a vector interaction[18]). The consequence of the nonmonotonic behavior of the energy-density relation is the appearance of *bound states* for the system; such bound states can be *local* [curve (B) in Fig. 3] and *absolute* [$\mu/m<1$, curve (C)]. Physical significance should be attached to the latter only, as will readily be pointed out.

Depending now on the value of the coupling

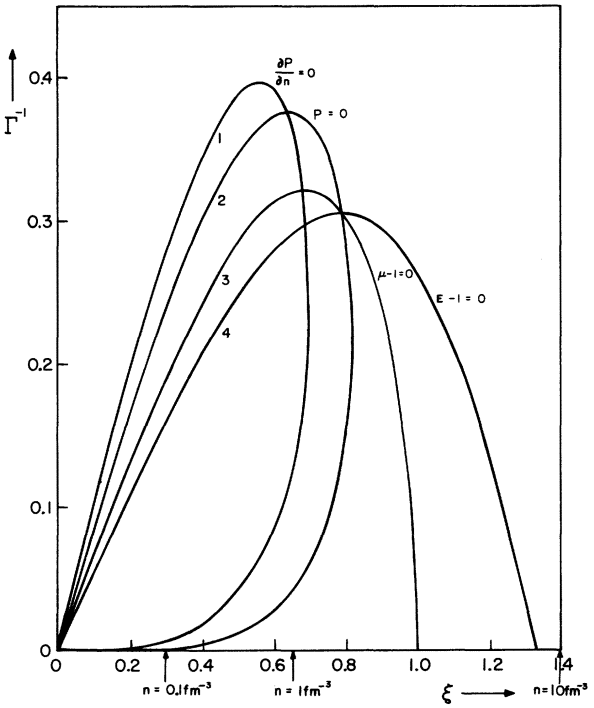

FIG. 5. Critical regions in the $\Gamma^{-1}-\xi$ phase plane, where

$$\Gamma^{-1}=\left[\frac{4}{\pi}\left(\frac{m}{\mu}\right)^2 g^2\right]^{-1},$$

$$\xi=p_0/m.$$

The curves labeled by 1, 2, 3, 4 represent gas-liquid phase transition, negative pressure, negative chemical potential, and bound-state boundaries, respectively.

TABLE I. The minimum Γ values below which no critical behavior takes place. 1: gas-liquid phase transition, 2: negative pressure, 3: negative chemical potential (relative to m), and 4: bound state. The required γ values ($\gamma=g^2/4\pi$ in the notation of Refs. 3–6; $\gamma=g^2$ in the notation of this paper) for α-, σ-, ϵ-, and δ-meson masses are also displayed. ξ_c is the value of the Fermi momentum where the minimum Γ occurs: Both lower and higher densities require higher coupling. ξ_m is the maximal value of the Fermi momentum, beyond which density no critical behavior takes place.

	Γ	γ_α	γ_σ	γ_ϵ	γ_δ	ξ_c	ξ_m
1	2.52	0.30	0.73	0.74	2.10	0.56	0.70
2	2.66	0.32	0.77	0.78	2.21	0.64	0.82
3	3.11	0.37	0.90	0.91	2.59	0.68	1
4	3.28	0.37	0.95	0.96	2.73	0.79	$\frac{4}{3}$

strength Γ, the over-all equilibrium behavior of the system can be classified as belonging to one of the five different categories below.

(1) If Γ is below a critical value Γ_1, both P and E (and consequently μ) are monotonically increasing functions of the density and the behavior of the system is not qualitatively different from that of a perfect Fermi gas.

(2) If $\Gamma<\Gamma_1$, the pressure becomes nonmonotonic, but for $\Gamma>\Gamma_2$ this does not affect the monotonic behavior of E for μ. This is tantamount to the incidence of a gas-liquid phase transition at critical densities depending on the actual value of Γ.

(3) For $\Gamma>\Gamma_2$ the energy ceases to be a monotonic function; as long, however, as $\Gamma<\Gamma_4$, one has a local minimum only.

(4) If Γ is further increased, μ changes its character and for $\Gamma>\Gamma_3$ ($\mu-m$) (i.e., the energy required to add a particle to the system) becomes negative.

(5) Finally, when $\Gamma>\Gamma_4$ is attained, the actual bound states ($E<m$) appear. No phase transition takes place beyond this limit.

The four critical curves displaying the boundaries of the critical regions and determined by $\partial P/\partial\xi=0$, $P=0$ ($\partial E/\partial\xi=0$), $\mu/m=1$, and $E/m=1$ can be calculated from the conditions

$$\Gamma=\frac{3-2x}{x\xi(x^2+\xi^2)^{1/2}}\ , \tag{48}$$

$$\Gamma=3\,\frac{2-3x+x^2}{\xi^3(x^2+\xi^2)^{1/2}}\ , \tag{49}$$

$$x^2+\xi^2=1\ , \tag{50}$$

$$\Gamma=\frac{3}{2}\,\frac{(1-x)^2}{\xi^3}\,\frac{1}{1-\frac{3}{8}xH(\xi/x)}\ , \tag{51}$$

where x is given by (34). Equations (45), (46), and

(47) are immediate consequences of (41), (37), and (38); (34) follows after some algebra from (41).

In Fig. 5 we present the critical curves in the Γ^{-1}-ξ plane, in which representation they show some resemblance to the customary critical curve in the temperature-density plane. The maxima of the curves correspond to the critical Γ values defined above; γ values, calculated for concreteness from the μ values for the δ (Ref. 5), ϵ (Ref. 5), and the proposed α (Ref. 3) and σ (Ref. 4) particles are given in Table I. These numerical values, together with the equivalent γ values for α-, δ-, ϵ-, and δ-meson mass ratios, are given in Table I.

The actual density dependence of the thermodynamic quantities is of course monotonic, as depicted in Fig. 6, with the aid of the conventional Maxwell construction. The local minima of the energy of Fig. 3 do not play any role; only the absolute minima ($E<m$) are significant. The corresponding phase diagram is given in Fig. 7. The left-hand boundary of the "bound" region is the locus of the absolute energy minima. Thus no density lower than the equilibrium value corresponding to this boundary can exist: This is the meaning of the "nonphysical" region for $\Gamma>3.28$. It is interesting to note that the intuitive argument that stronger coupling leads to high equilibrium density is verified only for $3.28<\Gamma<4.44$; for higher Γ values, contrary to intuition, *stronger* coupling results in *lower* equilibrium density. Compression beyond the equilibrium density increases the en-

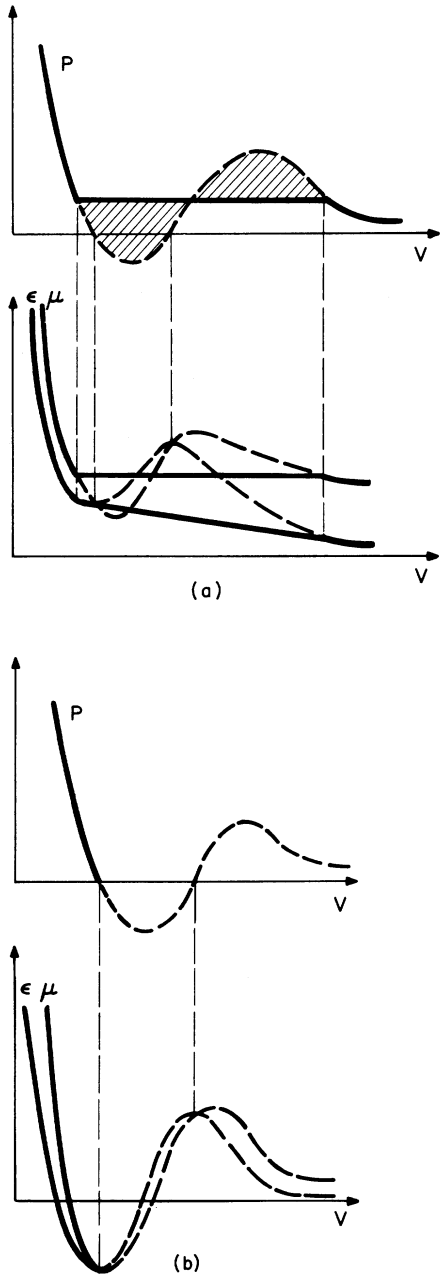

FIG. 6. The actual behavior of the energy, pressure, and chemical potential after the nonphysical negative-slope regions have been eliminated; (a) without, and (b) with bound regions.

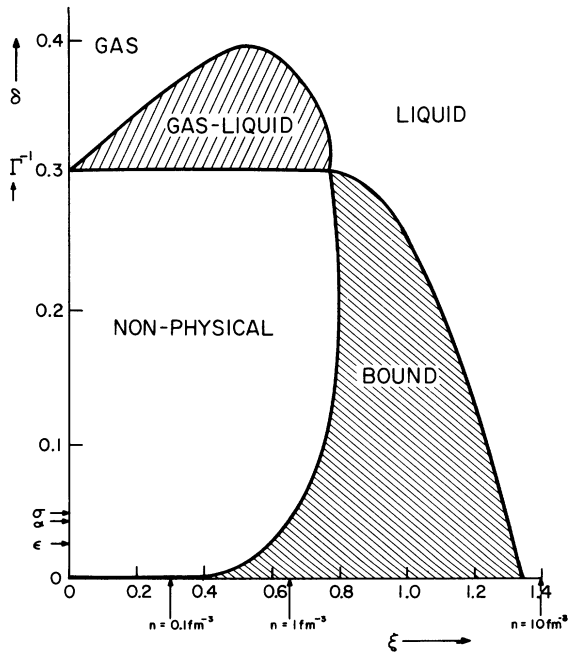

FIG. 7. Phase diagram in the Γ^{-1}-ξ plane, where

$$\Gamma^{-1}\left[\frac{4}{\pi}\left(\frac{m}{\mu}\right)^2 g^2\right]^{-1},$$

$\xi = p_0/m$.

The density scale and the Γ^{-1} values corresponding to σ, α, ϵ, and δ mesons are indicated.

ergy until it ceases to be negative at the right-hand boundary of the "bound" region. It is remarkable that irrespective of the coupling strength, no bound state exists for $\xi > \frac{4}{3}$. As for the gas-liquid phase transition, which is limited to regions of positive energy, such a transition can take place only in a narrow range of the coupling strengths $2.52 < \Gamma < 3.28$. It is unlikely that any physical system would fall in this region. Finally it should be observed that all the critical Γ values as exhibited in Table I correspond to fairly small values of the actual coupling, γ.

IV. BOSON BACKGROUND

The full Hartree ground-state wave function is the product of the filled Fermi-space c-fermion wave function and the boson vacuum wave function:

$$|G\rangle = \left(\prod_{\substack{p<p_0 \\ \text{spin}}} c_{\vec{p}}^{\dagger}|0\rangle\right)_{\text{fermion}} |0\rangle_{\text{boson}} . \tag{52}$$

Therefore the expectation value of the boson number in the Hartree ground state is 0. The question, however, whether there is or is not any real boson background is a rather moot one, since in order to evaluate the expectation value of the boson number to the same order as the Hartree energy, one certainly has to go to higher order than the order of the Hartree approximation. If the boson wave function is calculated to order g beyond the Hartree approximation, the expectation value of the boson number, to this order, becomes

$$\langle N_0\rangle = \frac{2\pi g^2}{\mu^3}\frac{1}{V}\sum_{\vec{p}\vec{q}}\frac{M^2}{\epsilon_{\vec{p}}\epsilon_{\vec{q}}} n_{\vec{p}} n_{\vec{q}} .$$

Only zero-momentum bosons are considered since finite-momentum bosons can be created at the expense of perturbing the Fermi sphere which we want to avoid. The boson background, however, can be taken into account explicitly from the outset, thus providing an alternative and somewhat more satisfactory description of the system. This can be done along the lines of the static source model: We take the boson ground state as a coherent mixture of N-boson, zero-momentum states, while the fermion ground state remains to be determined self-consistently:

$$|G\rangle = |\text{fermion}\rangle \sum_N c_N |N\rangle . \tag{53}$$

The Hamiltonian will be truncated to exclude all non-zero-momentum boson operators; it is also convenient to employ the fermion operators c, d defined by (15):

$$\mathcal{H} = \sum_{\vec{p}} \epsilon_{\vec{p}}(c_{\vec{p}}^{\dagger}c_{\vec{p}} + d_{\vec{p}}^{\dagger}d_{\vec{p}}) - \sum_{\vec{p}} [K(A_0^{\dagger}+A_0) - g\langle\Phi\rangle]\,\nu_0(\vec{p}) + \mu A_0^{\dagger}A_0 . \tag{54}$$

Here $A_0^{\dagger}$, A_0 are the zero-momentum boson operators, $\epsilon_{\vec{p}} \equiv \epsilon_{\vec{p}}(M)$,

$$K = \left(\frac{4\pi}{2\mu V}\right)^{1/2} g, \tag{55}$$

$$\nu_0(\vec{p}) = \frac{M}{\epsilon_{\vec{p}}}(c_{\vec{p}}^{\dagger}c_{\vec{p}} + d_{\vec{p}}^{\dagger}d_{\vec{p}}) + \frac{p}{\epsilon_{\vec{p}}}(c_{\vec{p}}^{\dagger}d_{-\vec{p}}^{\dagger} + d_{-\vec{p}}c_{\vec{p}})$$

[cf. Eq. (13)], and the Hartree coupling term has been added and subtracted in order to allow one to use the diagonalized fermion Hamiltonian. Since, however, the nondiagonal $\nu_0(\vec{p})$ term is now multiplied by the boson operators (rather than by a c number), the previously employed base states do not diagonalize the Hamiltonian in the fermion space. Nevertheless, diagonalization in the boson space can be accomplished by taking the fermion expectation value of (55), as will now be shown.

If one introduces the shifted boson operator

$$C_0 = A_0 - \frac{K}{\mu}\sum_{\vec{p}}\langle\nu_0(\vec{p})\rangle = A_0 - \frac{K}{\mu}\sum_{\vec{p}}\frac{M}{\epsilon_{\vec{p}}} n_{\vec{p}} , \tag{56}$$

the resulting Hamiltonian becomes

$$\mathcal{H} = \sum_{\vec{p}} \epsilon_{\vec{p}}(c_{\vec{p}}^{\dagger}c_{\vec{p}} + d_{\vec{p}}^{\dagger}d_{\vec{p}}) - \sum_{\vec{p}}\left[K(C_0^{\dagger}+C_0) + 2\frac{K^2}{\mu}\sum_{\vec{q}}\langle\nu_0(\vec{q})\rangle - g\langle\phi\rangle\right]\nu_0(\vec{p}) + \sum_{\vec{p}}\left[K(C_0^{\dagger}+C_0) + \frac{K}{\mu}\sum_{\vec{q}}\langle\nu_0(\vec{q})\rangle\right]\langle\nu_0(\vec{p})\rangle + \mu C_0^{\dagger}C_0 . \tag{57}$$

The boson Hamiltonian is obtained by taking the fermion expectation value of (57); the terms linear in C cancel and one is led to

$$\mathcal{H}_B = \langle\mathcal{H}\rangle_F = \sum \epsilon_{\vec{p}} n_{\vec{p}} - \frac{K^2}{\mu}\sum_{\vec{p},\vec{q}}\frac{M^2}{\epsilon_{\vec{p}}\epsilon_{\vec{q}}} n_{\vec{p}} n_{\vec{q}} + g\langle\phi\rangle\sum\frac{M}{\epsilon_{\vec{p}}} n_{\vec{p}} + \mu C_0^{\dagger}C_0 . \tag{58}$$

The ground state (58) is obviously the zero C-boson state, which is identical to the superposition of states of N physical bosons with

$$c_N = \frac{\kappa^N}{\sqrt{N!}} e^{-\kappa^2/2}, \quad \kappa = \frac{K}{\mu} \sum \frac{M}{\epsilon_{\vec{p}}} n_{\vec{p}} . \tag{59}$$

It should be kept in mind that in the context of the calculations of the present section $\langle\phi\rangle$ is still undetermined: It will be made definite by prescribing that the boson expectation value of the neglected off-diagonal term in (57) vanish. Evaluating the latter one finds

$$\langle \mathcal{H} \rangle_B = \sum_{\vec{p}} \epsilon_{\vec{p}} (c^\dagger_{\vec{p}} c_{\vec{p}} + d^\dagger_{\vec{p}} d_{\vec{p}}) - \sum_{\vec{p}} \left[2\frac{K^2}{\mu} \sum_{\vec{q}} \langle \nu_0(\vec{q}) \rangle - g\langle\phi\rangle \right] \nu_0(\vec{p}) + \frac{K^2}{\mu} \sum_{\vec{p},\vec{q}} \langle \nu_0(\vec{p}) \rangle \langle \nu_0(\vec{q}) \rangle . \tag{60}$$

The vanishing of the off-diagonal terms now is tantamount to requiring that the coefficient of $\nu_0(\vec{p})$ be zero:

$$g\langle\phi\rangle = 2\frac{K^2}{\mu} \sum_{\vec{p}} \langle \nu_0(\vec{p}) \rangle = \frac{4\pi g^2}{\mu^2} \frac{1}{V} \sum_{\vec{p}} \frac{M}{\epsilon_{\vec{p}}} n_{\vec{p}} , \tag{61}$$

which is the condition identical to (10). Using (61), Eq. (58) can be written as

$$\mathcal{H}_B = E_F + \mu C_0^\dagger C_0 , \tag{62}$$

where the expression for the fermion energy is identical to (24).

The expectation value for the boson ground state of the physical boson number is given immediately from (59):

$$\langle N \rangle_B = \sum c_N{}^2 N = \kappa^2 = \frac{2\pi g^2}{\mu^3} \frac{1}{V} \sum_{\vec{p}} \frac{M}{\epsilon_{\vec{p}}} n_{\vec{p}} . \tag{63}$$

It is now obvious that the explicit introduction of a "boson background"—at least in the Hartree approximation—does not lead to any new conclusions; it is only an alternative way of arriving at results which can be obtained otherwise.

V. CONCLUSIONS

The main conclusion emerging from this paper is that at high densities the consistent relativistic treatment of many-body systems leads to new effects which cannot even qualitatively be estimated by the extrapolation of nonrelativistic results. The crux of the matter is, of course, the coherent mixing of positive- and negative-energy states, which is easily seen not to be a peculiar feature of the scalar interaction. Indeed, recent recalculations of Gross[3] of relativistic scattering matrix elements indicate that the repulsive core is a common feature of all relativistic interactions—although the implications of this statement in the many-body context and the precise relativistic behavior of other than scalar couplings require further study. The present paper has not gone beyond the simple Hartree approximation, but it should be realized that the consistent relativistic Hartree approximation, in contrast to the nonrelativistic situation and more alike to what it means in atomic physics, is not equivalent to a lowest-order expansion in the coupling strength. This is certainly obvious from the rather complicated dependence of the results obtained on the coupling strength Γ, and can also be visualized by the diagrammatic expansion of Fig. 8. The Hartree approximation differs from its nonrela-

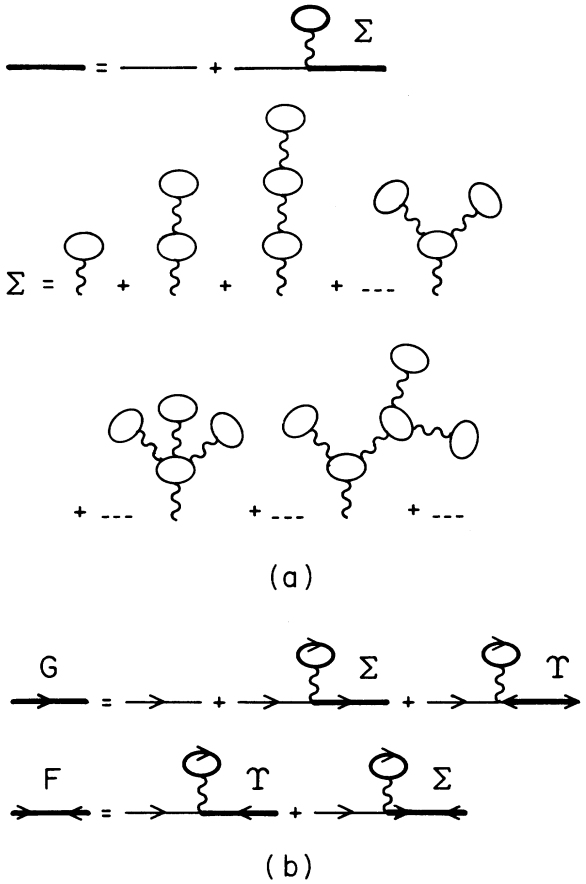

FIG. 8. Diagrammatic representation of the approximation scheme: (a) conventional Green's-function formalism; (b) alternate Green's-function formalism.

tivistic counterpart also in that the two-particle Hartree Hamiltonian

$$\mathcal{H}=\sum_{\vec{p}}\psi^{\dagger}_{\vec{p}}(\vec{\alpha}\cdot\vec{p}+\beta m)\psi_{\vec{p}}-\phi\sum_{\vec{p},\vec{q}}\psi^{\dagger}_{\vec{p}}\beta\psi_{\vec{p}}\psi^{\dagger}_{\vec{q}}\beta\psi_{\vec{q}}\,, \tag{64}$$

would not be diagonalized by the product of one-particle Hartree states; the solution of the problem represented by the model Hamiltonian (64) is a much more complex one than the solution of the one-particle Hartree approximation.

Insofar as the implications of the results of this paper for the behavior of high-density neutron matter are concerned, since the most important pseudoscalar and vector parts of the interaction are completely ignored, one should be rather reluctant to make actual comparisons or put forward predictions. Some feature can, however, be noted. As it is clear from Table I, all the previously suggested scalar-meson coupling strengths but that of the δ meson correspond to Γ values for which bound states could occur. The minimum densities for this to take place are in the range $0.6<\xi<0.7$, i.e., $1.28\times10^{15}<\rho_{\min}<2.03\times10^{15}$ g/cm^3. Calculating the corresponding $d\mu$ values (d = interparticle distance, $\frac{4}{3}\pi d^3 n=1$), one finds $2>d\mu>1.1$ (see Table II). For comparison, in this range $(d\mu)_\rho=2.62$; thus at the densities in question the scalar interaction should have a high relative weight. This is, of course, a nonrelativistic feature, which is properly included in any phenomenological, nonrelativistic nuclear potential—what is not included, however, is the behavior at the high-density boundary in Fig. 7. The maximum densities, where the scalar coupling ceases to bind the system, are in the range $1.29<\xi_{\max}<1.31$ or 12.7×10^{15} g/cm$^3<\rho_{\max}<13.3\times10^{15}$ g/cm^3, and the corresponding $d\mu$ values are $1>d\mu>0.5$ (see Table II). Thus the particles are deep within the scalar well, but, nevertheless, the interaction becomes rather ineffective. This should result in a stiffer equation of state than the one calculated by nonrelativistic methods, although much before this density range is reached the appearance of heavy baryons[19] renders the situation much more complex.

TABLE II. Estimated[3,4,5] γ ($=g^2/4\pi$ in the notation of these papers) values, meson-nucleon mass ratios for α, ϵ, and δ mesons, and the corresponding minimal and maximal Fermi momentum (ξ), density, and $d\mu$ (d = interparticle distance) values for bound-state regions.

	γ	μ/m	$\epsilon_{\min}$	$n_{\min}$(fm^{-3})	$(d\mu)_{\max}$	$\epsilon_{\max}$	$n_{\max}$(fm^{-3})	$(d\mu)_{\min}$
σ	5.40	0.61	0.67	1.097	1.75	1.29	7.84	0.97
α	2.41	0.38	0.65	1.00	1.12	1.29	7.84	0.57
ϵ	9.92	0.61	0.60	0.788	1.95	1.31	8.21	0.89
δ	1.14	1.03	...	...	...	...	...	...

APPENDIX: GREEN'S-FUNCTION FORMALISM

The usual many-body Green's-function formalism can be employed for relativistic systems with little modification. Fermion and boson Green's functions can be defined in the usual way:

$$\begin{aligned} G(xx')&=-i\langle T\psi(x)\,\overline{\psi}(x')\rangle\,,\\ D(xx')&=i\langle T\Phi(x)\,\Phi(x')\rangle\,. \end{aligned} \tag{A1}$$

The Green's functions and the spectral densities $\mathcal{G}$, $\mathfrak{D}$,

$$\begin{aligned} \mathcal{G}(xx')&=\langle\{\psi(x),\overline{\psi}(x')\}\rangle\,,\\ \mathfrak{D}(xx')&=\langle[\phi(x),\phi(x')]\rangle\,, \end{aligned} \tag{A2}$$

satisfy simple equations of motion:

$$\begin{aligned} &(\Lambda-\Sigma)\,G=1,\\ &(\Lambda-\vec{\Sigma})\,\mathcal{G}=0\,,\\ &(\Delta-\Pi)\,D=4\pi,\\ &(\Delta-\vec{\Pi})\,\mathfrak{D}=0, \end{aligned} \tag{A3}$$

which define the self-energy and polarization parts Σ and Π. For scalar interaction in particular

$$\begin{aligned} &\Sigma(xx')=-i\langle Tv(x)\,\overline{\psi}(x'')\rangle\;G^{-1}(x''x')\,,\\ &v(x)=-g\Phi(x)\,\psi(x)\,,\\ &\Pi(xx')=-4\pi ig\langle T\nu(x)\,\overline{\psi}(x'')\rangle\;D^{-1}(x''x')\,,\\ &\nu(x)=\overline{\psi}(x)\,\psi(x)\,. \end{aligned} \tag{A4}$$

The fermion Green's functions can conveniently be built up from the analytic pieces $G^{>}$, $G^{<}$:

$$\begin{aligned} G^{>AB}(xx')&=i\langle\psi^A(x)\,\overline{\psi}^B(x')\rangle\,,\\ G^{>AB}(\vec{p}\omega)&=-2\pi i\sum_{r,m}|\langle 0|a_{\vec{p}r}|m\rangle|^2\,\delta(\omega+E_m)\\ &\qquad\times u^A_r(\vec{p})\,u^B_r(\vec{p})\,, \end{aligned} \tag{A5}$$

$$\begin{aligned} G^{<AB}(xx')&=+i\langle\overline{\psi}^B(x')\,\psi^A(x)\rangle\,,\\ G^{<AB}(\vec{p}\omega)&=+2\pi i\sum_{r,m}|\langle m|a_{\vec{p}r}|0\rangle|^2\,\delta(\omega+E_m)\\ &\qquad\times u^A_r(\vec{p})\,u^B_r(\vec{p})\,. \end{aligned}$$

The notation is standard; r runs over both helicity and energy signature quantum numbers; the two-component spinor character of G^{AB} has explicitly been exhibited. Thus

$$G^{AB}(\vec{p}\omega) = \int d\omega' [\delta_+(\omega-\omega')G^{>AB}(\vec{p}\omega') + \delta_-(\omega-\omega')\,G^{<AB}(\vec{p}\omega')]$$

$$= -i\int d\omega' \left\{ \tfrac{1}{2}\delta(\omega-\omega')\,[\theta(\omega'-\mu) - \theta(\mu-\omega')] + \frac{i}{\pi} P \frac{1}{\omega-\omega'} \right\} \mathcal{G}^{AB}(\vec{p}\omega') \tag{A6}$$

follows, the second step by virtue of $G^{>}(\omega-\mu)$ carrying positive- and $G^{<}(\omega-\mu)$ negative-frequency parts only for the ground-state expectation value.

So far the formalism of nonrelativistic many-body theory remains unaffected. It should be noted, however, that the poles of the Green's function now are provided by the eigenvalue equation

$$|\Lambda^{AB}(\vec{p}\omega) - \Sigma^{AB}(\vec{p}\omega)| = 0 \tag{A7}$$

or equivalently by

$$|\omega \pm \epsilon_{\vec{p}} - \langle r[\vec{p}]\,|\,\Sigma(\vec{p}\omega)\,|\,r'[\vec{p}]\rangle\,| = 0\,,$$

which is tantamount to the required coherent mixing of positive- and negative-energy states discussed earlier.

The spectral density and the Green's function for the noninteracting system are easily calculated:

$$\mathcal{G}_0(\vec{p}\omega) = \frac{\pi}{\epsilon_{\vec{p}}}\{\delta(\omega-\epsilon_{\vec{p}})\,\Lambda(\epsilon_p) - \delta(\omega+\epsilon_{\vec{p}})\,\bar{\Lambda}(-\epsilon_{\vec{p}})\}\,,$$

$$G_0(\vec{p}\omega) = -i\frac{\pi}{\epsilon_{\vec{p}}}\{[\delta_+(\omega-\epsilon_{\vec{p}})(1-n_{\vec{p}s}) - \delta_-(\omega-\epsilon_{\vec{p}})\,n_{\vec{p}}]\,\bar{\Lambda}(\epsilon_{\vec{p}}) + \delta_-(\omega+\epsilon_{\vec{p}})\,\bar{\Lambda}(-\epsilon_{\vec{p}})\}\,, \tag{A8}$$

where $\bar{\Lambda}(\omega)$ is the adjoint Dirac operator.

The self-energy part Σ satisfies the usual Dyson equation; in the Hartree approximation it is a scalar, determined by

$$\Sigma = -ig^2 D_0(00)\frac{1}{V}\sum_{\vec{p}}\int\frac{d\omega}{2\pi}\,\mathrm{tr}\,G^{<}(\vec{p}\omega) \tag{A9}$$

together with

$$G^{-1}(\vec{p}\omega) = G_0^{-1}(\vec{p}\omega) - \Sigma\,. \tag{A10}$$

The self-consistent solution of (A9) and (A10) is

$$\Sigma = -\frac{4\pi g^2}{\mu^2}\frac{1}{V}\sum_{p}\frac{m+\Sigma}{\epsilon_{\vec{p}}(m+\Sigma)}n_{\vec{p}} \tag{A11}$$

in accordance with (10).

An alternate Green's-function formalism can be based on the Hamiltonian expressed in terms of the particle and antiparticle operators $a_{\vec{p}}$, $b_{\vec{p}}$. The two Green's functions

$$G(\vec{p}, t-t') = -i\langle T\,a_{\vec{p}}(t)\,a^{\dagger}_{\vec{p}}(t')\rangle\,,$$
$$\bar{G}(\vec{p}, t-t') = -i\langle T\,b_{\vec{p}}(t)\,b^{\dagger}_{\vec{p}}(t')\rangle \tag{A12}$$

can form the basis of the formalism. They present the advantage of not being four-spinor matrices anymore; the complexity of the system, on the other hand, is reflected through the fact that G and $\bar{G}$ become coupled to the additional Green's function F,

$$F(\vec{p}, t-t') = -i\langle T\,a^{\dagger}_{\vec{p}}(t)\,b^{\dagger}_{-\vec{p}}(t')\rangle\,,$$
$$F(\vec{p}, t-t') = i\langle T\,b_{-p}(t)\,a_{\vec{p}}(t')\rangle\,. \tag{A13}$$

The coupled Dyson equations for G and F now are

$$G = G_0 + G_0\,\Sigma G + G_0\,\Upsilon F\,,$$
$$F = G_0\,\Upsilon\,G + G_0\,\Sigma F\,, \tag{A14}$$

while the Hartree approximation for Σ and Υ yields

$$\Sigma(\vec{p}) = -ig^2 D_0(00)\frac{m}{\epsilon_{\vec{p}}}\frac{1}{V}\sum_{\vec{q}}\int\frac{d\omega}{2\pi}\frac{m}{\epsilon_{\vec{q}}}\{G(\vec{q}\omega) + \bar{G}(\vec{q}\,\omega)\}, \tag{A15}$$

$$\Upsilon(\vec{p}) = -ig^2 D_0(00)\frac{p}{\epsilon_p}\frac{1}{V}\sum_{\vec{q}}\int\frac{d\omega}{2\pi}\frac{m}{\epsilon_{\vec{q}}}\{G(\vec{q}\omega) + \bar{G}(\vec{q}\omega)\}\,.$$

The solution of the resulting eigenvalue equation is, of course, not different from what has been derived previously.

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