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# Solvable quantum cosmological model and the importance of quantizing in a special canonical frame

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By analyzing the simple cosmological model consisting of a real massless Klein-Gordon field with vanishing spatial derivatives in the Friedmann universe, we conclude that this model can be successfully quantized only by using an extrinsic time. If one attempts to quantize using an intrinsic time, one is faced with the problem of either not having a point of maximum expansion, which violates the correspondence principle, or a necessity to devise a new interpretation for a zero-normed quantum mechanics (in addition to the particle-antiparticle interpretation). However, if one uses an extrinsic time, none of these difficulties occur. In analyzing the distinction between these two quantization procedures, we have noted that there are two distinct types of quantum-mechanical tunneling. The first type is the usual quantum-mechanical tunneling which we call "coordinate-space tunneling," where the topology of the classical phase space is usually planar and the *phase space* has no classically forbidden regions, although for a fixed energy, there can exist certain regions of *coordinate space* that are classically forbidden. The second type occurs when the *phase space* has classically forbidden regions, and we call tunneling into these regions "phase-space tunneling." In terms of these two types of tunneling, quantization with an intrinsic time allows "phase-space tunneling" to occur, and it is the presence of this type of tunneling that gives this solution its undesirable features. On the other hand, quantization with a particular choice of extrinsic time absolutely forbids the occurrence of "phase-space tunneling," and it is the lack of this type of tunneling that gives this model its desirable features. Thus, based on this model and other general arguments, we propose that although "coordinate-space tunneling" is quantum-mechanically allowed, the distinctly different tunneling process, "phase-space tunneling," is not only classically forbidden, but also must be considered to be quantum-mechanically forbidden as well.

## I. INTRODUCTION

Since its inception, the problem of quantizing general relativity has forced reexamination of many of the more heuristic aspects of quantization, thereby placing them on a more definitive footing. For example, the operational procedure of replacing momenta by partial differentials with respect to their conjugate coordinates proved to be valid only if the coordinates were related to Killing directions in the space (i.e., Cartesian coordinates in flat Galilean space-time). Of course, in an arbitrary general relativistic system, one may not have any Killing directions, in which case it was found that the coordinate-invariant replacement scheme

$$p_i \to -i\hbar g^{1/4}\partial_{q_i} g^{-1/4} \tag{1.1}$$

could be used,[1] where the $q_i$'s are coordinates canonically conjugate to the momenta $p_i$, and $g$ is the determinant of the metric. Although this prescription is independent of the coordinate system in which it is applied, nevertheless the resulting quantum system is still highly dependent on the classical canonical frame in which one is working.

In order to investigate this dependence, we have treated an extremely simple minisuperspace model, consisting of a real massless Klein-Gordon field, with vanishing spatial derivatives, in a Friedmann universe. Although this model has been simplified to the point of being unphysical, nevertheless, it proves instructive in that it is exactly solvable, both classically and quantum-mechanically, and at the same time it illustrates the importance of finding a "proper" canonical frame in which to carry out the canonical quantization procedure.

It is in general relativity that the importance of carrying out the canonical quantization procedure in a special canonical frame becomes manifest, because the quadratic constraint tends to impose forbidden regions on the classical phase space. Explicitly, this constraint in the standard canonical frame where the metric components are considered as superspace coordinates has the basic form

$$-K_{\exp} + K_g - {}^{(3)}R + E_m = 0, \tag{1.2}$$

where $K_{\exp}$ is proportional to the square of the local rate of expansion and is positive-definite, $K_g$ is the "kinetic energy" of the gravitational radiation (the other two degrees of freedom of the gravitational field), also positive-definite, ${}^{(3)}R$ is the curvature scalar of the hypersurface, and

$E_m$ is the energy density of all matter present as well as its interaction energy with the gravitational field, which for realistic classical fields is positive-definite also. Now, consider the initial-value problem. Given the initial energy density of matter, the curvature scalar of the hypersurface, and $K_g$, we want to solve the constraint (1.2) for the expansion rate by taking the square root of $K_{\exp}$. But this can be done only if

$$^{(3)}R \leq K_g + E_m \,. \tag{1.3}$$

In other words, the *phase space* in this canonical frame has classically forbidden regions when the curvature scalar is chosen too large. The presence of these forbidden regions actually constitutes *additional* classical constraints on the system and these must be taken into account upon quantization. However, as (1.3) indicates, these additional constraints are inequalities and how one could cast such conditions into operator form is not at all clear.

It is this consideration that our model quite vividly illustrates. First, we quantize in the standard canonical frame (where the radius of the universe is the timelike variable) and find that the solution allows quantum-mechanical tunneling into the classically forbidden regions of phase space defined by $^{(3)}R > K_g + E_m$. When we attempt to interpret these solutions, we find that contrary to the classical solution, the quantum-mechanical solution does not exhibit a "bounce" at a maximum radius of expansion (the classical turning point), but rather once expansion has been initiated, the universe "tunnels" through this classical turning point and expands indefinitely into the classically forbidden region. Naturally, if the inequality constraint (1.3) could be properly quantized, this type of unphysical behavior would be forbidden. Although such procedures have yet to be developed, fortunately there exists an alternative method, and that is to first perform a canonical transformation on the classical system such that (1.3) reduces to a triviality. In other words, we find a classical canonical frame where the initial-value data are unconstrained.[2] Although we cannot say anything about the uniqueness of such a frame, we do find one such frame for our model. However, in doing so, we must use what is called an "extrinsic time" since the new timelike variable must depend not only on radius of the universe, but also on its conjugate momentum.

When we quantize in this new frame, all the previous difficulties vanish and the quantum-mechanical solution corresponds faithfully to the classical solution. Finally, we conclude with a brief discussion of our results.

## II. THE MASSLESS REAL KLEIN-GORDON FIELD WITH VANISHING SPATIAL DERIVATIVES IN THE FRIEDMANN UNIVERSE

The example which will now be considered is a real massless Klein-Gordon field, with vanishing spatial derivatives, in the Friedmann universe. With these restrictions, we have only two independent variables; the radius of the universe and the amplitude of the Klein-Gordon field. First, we will look at the classical theory and solve the equations of motion. Then we will quantize in the standard manner and attempt to interpret the results. We will see that the standard interpretation *cannot* be used in this case without violating the correspondence principle. In an attempt to rectify this, we perform a canonical transformation on the classical system and then requantize. Now, all difficulties with the old quantum theory have vanished and the correspondence principle is beautifully satisfied. As for our conventions, we take the space-time metric to be

$$[g_{\mu\nu}] = \text{Diagonal}\ [\bar{\beta}^2, -R^2, -R^2\sin^2 r, -R^2\sin^2 r\sin^2\theta], \tag{2.1}$$

where

$$\bar{\beta} \equiv 2\pi^2 R^3 \alpha, \tag{2.2}$$

the coordinates are $(t, r, \theta, \phi)$, and $\alpha$ and $R$ depend only on $t$.[3] We introduce Misner's "$\Omega$" variable[4] by letting

$$R = \frac{1}{\pi\sqrt{2}}\, e^{\Omega}, \tag{2.3}$$

and take our units to be such that $c = 1 = 4\pi G/3$. Then upon setting the mass of the Klein-Gordon field equal to zero, the total Lagrangian becomes

$$L = \frac{1}{2}\left(\frac{1}{\alpha}\dot{\phi}^2 - \frac{1}{\alpha}\dot{\Omega}^2 + \alpha e^{4\Omega}\right), \tag{2.4}$$

where $\alpha$ is the gauge variable and $(\Omega, \phi)$ are the minisuperspace variables.

### A. The Classical Theory

Following the Dirac procedure,[5] we have one primary constraint,

$$\Phi \equiv P_\alpha \simeq 0, \tag{2.5}$$

and the Hamiltonian is

$$H = \alpha\chi, \tag{2.6}$$

where the secondary constraint, $\chi$, is given by[6]

$$\chi \equiv \tfrac{1}{2}(P_\phi^2 - P_\Omega^2 - e^{4\Omega}) \simeq 0. \tag{2.7}$$

The classical solution of Hamilton's equations of motion can be obtained in terms of an invariant parameter, $\tau$, defined up to a constant by

$$d\tau \equiv \alpha\, dt. \quad (2.8)$$

The solution in parametric form is

$$P_\phi = \text{constant}, \quad (2.9a)$$

$$\phi = P_\phi \tau + \phi_0, \quad (2.9b)$$

$$\Omega = \tfrac{1}{2}\ln\left(\frac{|P_\phi|}{\cosh(2P_\phi \tau)}\right), \quad (2.9c)$$

where $\phi_0$ is a constant of the motion, and we have scaled $\tau$ so that $\tau = 0$ corresponds to the time of maximum expansion. This motion is quite simple and well known. The limit $\tau \to -\infty$ corresponds to the initial singular state from which the universe expands until a maximum radius of

$$R_m = \frac{1}{\pi}(\tfrac{1}{2}|P_\phi|)^{1/2}$$

is reached. Then the contraction phase begins and the universe again approaches the singular state as $\tau \to +\infty$. Meanwhile, the motion of $\phi$ is like that of a free particle, in that it linearly increases (decreases) its value for positive (negative) momentum, $P_\phi$. Now, let us see how well a quantum theory can describe the same system.

### B. The Standard Quantum Theory

In canonical quantization,[5] one considers constraints to be conditions on the state function, not operator identities, and it is postulated that they constitute all of the physical requirements on the system. Consequently, for our simple system, the usual procedure[7] of letting $p_\Omega \to i\hbar\partial_\Omega$, $p_\phi \to -i\hbar\partial_\phi$, and $p_\alpha \to -i\hbar\partial_\alpha$ in Eqs. (2.5) and (2.7) gives the quantum constraints of the standard quantum theory[8] to be

$$\Phi\Psi = -i\hbar\partial_\alpha\Psi = 0, \quad (2.10)$$

$$\chi\Psi = \tfrac{1}{2}(-\hbar^2\partial_\phi^2 + \hbar^2\partial_\Omega^2 - e^{4\Omega})\Psi = 0. \quad (2.11)$$

The general solution of Eqs. (2.10) and (2.11) is simply

$$\Psi = \int_{-\infty}^{\infty} \frac{e^{i\nu\phi}}{2(\sinh|\frac{1}{2}\pi\nu|)^{1/2}}\left[A_\nu I_{-i|\nu/2|}(e^{2\Omega}/2\hbar) + B_\nu I_{i|\nu/2|}(e^{2\Omega}/2\hbar)\right]d\nu, \quad (2.12)$$

where $I_{i|\nu/2|}(x)$ is the modified Bessel function[9] of imaginary order, $i|\frac{1}{2}\nu|$. Now, let us try to interpret this theory.

First, since the secondary quantum constraint is a two-dimensional Klein-Gordon-type equation with a time-dependent potential, we have the conserved vector current

$$j_\mu = \tfrac{1}{2}i\left[\Psi^*\partial_\mu\Psi - (\partial_\mu\Psi)^*\Psi\right]. \quad (2.13)$$

Thus, we can use the component of $j_\mu$ in the $\Omega$ direction to normalize the state vector. Defining

$$N \equiv \int_{-\infty}^{\infty} j_\Omega\, d\phi, \quad (2.14)$$

then from the known properties of Bessel functions[9]

$$N = \int_{-\infty}^{\infty}(A_\nu^* A_\nu - B_\nu^* B_\nu)\, d\nu. \quad (2.15)$$

Due to the indefinite nature of this norm, we could apply an interpretation scheme which exactly parallels that of special relativistic spin-zero bosons.[10] However, instead of interpreting the two degrees of freedom as particle-antiparticles, following Misner,[11] we could consider them as indexing the expansion and contraction phases of the universe.

To see if this system will properly describe the original classical system, let us look at some expectation values. By the statistical interpretation, the expectation value of an observable is simply the average of the measured values for a large number of identical ensembles. We define the expectation value of an operator $O$ by

$$\langle O\rangle \equiv \tfrac{1}{2}i\int_{-\infty}^{\infty}\left[\Psi^* O\partial_\Omega\Psi - (\partial_\Omega\Psi^*)O\Psi\right]d\phi. \quad (2.16)$$

Consider a wave packet where $B_\nu = 0$, and $A_\nu$ is sharply peaked about $\nu = \nu_0$. For this packet, the expectation value of $P_\phi$ remains centered about $h\nu_0$ for all $\Omega$, and thus is a constant of the motion in agreement with the classical theory. Meanwhile, since the spatial metric components depend only on $\Omega$, they are simply $c$ numbers with no quantum uncertainties. However, the expectation values of the operators $\phi$ and $P_\Omega$ do not correspond to the classical values for all $\Omega$. As $\Omega \to -\infty$, we have

$$\langle P_\Omega\rangle \simeq \hbar\nu_0, \quad (2.17a)$$

$$\langle\phi\rangle \simeq \frac{\nu_0}{|\nu_0|}\Omega \quad (2.17b)$$

as expected from the classical theory, but when $e^{2\Omega} >> \hbar\nu_0$, they approach

$$\langle P_\Omega\rangle \simeq -\frac{\hbar}{2\sinh|\pi\nu_0/2|}\exp(e^{2\Omega}/\hbar), \quad (2.18a)$$

$$\langle\phi\rangle \simeq -\hbar(\hbar\nu_0)\exp[e^{2\Omega}/\hbar - 4\Omega]. \quad (2.18b)$$

As one immediately notes, these are quantum corrections to the classical equations of motion. Obviously lacking is the turning point which classically occurs when $P_\Omega = 0$ or $e^{2\Omega} = |P_\phi|$. Thus, the implications of this theory are that upon the initial expansion of the Friedmann Universe, it expands according to the classical equations of motion until the turning point at $e^{2\Omega}=|P_\phi|$ is reached. Then, instead of contracting, the universe will "tunnel" through the barrier and continue its expansion.

As noted by Misner[11] in other related cases, one possibility for rectifying this difficulty would be to apply a boundary condition on the timelike variable and require $\Psi \to 0$ as $\Omega \to \infty$. This can be done by setting $B_\nu = -A_\nu$ in Eq. (2.12) but this results in a zero-norm state as seen from Eq. (2.15). Although this has now rectified the "runaway" as exhibited by Eqs. (2.18), it has created a new problem. First, we know of no physically consistent interpretation for zero-normed quantum mechanics, except the particle-antiparticle interpretation. But, this interpretation requires the possibility of physically separating the particle from the antiparticle, and performing measurements on each one individually. In our case, when this is done we again get the run-away for each mode (although the sum would vanish), so we have not really gained anything. Therefore, to completely eliminate this run-away one must postulate that it is impossible to separate the two modes, and to consider the solution $B_\nu = -A_\nu$ to be a single inseparable entity. But then one would have to devise a new interpretation for this type of zero-normed quantum mechanics.

Thus, in the absence of an alternate interpretation of zero-normed quantum mechanics one finds it difficult to escape from the conclusion that if Eq. (2.11) is the proper quantum constraint, then when the Friedmann universe reaches the classical turning point, it will *not* initiate contraction; rather, it will expand indefinitely. Of course, one finds this conclusion totally unacceptable, and we propose that Eq. (2.11) is not the correct quantum constraint.

### C. The Source of the Difficulty and Its Resolution

Returning to the quantum constraint, Eq. (2.11), one notes that the origin of the difficulty is the last term, which represents the curvature scalar of the Friedmann universe. If this term were absent or of the opposite sign, then $\Psi$ would not exhibit the exponential runaway and would remain bounded in magnitude. Thus, relative to $\Omega$, the curvature scalar of a Friedmann universe enters the constraint as a repulsive potential barrier creating a region of *phase* space which is classically forbidden. Now, a very crucial question is whether or not this region should also be considered quantum-mechanically forbidden.

At this point, we must emphasize that there is a distinct difference between *classically forbidden phase-space regions*, as illustrated above, and *classically forbidden coordinate-space regions*. To illustrate the distinction between these two, consider the Hamiltonian

$$H=\alpha(\tfrac{1}{2}P_\phi^2-\tfrac{1}{2}P_\Omega^2-\tfrac{1}{2}e^{4\Omega}) \tag{2.6$'$}$$

and the classical Hamiltonian

$$H=\alpha[\tfrac{1}{2}p^2+V(x)-E], \tag{2.19}$$

which is simply the Hamiltonian of a one-dimensional particle of unit mass, interacting with the potential $V(x)$, written in the constraint form.[12]

Now, by *classically forbidden coordinate-space regions* we mean the following. Consider any particular solution of the equations of motion. If there exist ranges of the coordinate in which the system will never be found classically, then these regions are the *classically forbidden coordinate-space regions*. In the case of (2.19), these are whenever $V(x)>E$, where $E$ is the energy of the particular solution being considered. On the other hand, (2.6′) possesses *no* such regions (except when $P_\phi=0$), as can be seen from the classical solution (2.9). But, consider now the initial-value problem in both cases. In the case of our model, we first specify the initial values of $\phi$ and $P_\phi$ at some initial value of $\Omega$, and in the other case, we specify the initial values of $x$ and $p$ at some initial value of $t$. Then in both cases, we are to use the secondary constraint to determine the conjugate momentum of the timelike variable. In the case of (2.19), this presents no difficulty whatsoever. But, in the case of (2.6′), if the value of $P_\phi$ is chosen too small, *no* value of $P_\Omega$ would satisfy the constraint. Thus, relative to the initial-value problem, the $(\phi, P_\phi)$ phase space has *classically forbidden phase-space regions*, whereas (2.19) has no such regions. Note that these are two distinct situations. Classically forbidden phase-space regions are defined by the constraint, while classically forbidden coordinate-space regions are defined by a particular solution and are energy dependent. Consequently, the existence of *coordinate-space tunneling* (which is well known to occur) need not necessarily imply that *phase-space tunneling* will also be observed. In fact, due to the difficulty implied by the standard quantization of this model, we propose that such regions must remain not only classically forbidden, but also quantum-mechanically forbidden as well. One way to ensure this is to somehow quantize the classical phase-space constraint

$$P_\phi^2 \geq e^{4\Omega}. \tag{2.20}$$

Clearly, this is a nontrivial classical constraint, but, since it is an inequality, just how one should cast it into operator form is not at all clear. Instead, following the ideas of York,[2] we will seek to find a canonical transformation on the classical system which will reduce (2.20) to a triviality.

One such transformation is

$$\Omega = \tfrac{1}{2}\ln[P_T/\cosh(2T)], \tag{2.21a}$$

$$P_\Omega = -P_T \tanh(2T), \tag{2.21b}$$

whose inverse is

$$P_T = (P_\Omega^2 + e^{4\Omega})^{1/2}, \tag{2.22a}$$

$$T = \Omega - \tfrac{1}{2}\ln[P_\Omega + (P_\Omega^2 + e^{4\Omega})^{1/2}]. \tag{2.22b}$$

Note that this new time, $T$, is an extrinsic (momentum-dependent) variable. Under the transformation (2.21), the constraint (2.17) becomes simply

$$\chi = \tfrac{1}{2}(P_\phi^2 - P_T^2), \tag{2.23}$$

i.e., a massless Klein-Gordon equation. Now, the $(\phi, P_\phi)$ phase space is no longer constrained. Consequently, one can now arbitrarily specify initial values for $\phi$ and $P_\phi$ at an initial value of $T$ (not $\Omega$), and one can always find a value of $P_T$ satisfying the secondary constraint, Eq. (2.23). The classical solutions for this system, again in terms of the invariant parameter $\tau$ defined by (2.2), are

$$P_\phi = \text{const}, \tag{2.24a}$$

$$P_T = +|P_\phi|, \tag{2.24b}$$

$$\phi = P_\phi \tau + \phi_0, \tag{2.24c}$$

$$T = |P_\phi|\tau + T_0, \tag{2.24d}$$

where $\phi_0$ and $T_0$ are arbitrary constants. Elimination of $\tau$ from (2.25c), (2.25d) gives the dependence of $\phi$ on $T$ as

$$\phi = \frac{P_\phi}{|P_\phi|}T + \text{const}. \tag{2.25}$$

The quantized version of this system is identical to the previous one except for the quadratic constraint which now reads

$$\hbar^2(\partial_T^2 - \partial_\phi^2)\psi = 0. \tag{2.26}$$

The general solution to this equation may be written as

$$\psi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d\nu}{(E_\nu/\hbar)}\left[A_\nu e^{i(\nu\phi - E_\nu T/\hbar)} + B_\nu e^{-i(\nu\phi - E_\nu T/\hbar)}\right], \tag{2.27}$$

where $E_\nu \equiv +\hbar\sqrt{\nu^2}$ and $A_\nu$, $B_\nu$ are arbitrary functions of $\nu$. The norm as defined by (2.13) and (2.14) is identical in form to (2.15) of the previous system. One can now easily see that this new quantized version faithfully mirrors the classical system. First, for the expectation value of the operator $\phi$, we have that in general

$$\begin{aligned}\langle\phi\rangle = {} & T\int_{-\infty}^{\infty} d\nu \frac{\hbar\nu}{E_\nu}(A_\nu^* A_\nu - B_\nu^* B_\nu) + \tfrac{1}{2}i\int_{-\infty}^{\infty} d\nu (A_\nu^* A_\nu' - A_\nu^{*\prime} A_\nu + B_\nu^* B_\nu' - B_\nu^{*\prime} B_\nu) \\ & + \tfrac{1}{2}i\int_{-\infty}^{\infty}\frac{d\nu}{\nu}(A_\nu B_{-\nu}^* e^{-2iE_\nu T/\hbar} - A_\nu^* B_{-\nu} e^{2iE_\nu T/\hbar}),\end{aligned} \tag{2.28}$$

where the primes denote differentiation with respect to $\nu$. Obviously, if $A_\nu$ or $B_\nu$ is zero, we have the classical result (2.25), while the third term is a quantum correction indicating interference between the (±) degrees of freedom. Note that the canonical transformation (2.21) has proven effective in that there is no longer any asymptotic difficulty with this expectation value; the correspondence principle is satisfied.

The expectation values of the conjugate momenta $P_\phi$ and $P_T$ are readily computed to be

$$\langle P_\phi\rangle = \hbar\int_{-\infty}^{\infty}\nu\, d\nu (A_\nu A_\nu^* + B_\nu B_\nu^*), \tag{2.29a}$$

$$\langle P_T\rangle = +\int_{-\infty}^{\infty} E_\nu d\nu (A_\nu A_\nu^* + B_\nu B_\nu^*). \tag{2.29b}$$

Thus, for well-localized packets, the $\phi$-conjugate momentum, $P_\phi$, has a constant expectation value whose absolute value is essentially equal to the expectation value of the $T$-conjugate momentum, $P_T$. These results correspond exactly to those of the classical theory given by (2.24a), (2.24b).

Other expectation values that one would be interested in are the spatial part of the space-time metric and the associated curvature scalar. Since our time, $T$, is no longer intrinsic, $\gamma_{ij}$ and ${}^{(3)}R$ are no longer $c$ numbers. For $\gamma_{11}$, Hermiticity requires the factor ordering

$$\gamma_{11} = R^2 - \frac{i\hbar}{2\pi^2}(\cosh 2T)^{-1}\partial_T, \tag{2.30}$$

$$\langle R^2\rangle = \frac{\int_{-\infty}^{\infty} d\nu\, E_\nu (A_\nu^* A_\nu + B_\nu^* B_\nu)}{2\pi^2\cosh(2T)}. \tag{2.31}$$

Now, we clearly have an oscillating universe. The expectation value of the square of the radius of the universe is positive-definite, and as $T$ ranges from $-\infty$ to $+\infty$, $\langle R^2\rangle$ goes from zero to a maximum value given by

$$\langle R^2\rangle_{\max} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\nu\, E_\nu (A_\nu A_\nu^* + B_\nu B_\nu^*)\,, \qquad (2.32)$$

and back to zero. Thus, the universe oscillates with a maximum expansion directly related to the amount of energy contained within it.

The curvature scalar, ${}^{(3)}R$, for a closed Friedmann universe is given by (recall units $\frac{4}{3}\pi G = c = 1$)

$$ {}^{(3)}R = \frac{1}{2R^2}\,, \qquad (2.33)$$

which upon use of (2.21a) and (2.3) becomes

$$ {}^{(3)}R = \pi^2(\cosh 2T)(P_T)^{-1}\,, \qquad (2.34)$$

where the factor ordering follows from the Hermiticity of the operator $R$. Computing the expectation value of this is now straightforward, giving

$$\langle {}^{(3)}R\rangle = \pi^2 \cosh(2T)\int_{-\infty}^{\infty}\frac{d\nu}{E_\nu}(A_\nu A_\nu^* + B_\nu B_\nu^*). \qquad (2.35)$$

Hence, the intrinsic curvature is positive-definite, reaching a minimum value at the turning point ($T=0$) and increasing without bound in the limits $T\to\pm\infty$.

It should be mentioned that the operator $P_T$ is Hermitian and its eigenfunctions are complete in that they can be used to form all possible solutions of the Klein-Gordon-type equation (2.27). Thus, $P_T$ is an observable in the usual sense and, by the operator equivalent of Eq. (2.32), so is $R^2$. Hence, the expression (2.32) gives the physically measurable value of $R^2$ and since it has a definite maximum as a function of $T$, there is no tunneling into the classically forbidden phase-space regions. This is in contrast to the exponentially decaying behavior in this region exhibited by the bounded ($K_{i|\nu/2|}$) solution in the previous system. Also note that in this case, $R^2$ is an operator ($q$ number) with an associated spread (uncertainty), whereas in the previous system $e^{2\Omega}$ was simply a scalar ($c$ number).

## III. DISCUSSION

As shown by our example, when the classical phase space is restricted in any manner, straightforward application of (1.1) does not necessarily produce an acceptable quantum system. Actually, the fault does not lie in the prescription (1.1) but in failure to take into account the additional constraints on the system implied by any restrictions or redundancies in the phase space. That this is really nothing new can be seen by investigating the effects of canonical transformations on various simple systems. Consider the harmonic oscillator where $H=\frac{1}{2}(p^2+q^2)$. If we apply the canonical transformation

$$p=\sqrt{2P}\,\cos Q\,, \qquad (3.1a)$$

$$q=\sqrt{2P}\,\sin Q\,, \qquad (3.1b)$$

the new Hamiltonian is simply $H=P$, but the new phase space is constrained by

$$P\geq 0\,, \qquad (3.2)$$

since in the phase space of $(Q,P)$, points with $P<0$ do not correspond to any classically *allowed* state of the harmonic oscillator. If we tried to quantize the harmonic oscillator in this new canonical frame and ignore the inequality constraint (3.2), thereby *allowing phase-space tunneling*, we would come to the conclusion that one could find the quantized harmonic oscillator in a state of *negative* energy ($E=P<0$). Since this has never been observed experimentally, we have one more reason for postulating that these states must be quantum-mechanically as well as classically forbidden.

In special-relativistic theories, these considerations are essentially academic because the original canonical frame always possesses a phase space with planar topology. However, as mentioned in the Introduction, this issue is forced in general-relativistic treatments because the quadratic constraint may impose forbidden regions in the phase space. This will occur in any space whose intrinsic curvature disobeys the phase-space constraint

$$ {}^{(3)}R \leq K_g + E_m\,. \qquad (1.2')$$

Unfortunately, a general technique for handling inequality constraints is yet to be formulated and the simple general-relativistic example treated in this paper amply illustrates that pathology can arise from ignoring these restrictions. A possible alternative is to canonically transform to a frame in which the phase constraints become trivialities. This procedure was successfully applied to the example treated here and standard canonical quantization in the new frame resulted in a system which, in the proper limit, faithfully reproduced the classical behavior.

Also, it is important to note that our model definitely favors the use of an extrinsic timelike variable because it was found to be impossible to quantize using the intrinsic time variable ($\Omega$), while on the other hand, introduction of the extrinsic time variable ($T$) produced a very reasonable quantum theory. In fact, this result can be easily extended to a large class of general-relativistic systems. Consider the general form of the quadratic constraint as given by (1.2). In terms of any intrinsic time, whenever ${}^{(3)}R > 0$, there will be forbidden regions in phase space. If one now accepts the postulate that classically forbidden phase-space regions (relative to the initial value problem) are also to be quantum-mechanically forbidden, then it follows that it will be necessary to transform to a new canonical frame in which $[K_{\text{exp}} + {}^{(3)}R]$ will be a positive-definite function of the conjugate momentum of the new time variable, with a range from zero to infinity. Clearly, the application of such a canonical transformation will automatically introduce an extrinsic timelike variable.

It should be noted that other attempts to quantize the Friedmann universe have, in a sense, also used an extrinsic time. DeWitt[13] was one of the first to obtain some reasonable results from this model and attempt to treat this problem of quantum-mechanical tunneling and indefinite norm. He approached it by considering the WKB solution for a Friedmann universe, and his analysis of the WKB solution showed that along an $R$ = constant hypersurface, as expected, probability flowed in both directions. The expansion phase was represented by probability flux in the increasing $R$ direction, and the contraction phase was also present, and was represented by a probability flux in the decreasing $R$ direction. Consequently, along any $R$ = constant hypersurface, the total flux passing through the hypersurface was identically zero, implying a zero norm, if one would define the inner product in the usual manner. However, DeWitt noted that, at least in certain special cases, if one defined the inner product as an integral over a well-chosen subset, then the norm not only would be nonzero, but also would be positive-definite. This could be done by choosing the subset such that only the probability flux corresponding to increasing $R$ would be intercepted, while excluding that flux corresponding to decreasing $R$. Similar difficulties have also been noted by Misner and others[4,11] in various cosmological models they have investigated, and they have suggested similar methods for handling the problem. Basically, what one seeks to do is to intercept only the flux corresponding to $P_\Omega > 0$, while excluding that corresponding to $P_\Omega < 0$. But, this is exactly what one does when one uses an extrinsic time. Consider Eq. (2.21b). When $T < 0$, we have $P_\Omega > 0$, and when $T > 0$, $P_\Omega < 0$. Thus, if the norm is defined along the hypersurface where $T$ = constant, then for $T < 0$, only the positive flux will be intercepted. Consequently, any definition of a norm which depends on the sign of $P_\Omega$ is actually introducing an extrinsic time.

Although attempts to quantize several cosmological models have been made,[14] reasonable quantum solutions for almost all of these are yet to be found. At best, a wave function is found, and "phase shifts" at a bounce are determined, but what do these quantities mean? When the norm of the wave function vanishes identically, how can the theory be interpreted, how can one evaluate matrix elements, and how does one correlate these matrix elements to experimental measurements? On the other hand, if one does not require a vanishing norm, then one is faced with the problem of these "run-away" solutions which do not have a classical limit. Obviously, something is wrong with our present so-called "quantization procedure," in that when we attempt to apply it to general relativity, we are unknowingly violating a fundamental and unknown principle. Based on our results presented here, we find that this principle is that *phase-space tunneling is quantum mechanically as well as classically forbidden*. As far as we can ascertain, this is a new principle and has not been discussed previously; but then again, until now there has been no need to be concerned about violating this principle. In nonrelativistic and special-relativistic theories, one had only the simplest of possible examples of forbidden or cyclic phase-space regions (such as a particle in a box, rigid rotators, etc.), and it was obvious how these systems should be quantized. What made these systems so simple was that the appropriate boundaries in phase space were a function of only one variable, and were *not* a function of both $p$'s and $q$'s, as is the case in general relativity [see Eq. (1.2)]. In these previous cases, the phase of the wave function would satisfy the Hamilton-Jacobi equation (in the limit of $\hbar \to 0$) and therefore be stationary along the classical path by simply replacing the $p$'s and $q$'s by operators. Thus, the correspondence principle would be satisfied. But, in general relativity, there is an additional difficulty in that the classical path can actually "touch" a phase-space boundary (in the Friedmann universe, this is the point of maximum expansion). Now, simply replacing the $p$'s and $q$'s by the usual operators is no longer adequate because the stationary point of the phase of the wave function will then touch, and unless care is taken could pass through, the boundary into a forbidden region of phase space. When this happens, no longer will the correspondence principle be satisfied, and the quantum path will deviate wildly from the classical path.

At the present time, the most practical solution to this difficulty seems to be to find canonical transformation which will "unconstrain" the original phase space by mapping it into another, which has a more practical topology. How this can be done in the general case of general relativity has recently been treated by York.[2]

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[1]Bryce S. DeWitt, Rev. Mod. Phys. 29, 377 (1957).

[2]It must be mentioned that this is exactly the same idea being developed in more general cases by York and Kuchař. See in particular James W. York, Jr., Phys. Rev. Lett. 26, 1656 (1971); 28, 1082 (1972); J. Math. Phys. 13, 125 (1972); 14, 456 (1973); Karel Kuchař, Phys. Rev. D 4, 955 (1971); J. Math. Phys. 13, 768 (1972).

[3]By defining $g_{44} \equiv (2\pi^2 R^3 \alpha)^2$, we are either (1) simply performing a conformal transformation on our minisuperspace [C. W. Misner, in *Magic Without Magic: John Archibald Wheeler, a Collection of Essays in Honor of His 60th Birthday*, edited by J. Klauder (Freeman, San Francisco, 1972)], or (2) performing an $\alpha$ transformation on our constraint [David J. Kaup, Gen. Relativ. Gravit. 2, 247 (1971)].

[4]C. W. Misner, in *Relativity*, edited by M. Carmeli *et al.*, (Plenum, New York, 1970), p. 58.

[5]P. A. M. Dirac, Can. J. Phys. 2, 129 (1950).

[6]It should be noted that except for interpretation and numerical factors, this Hamiltonian is exactly the same as that for the Kantowski-Sachs universe. (See reference in footnote 3.) This universe has been studied by L. Fishbone (unpublished) using the Arnowitt-Deser-Misner procedure, and his results are discussed by M. Ryan, in *Hamiltonian Cosmology, Lecture Notes in Physics, No. 13*, edited by J. Ehlers *et al.* (Springer, New York, 1972). However, Fishbone's treatment of the quantum problem is restricted to the zero-norm case and no attempt at interpretation is made.

[7]Note the sign chosen for the operator $P_\Omega$. In this paper, we shall always change the sign of the conjugate momentum of the timelike variable so that it will be intrinsically positive, not negative.

[8]In the case of a minisuperspace constraint, a very simple quantization procedure has been given by Misner (see footnote 3), which gives the same result in this limit as the more general case treated by Kaup (see footnote 3).

[9]*Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), p. 374.

[10]Herman Feshbach and Felix Villars, Rev. Mod. Phys. 30, 24 (1958).

[11]Cf. C. W. Misner, in *Magic Without Magic: John Archibald Wheeler, a Collection of Essays in Honor of His 60th Birthday*, edited by J. Klauder (Freeman, San Francisco, 1972).

[12]Upon inserting the proper operators, Eq. (2.19) becomes the Schrödinger equation. See also footnote 7.

[13]Bryce S. DeWitt, Phys. Rev. 160, 1113 (1967).

[14]M. Ryan, in *Hamiltonian Cosmology, Lecture Notes in Physics, No. 13*, edited by J. Ehlers *et al.* (Springer, New York, 1972).