Gravitational Slavnov-Ward identities

D. M. Capper and M. Ramón Medrano*
International Centre for Theoretical Physics, Trieste, Italy
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Functional techniques are used to derive the general Slavnov-Ward identities for the quantized gravitational field. These identities are verified to second order in the coupling constant, employing the technique of dimensional regularization.

I. INTRODUCTION

Despite much theoretical work on quantum gravity,[1-3] there have been only a few brief considerations of the gravitational Ward identities,[1,4] although such identities are an essential step in deciding if the theory is renormalizable.

In this paper we derive a general formula containing all the gravitational Slavnov-Ward identities and indicate how the individual identities may be extracted in a simple way. We then use the results of a previous paper[5] (henceforth referred to as I) to show that the lowest-order graviton and fictitious-particle self-energy and vertex parts are consistent with these identities.

In order to verify these Slavnov-Ward identities, we employ the technique of dimensional regularization, which is particularly useful for gauge theories.[6,7]

II. THE GENERAL GRAVITATIONAL SLAVNOV-WARD IDENTITIES

The Feynman rules are obtained by considering the generating functional

$$Z[j_{\mu\nu}]=\int d[\tilde g^{\mu\nu}]\Delta[\tilde g^{\mu\nu}]\exp\left\{i\int dx\left[\mathfrak{L}+\frac{1}{K}\tilde g^{\mu\nu}j_{\mu\nu}-\frac{1}{K^2\alpha}\partial_\mu\tilde g^{\mu\nu}\partial_\alpha\tilde g^{\alpha\beta}\delta_{\nu\beta}\right]\right\},\tag{2.1}$$

whose origin is explained in detail, together with the notation, in I. We note here that Δ can be interpreted in terms of fictitious particles and is given by

$$(\Delta[\tilde g^{\mu\nu}]^{-1})=\int d[\xi_\lambda]d[\eta_\nu]\exp\left\{i\int dx\,\eta_\nu[\delta_{\nu\lambda}\Box-K(\phi_{\mu\nu,\lambda\mu}-\phi_{\mu\rho}\delta_{\nu\lambda}\partial_\mu\partial_\rho-\phi_{\mu\rho,\mu}\delta_{\nu\lambda}\partial_\rho+\phi_{\mu\nu,\mu}\partial_\lambda)]\xi_\lambda\right\},\tag{2.2}$$

where we have defined the graviton field $\phi^{\alpha\beta}$ by

$$\tilde g^{\alpha\beta}\equiv\delta^{\alpha\beta}+K\phi^{\alpha\beta}.\tag{2.3}$$

The indices on $\phi^{\alpha\beta}$ are raised and lowered by means of the flat-space metric $\delta_{\alpha\beta}$. $\mathfrak{L}$ is the pure gravitational Lagrangian which, in an n-dimensional space, is given (in terms of the weight-one tensor density $\tilde g^{\alpha\beta}$) by

$$\mathfrak{L}=\frac{1}{2K^2}\left(\tilde g^{\rho\sigma}\tilde g_{\lambda\mu}\tilde g_{\kappa\nu}-\frac{1}{n-2}\tilde g^{\rho\sigma}\tilde g_{\mu\kappa}\tilde g_{\lambda\nu}-2\delta^\sigma_\kappa\delta^\rho_\lambda\tilde g_{\mu\nu}\right)\tilde g^{\mu\kappa}{}_{,\rho}\tilde g^{\lambda\nu}{}_{,\sigma}.\tag{2.4}$$

We now derive the general Ward identities, following the technique of Slavnov,[8] by considering the effect of an infinitesimal gauge transformation,

$$x'^\mu=x^\mu+\xi^\mu(x,\phi),\tag{2.5}$$

on the generating functional $Z[j_{\mu\nu}]$ of Eq. (2.1), where, in general, ξ^μ can be a functional of $\phi^{\alpha\beta}$ as well as a function of x. The gauge transformation (2.5) induces the following change in $\tilde g^{\mu\nu}(x)$:

$$\delta\tilde g^{\mu\nu}(x)=-\xi^\lambda(x)\partial_\lambda\tilde g^{\mu\nu}(x)+\xi^\mu{}_{,\rho}(x)\tilde g^{\rho\nu}(x)+\xi^\nu{}_{,\sigma}(x)\tilde g^{\mu\sigma}(x)-\xi^\alpha{}_{,\alpha}(x)\tilde g^{\mu\nu}(x).\tag{2.6}$$

The functional $Z[j_{\mu\nu}]$ is invariant under changes in the integration variable and hence, in terms of the field $\phi_{\alpha\beta}$ introduced in Eq. (2.3), we obtain

$$\delta Z=i\int dx\,d[\phi_{\alpha\beta}]\,\bar Z[\phi_{\alpha\beta},j_{\mu\nu}]\times\left\{j_{\mu\nu}(x)\delta\phi_{\mu\nu}(x)-\frac{2}{\alpha}\partial_\lambda\phi_{\lambda\beta}(x)\delta(\partial_\mu\phi_{\mu\beta}(x))+[J+\delta\Delta]\right\}=0,\tag{2.7}$$

where

$$\bar Z[\phi_{\alpha\beta},j_{\mu\nu}]=\Delta\exp\left\{i\int dx\left[\mathfrak{L}+\frac{1}{K}\tilde g^{\mu\nu}j_{\mu\nu}-\frac{1}{K^2\alpha}\partial_\mu\tilde g^{\mu\nu}\partial_\alpha\tilde g^{\alpha\beta}\delta_{\nu\beta}\right]\right\},\tag{2.8}$$

and J is the Jacobian of the transformation (2.6). Using Eqs. (2.3) and (2.6), we obtain

$$\delta\phi_{\mu\nu}(x) = -\xi_\lambda\phi_{\mu\nu,\lambda} + \xi_{\mu,\rho}\phi_{\rho\nu} + \xi_{\nu,\rho}\phi_{\mu\rho} - \xi_{\rho,\rho}\phi_{\mu\nu} + K^{-1}(\xi_{\mu,\nu} + \xi_{\nu,\mu} - \xi_{\rho,\rho}\delta_{\mu\nu}) \quad (2.9)$$

and

$$\delta(\partial_\mu\phi_{\mu\nu}(x)) = -\xi_\lambda\phi_{\mu\nu,\lambda\mu} + \xi_{\nu,\rho\mu}\phi_{\mu\rho} + \xi_{\nu,\rho}\phi_{\mu\rho,\mu} - \xi_{\rho,\rho}\phi_{\mu\nu,\mu} + K^{-1}\xi_{\nu,\mu\mu} \quad (2.10)$$

or, more compactly,

$$\delta\phi_{\mu\nu}(x) = A_{\mu\nu\lambda}(x)\xi_\lambda(x) \quad (2.11)$$

and

$$\delta(\partial_\mu\phi_{\mu\nu}(x)) = M_{\nu\lambda}(x)\xi_\lambda(x) \equiv \chi_\nu(x), \quad (2.12)$$

where the new operators $A_{\mu\nu\lambda}$ and $M_{\nu\lambda}$ are defined implicitly through Eqs. (2.9) to (2.12). Equation (2.12) leads to the following solution for $\xi_\lambda(x)$:

$$\xi_\lambda(x) = \int N_{\lambda\nu}(x,y)\chi_\nu(y)dy, \quad (2.13)$$

where $N_{\lambda\nu}$ is defined by the *operator* relation

$$M_{\alpha\lambda}(x)N_{\lambda\nu}(x,y) = \delta_{\alpha\nu}\,\delta(x-y). \quad (2.14)$$

We now consider $\chi_\nu(y)$ as an arbitrary function and hence ξ_λ is implicitly a functional of $\phi_{\alpha\beta}$. It is important to know the effect of such a nonlinear transformation on the integration measure $d[\tilde{g}]$ and the fictitious-particle functional Δ. A direct calculation (see Appendix A) shows that the product $d[\tilde{g}]\Delta$ is invariant, as in the Yang-Mills case.[8] Equations (2.11) to (2.13) now enable us to express the variation of the generating functional Z in the convenient form

$$\delta Z = i\int d[\phi_{\alpha\beta}]dx\,dy\,\bar{Z}[\phi_{\alpha\beta}, j_{\mu\nu}] \times\left[j_{\mu\nu}(x)A_{\mu\nu\lambda}(x)N_{\lambda\beta}(x,y) - \frac{2}{\alpha}\delta(x-y)\partial_\lambda\phi_{\lambda\beta}(x)\right]\chi_\beta(y) = 0. \quad (2.15)$$

Since $\chi_\beta(y)$ can be considered an arbitrary function of y, Eq. (2.15) leads immediately to the following formal expression for the generalized Slavnov-Ward identities:

$$\int dx\,d[\phi_{\alpha\beta}]\,\bar{Z}[\phi_{\alpha\beta}, j_{\mu\nu}] \times\left\{j_{\mu\nu}(x)A_{\mu\nu\lambda}(x)N_{\lambda\beta}(x,y) - \frac{2}{\alpha}\delta(x-y)\partial_\lambda\phi_{\lambda\beta}(x)\right\} = 0. \quad (2.16)$$

To extract the Slavnov-Ward identities for the individual Green's functions, we take the appropriate number of functional derivatives with respect to $j_{\mu\nu}$ and then put $j_{\mu\nu}=0$. The simplest identity is

$$\langle\partial_\lambda\phi_{\lambda\beta}\rangle = 0, \quad (2.17)$$

which is just a statement of translation invariance. Another identity is obtained by taking the functional derivative $\delta/\delta j_{\mu\nu}$ of Eq. (2.16) and putting $j_{\mu\nu}=0$. We thus obtain [using also Eq. (2.17)]

$$\frac{2i}{\alpha}\langle T(\phi_{\mu\nu}(z)\phi_{\lambda\beta,\lambda}(y))\rangle - \langle T(A_{\mu\nu\lambda}(z)N_{\lambda\beta}(z,y))\rangle = 0, \quad (2.18)$$

where T is the covariant time-ordering operator. The Slavnov-Ward identity (2.18) imposes conditions on the total graviton propagator, but for it to be of any use we need to know $N_{\lambda\beta}$. In principle, $N_{\lambda\beta}$ can be determined from Eqs. (2.14), (2.12), and (2.10) as an infinite nonlocal power series in the fields $\phi_{\alpha\beta}$, but this is both unnecessarily complicated and obscures the diagrammatic interpretation. Instead, we employ the fact that the operator $M_{\alpha\lambda}(x)$ [Eqs. (2.10) and (2.12)] is the same as that appearing in the fictitious-particle functional [Eq. (2.2)]. If we make the replacement (see Appendix B)

$$N_{\mu\lambda}(x,y) \to -iK\xi_\mu(x)\eta_\lambda(y) \quad (2.19)$$

whenever $N_{\mu\lambda}$ appears *inside* functional integrals, then it is readily verified that this is consistent with the definition (2.14). To prove this we consider

$$\Delta N_{\mu\lambda}(x,y) \equiv -iK\int d[\eta_\alpha]d[\xi_\beta]\,\xi_\mu(x)\eta_\lambda(y) \times\exp\left\{iK\int\eta_\gamma(z)M_{\gamma\beta}(z)\xi_\beta(z)dz\right\}. \quad (2.20)$$

Since $M_{\alpha\mu}(x)$ is an operator only dependent on $\phi_{\nu\lambda}(x)$, we may write

$$\Delta M_{\alpha\mu}(x)N_{\mu\lambda}(x,y)$$
$$= -iK\int d[\eta_\alpha]d[\xi_\beta]\eta_\lambda(y)M_{\alpha\mu}(x)\xi_\mu(x) \times \exp\left\{iK\int \eta_\gamma(z)M_{\gamma\beta}(z)\xi_\beta(z)dz\right\}$$
$$= -\int d[\eta_\alpha]d[\xi_\beta]\eta_\lambda(y)\frac{\delta}{\delta\eta_\alpha(x)} \times \exp\left\{iK\int \eta_\gamma(z)M_{\gamma\beta}(z)\xi_\beta(z)dz\right\}, \tag{2.21}$$

which, using integration by parts, reduces to

$$\int d[\eta_\alpha]d[\xi_\beta]\delta_{\alpha\lambda}\delta(y-x) \times \exp\left\{iK\int \eta_\gamma(z)M_{\gamma\beta}(z)\xi_\beta(z)dz\right\} = \Delta[\tilde{g}^{\alpha\beta}]. \tag{2.22}$$

This completes our proof.

We must emphasize here that the two fictitious particles are not equivalent and ξ_μ and η_λ cannot be interchanged in Eq. (2.19). Having expressed $N_{\mu\lambda}$ in terms of the fictitious-particle fields, the Slavnov-Ward identity (2.18) can be interpreted as giving conditions on various graviton and fictitious-particle Green's functions.

III. VERIFICATION OF THE SLAVNOV-WARD IDENTITY

We shall now restrict our attention to the Slavnov-Ward identity given in Eq. (2.18). We first quote the appropriate *Euclidean* space Feynman rules which were given in I. We employ a fictitious-particle ξ-η propagator given, in momentum space, by

$$\langle T\xi_\lambda\eta_\mu\rangle = \frac{\delta_{\lambda\mu}}{p^2} \tag{3.1}$$

and a graviton η-ξ vertex given by (see Fig. 1)

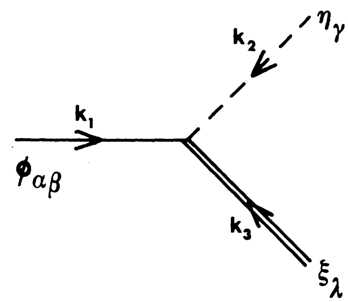

FIG. 1. Graviton–fictitious-particle vertex. The two different fictitious particles ξ and η have momentum labels k_3 and k_2, and polarization labels λ and γ, respectively.

$$V_{\alpha\beta,\gamma,\lambda}(k_1,k_2,k_3) = K\{-\delta_{\gamma(\alpha}k_{1\beta)}k_{2\lambda} + \delta_{\gamma\lambda}k_{2(\alpha}k_{3\beta)}\}, \tag{3.2}$$

where we have defined

$$A_{(\alpha}B_{\beta)} = \tfrac{1}{2}(A_\alpha B_\beta + A_\beta B_\alpha). \tag{3.3}$$

If we choose the gauge parameter $\alpha=-1$ in the functional (2.1), the graviton propagator is given by

$$D_{\alpha\beta,\lambda\mu}(p) = \frac{1}{2p^2}(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\beta\lambda}\delta_{\alpha\mu} - \delta_{\alpha\beta}\delta_{\lambda\mu}). \tag{3.4}$$

Using Eqs. (2.9) and (2.19), we may write Eq. (2.18) as

$$\frac{2}{\alpha}\frac{\partial}{\partial y_\lambda}\langle T\phi_{\mu\nu}(x)\phi_{\lambda\gamma}(y)\rangle = -\left(\delta_{\lambda\mu}\frac{\partial}{\partial x_\nu} + \delta_{\lambda\nu}\frac{\partial}{\partial x_\mu} - \delta_{\mu\nu}\partial_\lambda\right)\langle T\xi_\lambda(x)\eta_\gamma(y)\rangle - K\left\langle T\left[-\phi_{\mu\lambda,\lambda}(x) + \phi_{\rho\nu}(x)\delta_{\mu\lambda}\frac{\partial}{\partial x_\rho} + \phi_{\mu\rho}(x)\delta_{\nu\lambda}\frac{\partial}{\partial x_\rho} - \phi_{\mu\nu}(x)\frac{\partial}{\partial x_\lambda}\right]\xi_\lambda(x)\eta_\gamma(y)\right\rangle. \tag{3.5}$$

This Slavnov-Ward identity (3.5) is now directly interpretable in terms of Feynman diagrams. On the left-hand side $\langle T\phi_{\mu\nu}(x)\phi_{\lambda\gamma}(y)\rangle$ is the total graviton propagator, while the first term on the right-hand side $\langle T\xi_\lambda(x)\eta_\gamma(y)\rangle$ is the total fictitious-particle propagator. The remaining term is a fully corrected graviton $-\xi$-η "vertex," with the graviton and ξ lines going to the same point and various derivatives acting on internal lines. (See Fig. 2 for the lowest-order contribution; the notation is explained in Fig. 1.)

A. The Slavnov-Ward identity to order K^0

To lowest order the Slavnov-Ward identity (3.5) merely relates the bare propagators. For $\alpha=-1$ Eq. (3.5) then reads (although it is straightforward to verify this equation for general α)

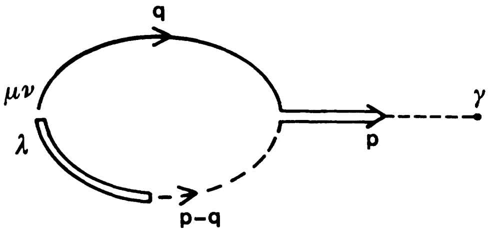

FIG. 2. The lowest-order "vertex" contribution to $\langle TA_{\mu\nu\lambda}N_{\lambda\gamma}\rangle$.

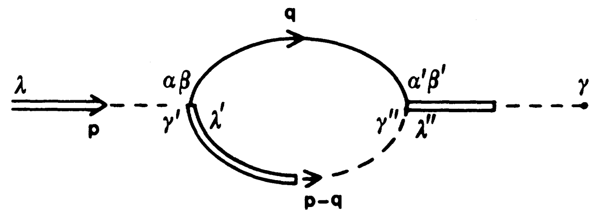

FIG. 3. Lowest-order correction to the fictitious-particle propagator.

$$2\frac{\partial}{\partial y_\lambda}D_{\mu\nu\lambda\gamma}(x-y)=\left[\frac{1}{K}\left(\delta_{\lambda\mu}\frac{\partial}{\partial x_\nu}+\delta_{\lambda\nu}\frac{\partial}{\partial x_\mu}-\delta_{\mu\nu}\frac{\partial}{\partial x_\lambda}\right)\right][K\delta_{\lambda\gamma}D(x-y)], \tag{3.6}$$

where $D(x)$ and $D_{\mu\nu\lambda\gamma}$ are the x-space scalar and spin-two massless propagators, respectively. Converting Eq. (3.6) to p space, we obtain

$$2p_\lambda D_{\mu\nu\lambda\gamma}(p)=(\delta_{\gamma\mu}p_\nu+\delta_{\gamma\nu}p_\mu-\delta_{\mu\nu}p_\gamma)D(p). \tag{3.7}$$

From Eqs. (3.1) and (3.4) we see that (3.7) is indeed satisfied by our Feynman rules (3.1)–(3.4).

B. The Slavnov-Ward identity to order K^2

As a more interesting example, consider the contributions of order K^2 to Eq. (3.5) (see Figs. 2 and 3). In order to handle the divergent Feynman integrals which arise from the loops, we employ the technique of dimensional regularization[6] as it is particularly useful for gauge theories. We refer the reader to Refs. 5, 6, and 7 for details of this technique, and merely note here that the basic idea is to work in a space-time of 2ω dimensions, where ω is a *complex* regulating parameter. In I we applied this technique to the order-K^2 contributions to the graviton propagator [which we define as $Q_{\mu\nu\lambda\gamma}(p)$] (see Fig. 4) and obtained

$$p_\lambda Q_{\mu\nu\lambda\gamma}(p)=\frac{1}{4(p^2)^2}\{p_\mu p_\nu p_\gamma[2p^2T_1-4(\omega-1)T_4]+\delta_{\mu\nu}p_\gamma[-(p^2)^2T_1+4(\omega-1)^2T_2+4(\omega-2)T_3-4p^2T_5]$$
$$+[\delta_{\gamma\nu}p_\mu+\delta_{\gamma\mu}p_\nu][4T_3+4p^2T_5]\}, \tag{3.8}$$

where

$$T_1=[8(4\omega^2-1)]^{-1}(+2\omega^4-5\omega^3+35\omega^2+16\omega)I_1, \tag{3.9a}$$

$$T_2=[32(\omega-1)^2(4\omega^2-1)]^{-1}(-14\omega^4-7\omega^3+36\omega^2+9\omega)(p^2)^2I_1, \tag{3.9b}$$

$$T_3=[32(4\omega^2-1)]^{-1}(+16\omega^3+18\omega^2-15\omega-8)(p^2)^2I_1, \tag{3.9c}$$

$$T_4=[32(\omega-1)(4\omega^2-1)]^{-1}(+4\omega^4-10\omega^3+38\omega^2+32\omega+8)p^2I_1, \tag{3.9d}$$

$$T_5=[32(4\omega^2-1)]^{-1}(-16\omega^3-18\omega^2+15\omega+8)p^2I_1, \tag{3.9e}$$

and

$$I_1=\int\frac{d^{2\omega}q}{q^2(p-q)^2}$$
$$=\frac{\pi^\omega\Gamma(2-\omega)\Gamma(\omega-1)\Gamma(\omega-1)}{\Gamma(2\omega-2)}(p^2)^{\omega-2}. \tag{3.10}$$

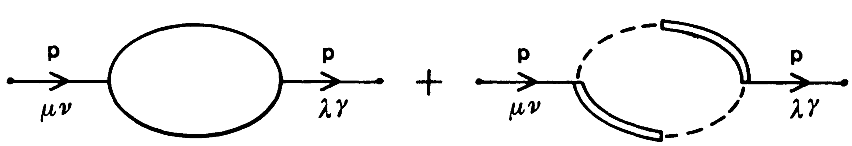

FIG. 4. Lowest-order corrections to the graviton propagator.

The $O(K^2)$ contribution to $\langle T\xi_\lambda\eta_\gamma\rangle$ is shown in Fig. 3 and is given in momentum space by [see Eqs. (3.1) to (3.4)]

$$\mathcal{F}_{\lambda\gamma}(p)=\int dq\, V_{\alpha\beta,\gamma',\lambda'}(-q,p,q-p)D_{\alpha\beta,\alpha'\beta'}(q)\frac{\delta_{\lambda'\gamma''}}{(p-q)^2}V_{\alpha'\beta',\gamma'',\lambda''}(q,p-q,-p)\frac{\delta_{\lambda\gamma'}}{p^2}\frac{\delta_{\lambda''\gamma}}{p^2}. \tag{3.11}$$

Using precisely the same techniques[5] as in I, one readily obtains

$$\mathcal{F}_{\lambda\gamma}(p)=\frac{-1}{p^2}\left\{p_\lambda p_\gamma\frac{(\omega-1)}{4(2\omega-1)}+p^2g_{\lambda\gamma}\frac{1}{8(2\omega-1)}\right\}I_1, \tag{3.12}$$

and hence

$$[\delta_{\lambda\mu}p_\nu+\delta_{\lambda\nu}p_\mu-\delta_{\mu\nu}p_\lambda]\mathcal{F}_{\lambda\gamma}(p)=\frac{-1}{p^2}\left\{p_\mu p_\nu p_\gamma\frac{(\omega-1)}{2(2\omega-1)}+p^2\delta_{\mu\nu}p_\gamma[-\tfrac{1}{8}]+p^2[\delta_{\gamma\nu}p_\mu+\delta_{\gamma\mu}p_\nu]\left[\frac{1}{8(2\omega-1)}\right]\right\}I_1. \tag{3.13}$$

We have thus found the first term on the right-hand side of Eq. (3.5) to order K^2. The remaining term of (3.5) is

$$W_{\mu\nu\gamma}(x-y)\equiv\left\langle T\left[-\phi_{\mu\nu,\lambda}(x)+\phi_{\rho\nu}(x)\delta_{\mu\lambda}\frac{\partial}{\partial x_\rho}+\phi_{\mu\rho}(x)\delta_{\nu\lambda}\frac{\partial}{\partial x_\rho}-\phi_{\mu\nu}(x)\frac{\partial}{\partial x_\lambda}\right]\xi_\lambda(x)\eta_\gamma(y)\right\rangle. \tag{3.14}$$

In a similar way (see Fig. 2) this gives in momentum space (to order K^2)

$$W_{\mu\nu\gamma}(p)=\frac{-1}{p^2}\left\{p_\mu p_\nu p_\gamma\frac{(2-3\omega)}{2(2\omega-1)}+p^2\delta_{\mu\nu}p_\gamma[\tfrac{5}{8}]+p^2[\delta_{\gamma\nu}p_\mu+\delta_{\gamma\mu}p_\nu]\left[\frac{-1}{8(2\omega-1)}\right]\right\}I_1. \tag{3.15}$$

Using Eqs. (3.8), (3.13), and (3.15) we can thus write Eq. (3.5) in momentum space as

$$\frac{1}{2(p^2)^2}\{p_\mu p_\nu p_\gamma[2p^2T_1-4(\omega-1)T_4]+\delta_{\mu\nu}p_\gamma[-(p^2)^2T_1+4(\omega-1)^2T_2+4(\omega-2)T_3-4p^2T_5]+[\delta_{\gamma\nu}p_\mu+\delta_{\gamma\mu}p_\nu][4T_3+4p^2T_5]\}$$
$$=[\delta_{\lambda\mu}p_\nu+\delta_{\lambda\nu}p_\mu-\delta_{\mu\nu}p_\lambda]\mathcal{F}_{\lambda\gamma}(p)+W_{\mu\nu\gamma}(p). \tag{3.16}$$

From Eqs. (3.13) and (3.15) we obtain

$$T_3=-p^2T_5, \tag{3.17}$$

$$2p^2T_1-4(\omega-1)T_4=p^2I_1, \tag{3.18}$$

$$-(p^2)^2T_1+4(\omega-1)^2T_2+4(\omega-2)T_3-4p^2T_5=-(p^2)^2I_1. \tag{3.19}$$

Eliminating I_1 from Eqs. (3.17) to (3.19), we find

$$(p^2)^2T_1+4(\omega-1)^2T_2+4(\omega-1)(T_3-p^2T_4)=0. \tag{3.20}$$

Hence these Slavnov-Ward identities are consistent with the more restricted identities[8] [i.e., Eqs. (3.17) and (3.20)] obtained and verified in I. Moreover, directly substituting the expressions for T_1 to T_5 [Eqs. (3.9)] into Eqs. (3.17) to (3.19), it can be verified that all the Slavnov-Ward identities to order K^2 are indeed satisfied by the various graviton and fictitious-particle one-loop diagrams given in Figs. 2, 3, and 4.

IV. CONCLUSION

We have derived the gravitational Slavnov-Ward identities and shown in general how to give a diagrammatic interpretation. Furthermore, by employing the technique of dimensional regularization we have been able to verify these identities to order K^2. The gravitational Slavnov-Ward identities derived here are stronger than those identities previously derived in I. It is interesting to note that some of the diagrams which are needed to verify these identities [arising from $A_{\mu\nu\lambda}(z)N_{\lambda\beta}(z,p)$] do not appear in some of the "simplest" calculations such as graviton-graviton scattering.

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APPENDIX A

The fictitious-particle factor Δ, as used in actual calculations, is not gauge-invariant since the integration is not performed over the entire gauge group. Also the integration measure $d[\tilde{g}^{\mu\nu}]$ is only invariant with respect to c-number gauge transformations. However, in this appendix we show that, as in the Yang-Mills case,[8] the combination $d[\tilde{g}]\Delta$ is invariant with respect to the particular nonlinear gauge transformations given by Eq. (2.12).

We employ the notation of DeWitt[9] in that we sum over repeated indices (μ, a, etc.) and integrate over repeated space-time points (indicated by $'$, $''$, $\hat{}$, $\bar{}$, etc.). Latin letters are used to represent an index pair [e.g., $a=(\mu\nu)$].[10] Equation (2.6) can thus be rewritten

$$\delta\tilde{g}^{a}=G^{a}{}_{b'\lambda''}\tilde{g}^{b'}\xi^{\lambda''}, \tag{A1}$$

where

$$\begin{aligned}G^{(\mu\nu)}{}_{(\alpha'\beta')\lambda''}=[&\delta^{\mu}_{\lambda}\delta^{\rho\nu}_{\alpha\beta}\delta(x-x')\delta_{,\rho}(x-x'')\\&+\delta^{\nu}_{\lambda}\delta^{\rho\mu}_{\alpha\beta}\delta(x-x')\delta_{,\rho}(x-x'')\\&-\delta^{\mu\nu}_{\alpha\beta}\delta(x-x')\delta_{,\lambda}(x-x'')\\&-\delta^{\mu\nu}_{\alpha\beta}\delta_{,\lambda}(x-x')\delta(x-x'')].\end{aligned} \tag{A2}$$

From Eqs. (2.12) to (2.14) we also have

$$M^{\nu}{}_{\rho''}=\partial_{\mu}G^{\mu\nu}{}_{b'\rho''}\tilde{g}^{b'}, \tag{A3}$$

$$M^{\nu}{}_{\lambda'}N^{\lambda'}{}_{\mu''}=N^{\nu}{}_{\lambda'}M^{\lambda'}{}_{\mu''}=\delta^{\nu}{}_{\mu''}, \tag{A4}$$

$$\xi^{\lambda''}=N^{\lambda''}{}_{\gamma'''}\chi^{\gamma'''}. \tag{A5}$$

(i) The Jacobian. The Jacobian J of the transformation (A1) is given by

$$\begin{aligned}J&=\operatorname{Det}\frac{\delta}{\delta\tilde{g}^{\bar{b}}}(\tilde{g}^{a}+\delta\tilde{g}^{a})\\&\approx\operatorname{Tr}\left[G^{a}{}_{b'\lambda''}\frac{\delta\tilde{g}^{b'}}{\delta\tilde{g}^{\bar{b}}}\xi^{\lambda''}+G^{a}{}_{b'\lambda''}\tilde{g}^{b'}\frac{\delta\xi^{\lambda''}}{\delta\tilde{g}^{\bar{b}}}\right].\end{aligned} \tag{A6}$$

The trace $G^{a}{}_{a\lambda''}$ is formally zero and employing Eqs. (A3) to (A5), we have

$$\frac{\delta\xi^{\lambda''}}{\delta\tilde{g}^{\bar{b}}}=\frac{\delta N^{\lambda''}{}_{\gamma'''}}{\delta\tilde{g}^{\bar{b}}}\chi^{\gamma'''} \tag{A7}$$

and

$$\frac{\delta N^{\lambda''}{}_{\gamma'''}}{\delta\tilde{g}^{\bar{b}}}=-N^{\lambda''}{}_{\hat{\rho}}\frac{\delta M^{\hat{\rho}}{}_{\beta}}{\delta\tilde{g}^{\bar{b}}}N^{\beta}{}_{\gamma'''}, \tag{A8}$$

$$\frac{\delta M^{\hat{\rho}}{}_{\bar{\beta}}}{\delta\tilde{g}^{\bar{b}}}=\partial_{\hat{\mu}}G^{\hat{\mu}\hat{\rho}}{}_{\bar{b}\bar{\beta}}, \tag{A9}$$

where there is *no* integration over $\hat{x}$ in Eq. (A9). Hence

$$J=-G^{\bar{b}}{}_{b'\lambda''}\partial_{\hat{\mu}}G^{\hat{\mu}\hat{\rho}}{}_{\bar{b}\bar{\beta}}[\tilde{g}^{b'}N^{\lambda''}{}_{\hat{\rho}}\xi^{\bar{\beta}}]. \tag{A10}$$

(ii) The variation in Δ. From Eq. (2.2) (see also Ref. 5), Δ is given by the functional determinant

$$\Delta=\operatorname{Det}M. \tag{A11}$$

Hence, if M undergoes a transformation parametrized by χ [Eq. (2.12)], then

$$\begin{aligned}\delta\Delta&=\operatorname{Tr}M(\chi)M^{-1}\\&=\operatorname{Tr}\frac{\delta M^{\nu}{}_{\lambda'}}{\delta g^{\bar{b}}}\delta\tilde{g}^{\bar{b}}N^{\lambda'}{}_{\beta'''}.\end{aligned} \tag{A12}$$

Using Eqs. (A1) and (A9) we thus obtain

$$\delta\Delta=G^{\bar{b}}{}_{b'\bar{\beta}}\partial_{\hat{\mu}}G^{\hat{\mu}\hat{\rho}}{}_{\bar{b}\lambda''}[\tilde{g}^{b'}N^{\lambda''}{}_{\hat{\rho}}\xi^{\bar{\beta}}]. \tag{A13}$$

(iii) The invariance of $d[\tilde{g}]\Delta$. From Eqs. (A10) and (A13) we have that the variation in $d[\tilde{g}]\Delta$ is given by

$$\begin{aligned}I&\equiv J+\delta\Delta\\&=[\tilde{g}^{b'}N^{\lambda''}{}_{\hat{\rho}}\xi^{\bar{\beta}}]\partial_{\hat{\mu}}\{G^{(\hat{\mu}\hat{\rho})}{}_{\bar{b}\lambda''}G^{\bar{b}}{}_{b'\bar{\beta}}\\&\qquad-G^{(\hat{\mu}\hat{\rho})}{}_{\bar{b}\bar{\beta}}G^{\bar{b}}{}_{b'\lambda''}\}.\end{aligned} \tag{A14}$$

But $G^{a}{}_{b'\lambda''}$ is a matrix element of the generator $G_{\lambda''}$ of the group and satisfies the commutation relation (see Ref. 9)

$$[G_{\mu},G_{\nu'}]=G_{\sigma''}C^{\sigma''}{}_{\mu\nu'}. \tag{A15}$$

$C^{\sigma''}{}_{\mu\nu'}$ in Eq. (A15) is the structure constant of the group and, in fact, is given by[9]

$$\begin{aligned}C^{\mu}{}_{\nu'\sigma''}=&\,\delta^{\mu}{}_{\nu}\delta_{,\sigma}(x-x')\delta(x-x'')\\&-\delta^{\mu}{}_{\sigma}\delta_{,\nu}(x-x'')\delta(x-x').\end{aligned} \tag{A16}$$

Employing Eqs. (A3), (A4), (A14), and (A15), we finally obtain

$$I=C^{\lambda''}{}_{\lambda''\beta}\xi^{\beta}. \tag{A17}$$

However, the trace of the structure constant is formally zero and thus the variation in the fictitious-particle term is exactly canceled by the Jacobian for the transformations of Eq. (A1). Clearly this proof is not limited to gravity or the Yang-Mills fields and can readily be extended to any theory involving a similar kind of gauge-breaking term.

Finally, we must emphasize that the proof is purely formal, in that it involves products of derivatives of δ functions, and is only strictly valid in the context of some regularizing technique.[6,7]

APPENDIX B

In this appendix we briefly outline how a similar calculation to order K^2 verifies that interpreting $N_{\mu\lambda}(x,y)$ as $-iK\xi_\mu(x)\eta_\lambda(y)$ is indeed correct. We simply write

$$\begin{aligned}\langle TM_{\alpha\lambda}(\phi(x))N_{\lambda\beta}(x,y)\rangle &= -iK\langle TM_{\alpha\lambda}(\phi(x))\xi_\lambda(x)\eta_\beta(y)\rangle \\ &= \delta_{\alpha\beta}\delta(x-y) + \frac{\partial^2}{\partial x^2}F_{\alpha\beta}(x-y) \\ &\quad -iK\langle T[-\phi_{\mu\alpha,\lambda\mu} + \delta_{\alpha\lambda}\phi_{\mu\rho}\partial_\mu\partial_\rho + \delta_{\alpha\lambda}\phi_{\mu\rho,\mu}\partial_\rho - \phi_{\mu\alpha,\mu}\partial_\lambda]\xi_\lambda(x)\eta_\beta(y)\rangle\,, \end{aligned} \tag{B1}$$

where $F_{\alpha\beta}(x-y)$ consists of all the higher-order corrections to the fictitious-particle propagator. To order K^2, $F_{\alpha\beta}(x-y)$ corresponds in momentum space to $\mathfrak{F}_{\alpha\beta}(p^2)$ [see Eq. (3.12) and Fig. 3]. In order to verify that $N_{\lambda\beta}(x,y)$ really is the right inverse of $M_{\alpha\lambda}(x)$, we merely have to show that the last two terms in Eq. (B1) cancel. We already have an expression for $\mathfrak{F}_{\alpha\beta}(p)$ [Eq. (3.12)] and the remaining term in Eq. (B1) can be evaluated in exactly the same way as that of Eq. (3.12). In this way (2.19) can be verified to order K^2.

*On leave of absence from Departamento de Física Teórica, Universidad Complutense de Madrid, Spain.

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[10]Note that in Appendix A we consistently use upper and lower indices, since we are concerned with the covariant object $\tilde{g}^{\mu\nu}$ rather than its noncovariant decomposition $\phi_{\mu\nu}$.