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Model-independent tests of the on-shell pion-nucleon Ward identities

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A dispersion-theoretic method is developed to determine the axial-vector background amplitudes appearing in the on-shell πN Ward identities. These current-algebra constraints are then found to be completely consistent with low-energy data and a nucleon σ term of 73 ± 21 MeV. All threshold s- and p-wave scattering lengths are correctly predicted by our model-independent approach.

I. INTRODUCTION

Up to now, the current algebra constraints on the on-mass-shell pion-nucleon scattering amplitude[1-4] have been tested in a model-dependent fashion. The Ward identities of axial-vector nucleon scattering have been saturated with the s- and u-channel nucleon and $\Delta(1231)$ field-theory[1,5] or "dispersion-theory"[3] poles. Alternatively, the phenomenological nonlinear (current algebra) Lagrangian has been approximated by s- and u-channel nucleon and $\Delta(1231)$ field-theory poles along with t-channel ρ and σ poles and contact terms.[2,6,7] Unfortunately the spin-$\frac{3}{2}$ field-theoretic propagator is not unique; likewise the corresponding dispersion-theory spin-$\frac{3}{2}$ projection operator has thus far been accompanied by the arbitrary and nonunique assumption of unsubtracted dispersion relations (USDR) for all axial-vector–nucleon invariant amplitudes.[3] The situation has been well summarized by Höhler, Jakob, and Strauss (HJS),[8] who compare the various models with the HJS low-energy phenomenological parameters extracted from experiment. In effect, the current-algebra Ward identities have been used to test these field-theoretic and dispersion-theoretic models.

In this paper we develop a dispersion-theoretic expansion of axial-vector–nucleon amplitudes which minimizes the number of USDR assumptions by using current-conserving covariants whenever possible and accounting for the resulting kinematic singularities thus induced. This method is an off-forward generalization of Bjorken's[9] technique for forward vector (or axial-vector) nucleon Compton scattering. The USDR assumption will be applied only to amplitudes which obviously vanish asymptotically in the lab energy variable ν; in this sense our approach is model-independent. Upon reexamining the πN Ward identities we conclude that the isotopic-odd Adler[10]-Weisberger[11] forward constraint as well as the spin-flip current-algebra constraint notably agree with experiment.

It is then natural to extract the nucleon σ term from the Ward identity for the isotopic-even forward amplitude. Here the background axial-vector amplitude can be shown to be small in a particular kinematic configuration[1] which we verify in our approach. However a new problem arises: to determine the exact low-energy experimental πN amplitude. Previously there has been a discrepancy between the σ-term determinations of Cheng and Dashen (CD)[12] and Altarelli, Cabibbo, and Maiani (ACM)[13] of $\sigma_N \sim 110$, 80 MeV with the much smaller value found by HJS[14] of $\sigma_N \sim 40$ MeV. We have reexamined the data analysis of HJS leading to their low-energy parameters and find that these are somewhat closer to the CD values than had been formerly claimed.[8,14] In fact we find that the HJS and ACM approaches are very similar; the former leading to the value of $\sigma_N = 73 \pm 21$ MeV. The final discrepancy of 30 MeV between our value and that of CD is due to an overemphasis of the low-energy contribution by their broad area subtraction technique, as explained by HJS.[14]

The threshold scattering lengths play a central role in determining the HJS low-energy parameters. We have found that the effect of the threshold cusp does not significantly alter the extrapolation of any of the two s-wave or four p-wave scattering lengths down to the symmetry point $\nu = t = 0$ or current-algebra (CD) point $\nu = q' \cdot q = 0$. Conversely we can solve for these scattering lengths in terms of the current-algebra Ward identities. Using our dispersion-theory model-independent method to determine the axial-vector–nucleon background, excellent agreement is found in all six cases. Previous model-dependent calculations yielded much too small a value for the isotopic-odd p-wave $J = \frac{1}{2}$ scattering length $a_{11} - a_{31}$.

We review the current-algebra constraints for πN scattering in Sec. II and apply the dispersion theory of axial-vector–nucleon scattering developed in the Appendix to resonance exchanges in Sec. III. In Sec. IV we discuss the HJS low-energy parameters obtained from scattering-length data and once-subtracted dispersion relations, and in Sec. V we test the πN Ward identities against these parameters while finding $\sigma_N = 73 \pm 21$ MeV.

We conclude in Sec. VI with a discussion of the significance of such a large σ term, both for the σ model and for the $SU_3 \times SU_3$ breaking scheme of Gell-Mann, Oakes, and Renner.[15]

II. REVIEW

We begin by reviewing the kinematics and current-algebra constraints on the scattering process $\pi^j(q)+N(p)\to\pi^i(q')+N(p')$. We define the amplitude[16] as

$$T^{ij}=T^{(+)}\delta^{ij}+T^{(-)}i\epsilon^{ijk}\tau^k\,, \tag{1}$$

where $T^{(+,-)}$ are the t-channel isotopic-even and odd amplitudes related to the s-channel isotopic amplitudes by

$$T^{(+)}=\tfrac{1}{3}(T_{1/2}+2T_{3/2})\,, \tag{2}$$

$$T^{(-)}=\tfrac{1}{3}(T_{1/2}-T_{3/2})\,. \tag{3}$$

The momentum decomposition is ($\not{Q}\equiv\gamma\cdot Q$)

$$T^{(\pm)}=\bar{u}(p')[A^{(\pm)}(\nu,t)+B^{(\pm)}(\nu,t)\not{Q}]u(p) \tag{4}$$

$$=\bar{u}(p')\left\{F^{(\pm)}(\nu,t)+B^{(\pm)}(\nu,t)\left(-\frac{1}{4m}\right)\times[\not{q}',\not{q}]\right\}u(p)\,, \tag{5}$$

where the generalized forward amplitude F is related to the more commonly used amplitudes A and B by

$$F^{(\pm)}(\nu,t)=A^{(\pm)}(\nu,t)+\nu B^{(\pm)}(\nu,t)\,, \tag{6}$$

with $P=\frac{1}{2}(p'+p)$, $Q=\frac{1}{2}(q'+q)$, $\Delta=q-q'=p'-p$, and $t=\Delta^2$, $\nu=P\cdot Q/m$, $m_N=m$, $m_\pi=\mu$.

With these definitions the contributions due to the s- and u-channel nucleon pseudoscalar poles are ($H_{\pi NN}=g\gamma_5\tau^i$)

$$A_P^{(\pm)}(\nu,t)=0\,,$$

$$\nu^{-1}B_P^{(+)}(\nu,t)=\frac{g^2}{m}\frac{1}{\nu_B^2-\nu^2}\,,\quad B_P^{(-)}(\nu,t)=\frac{g^2}{m}\frac{\nu_B}{\nu_B^2-\nu^2}\,, \tag{7}$$

$$F_P^{(+)}(\nu,t)=\frac{g^2}{m}\frac{\nu_B^2}{\nu_B^2-\nu^2}\,,\quad \nu^{-1}F_P^{(-)}(\nu,t)=\frac{g^2}{m}\frac{\nu_B}{\nu_B^2-\nu^2}\,,$$

where $\nu_B=-q'\cdot q/2m=(t-2\mu^2)/4m$ is the value of $\pm\nu$ at the s- (u-) channel nucleon pole. It is important to realize that these pole amplitudes are defined in the dispersion theory sense that the numerators of the poles are evaluated at $\nu=\nu_B$, except for one power of ν in $B_P^{(+)}$ and $F_P^{(-)}$ which displays their crossing-odd nature. Thus $F_P^{(-)}$ can be obtained from $B_P^{(-)}$ according to (6), but $F_P^{(+)}$ instead is given by

$$F_P^{(+)}(\nu,t)=A_P^{(+)}(\nu,t)+\nu B_P^{(+)}(\nu,t)+g^2/m\,. \tag{8}$$

Consequently the remainder background amplitudes defined as

$$\bar{A}^{(\pm)}(\nu,t)=A^{(\pm)}(\nu,t)\,,$$
$$\bar{B}^{(\pm)}(\nu,t)=B^{(\pm)}(\nu,t)-B_P^{(\pm)}(\nu,t)\,, \tag{9}$$
$$\bar{F}^{(\pm)}(\nu,t)=F^{(\pm)}(\nu,t)-F_P^{(\pm)}(\nu,t)\,,$$

are related by

$$\bar{F}^{(-)}(\nu,t)=\bar{A}^{(-)}(\nu,t)+\nu\bar{B}^{(-)}(\nu,t)\,, \tag{10}$$

but (6) and (8) imply

$$\bar{F}^{(+)}(\nu,t)=\bar{A}^{(+)}(\nu,t)+\nu\bar{B}^{(+)}(\nu,t)-g^2/m\,. \tag{11}$$

We note that the Adler consistency condition[17] states

$$\bar{A}^{(+)}(0,\mu^2;q^2=0,q'^2=\mu^2)=gg(0)/m\sim 27\mu^{-1}\,, \tag{12}$$

whereas

$$\bar{F}^{(+)}(0,\mu^2;q^2=0,q'^2=\mu^2)=gg(0)/m-gg(0)/m=0\,. \tag{13}$$

Thus for on-shell amplitudes it will prove more convenient to use $\bar{F}^{(+)}$ which will be $\sim 1\mu^{-1}$ in the low-energy region.[18]

Next we list the constraints on the πN amplitudes due to the algebra of currents. With the definitions

$$\langle 0|A_\mu^i|\pi^j(q)\rangle=if_\pi q_\mu\delta^{ij}\,, \tag{14}$$

$$\langle N(p')|A_\mu^i|N(p)\rangle=\bar{N}\tfrac{1}{2}\tau^i i[g_A((p'-p)^2)\gamma_\mu\gamma_5+h_A((p'-p)^2)(p'-p)_\mu\gamma_5]N\,, \tag{15}$$

$$\langle N(p')|V_\mu^i|N(p)\rangle=\bar{N}\tfrac{1}{2}\tau^i[F_1^V((p'-p)^2)\gamma_\mu+F_2^V((p'-p)^2)[\gamma\cdot(p'-p),\gamma_\mu]/4m]N\,, \tag{16}$$

where[19] $f_\pi=91.7$ MeV $=0.657\ \mu$ and[10,20] $g_A(0)=g_A=1.24$, with $F_1^V(0)=1$, $F_2^V(0)=\kappa^V=3.7$, the Adler-Weisberger low-energy theorems are

$$\nu^{-1}\bar{F}^{(-)}(0,0;q^2=q'^2=0)=\frac{1}{2f_\pi^2}(1-g_A^2)\,, \tag{17}$$

$$\bar{F}^{(+)}(0,0;q^2=q'^2=0)=-\frac{\sigma_N}{f_\pi^2}\,, \tag{18}$$

where the nucleon σ term is defined as

$$\sigma_N=\left\langle N\left|\left[\int d^3x\,A_0^3(\vec{x}),i\partial\cdot A^3\right]\right|N\right\rangle\,. \tag{19}$$

The analogous theorem for on-shell pions involves the "Compton" amplitude $M_{\mu\nu}$ (where $T_{\mu\nu} = \bar{u} M_{\mu\nu} u$) for axial-vector currents with their pion poles removed.[1] Defining the background amplitudes $C^{(\pm)}$ and $D^{(\pm)}$ by

$$q'^{\mu} M^{(\pm)}_{\mu\nu} q^{\nu} = f_\pi^{\,2} \Big\{ F^{(\pm)}_{N,PV}(\nu, t) + C^{(\pm)}(\nu, t) + [B^{(\pm)}_{N,PV}(\nu, t) + D^{(\pm)}(\nu, t)] \times \left(\frac{-1}{4m}\right)[\not{q}', \not{q}] \Big\} , \tag{20}$$

the on-shell current algebra constraints are[1-5,7]

$$\bar{F}^{(+)}(\nu, t) = \frac{\sigma_N}{f_\pi^{\,2}} + C^{(+)}(\nu, t) , \tag{21}$$

$$\nu^{-1} \bar{F}^{(-)}(\nu, t) = \frac{F_1^V(t)}{2 f_\pi^{\,2}} - \frac{g^2}{2m^2} + \nu^{-1} C^{(-)}(\nu, t) , \tag{22}$$

$$\nu^{-1} \bar{B}^{(+)}(\nu, t) = \nu^{-1} D^{(+)}(\nu, t) , \tag{23}$$

$$\bar{B}^{(-)}(\nu, t) = \frac{1}{2 f_\pi^{\,2}} [F_1^V(t) + F_2^V(t)] - \frac{g^2}{2m^2} + D^{(-)}(\nu, t) . \tag{24}$$

Notice the change in sign of the σ-term contribution in (21) as compared with (18). It arises from a factor $q^2\mu^{-2} + q'^2\mu^{-2} - 1$ multiplying σ_N dictated by the Adler consistency condition.[12] Note too that g_A/f_π in (17) is replaced by its Goldberger-Treiman analog g/m in (22) and (24).[1] Our dispersion-theory method will give slight corrections to the nonpole nucleon contributions of (21)–(24), which we shift into the background amplitudes $C^{(\pm)}$ and $D^{(\pm)}$. This will be explained in more detail in Sec. III.

III. RESONANCE SATURATION OF THE AXIAL-VECTOR–NUCLEON AMPLITUDES

In this section we briefly describe our method and give our formulas for calculating nucleon and decuplet contributions to C and D. Attempts have been made to calculate C and D by both field theory and dispersion theory. However, the consistency of a theory of the spin-$\frac{3}{2}$ field is at best controversial, and it is well known that the propagator is ambiguous. Practical difficulties also appear in calculating the cross-channel contributions in an interference model.[8] Furthermore, a straightforward approach to dispersion theory[3] has proven to be little better. Calculation of $C^{(\pm)}$ involves many amplitudes of comparable magnitude, for each of which a USDR must be assumed. Since each amplitude contains a mixture of spin-1 and spin-0 axial-vector current contributions, it is difficult to analyze the USDR assumptions, which are the keys to resolving the ambiguity in dispersion theory. We have found a set of 20 covariants for $M_{\mu\nu}$ such that 18 have the property $q'^{\mu} M_{\mu\nu} q^{\nu} = 0$, and all kinematic singularities arising from divergence constraints are properly taken into account. Thus, in calculating C and D, only the two spin-0 scattering covariants and some contributions arising from singularities remain out of the twenty original covariants.

The dispersion approach of Schnitzer[3] does not take account of two "equivalence theorems"[21] which relate two of the possible covariants to the others. Goldberg and Gross[22] and Gerstein[23] implement these constraints in order to find the spin-flip sum rule in the soft limit. These equivalence theorems are isolated in the spin-1 transitions for the axial-vector–nucleon amplitude and can already be seen in on-shell photon-nucleon Compton scattering.[24] Therefore our choice of current-conserving covariants will essentially eliminate their role in the πN spin-0 transitions.

Our working assumption will be that all relevant amplitudes can be approximated by an unsubtracted dispersion integral over the direct-channel resonances; e.g., that the crossing-even coefficient $A_{19}^{(+)}$ of $q'_\mu q_\nu$ in the expansion of $M_{\mu\nu}$ can be written

$$A_{19}^{(+)}(\nu^2, t) = (A_{19}^{(+)})_N + \frac{1}{\pi} \int \frac{d\nu'^2}{\nu'^2 - \nu^2} \operatorname{Im} A_{19}^{(+)}(\nu'^2, t) \Big|_{\text{resonances}} . \tag{25}$$

This assumption comes into question for three of the amplitudes contributing to $C^{(+)}$, only one of which contributes at $q' \cdot q = 0$. We find that the presence of subtraction constants would do little harm to the predictive power of the theory, since they can be treated as part of an unknown background which we must deal with anyway.

We now present our results for the nucleon and $\Delta(1231)$ resonance contributions to C and D, postponing the detailed exposition of our method until the Appendix.

To obtain the on-shell Ward identity relations (21)–(24), the pion-pole contribution to A_μ must be extracted after being defined as follows[1]:

$$\langle N(p') | A_\mu^i | N(p) \rangle = i \tfrac{1}{2} \tau^i \bar{u}(p') \Big\{ f_\pi g(q^2) \left(\frac{\gamma_\mu}{m} - \frac{2 q_\mu}{q^2 - \mu^2} \right) + h'_A(q^2) [\gamma_\mu q^2 - q_\mu \gamma \cdot q] \Big\} \gamma_5 u(p) , \tag{26}$$

where $q \equiv p' - p$. Evidently $g(\mu^2)$ is the pion-nucleon coupling constant and $f_\pi g(0)/m = g_A$. Then $h'_A(q^2)$ always drops out of $q'_\mu M^{\mu\nu} q_\nu$. To calculate $C_N^{(\pm)}$ and $D_N^{(\pm)}$ we expand the relevant s-channel on-shell spin-$\frac{1}{2}$ exchange structure,

$$\frac{g(q^2)^2}{4m^2}\gamma_\mu\gamma_5(\not P+\not Q+m)\gamma_\nu\gamma_5$$
$$=\frac{g(q^2)^2}{4m^2}\{-2mg_{\mu\nu}-m[\gamma_\mu,\gamma_\nu]+\tfrac{1}{2}(\gamma_\mu\not Q\gamma_\nu-\gamma_\nu\not Q\gamma_\mu)+\gamma_\mu P_\nu+P_\mu\gamma_\nu+(q_\mu\gamma_\nu+\gamma_\mu q'_\nu)+\frac{1}{4m}g_{\mu\nu}[\not q',\not q]-\nu g_{\mu\nu}\}, \tag{27}$$

in terms of our covariants, find the s-channel nucleon-pole contributions to the corresponding invariants, multiply by $q'^\mu(\;)q^\nu$, and identify contributions to $C^{(\pm)}$ and $D^{(\pm)}$ using crossing symmetry. (This procedure automatically incorporates the u-channel contributions.) If we define

$$[g(q^2)]^2 \equiv h_0+h_1q^2+h_2(q^2)q^4\,, \tag{28}$$

our resulting corrections to the pseudovector nucleon contributions from field theory are (using Table IV in the Appendix)

$$C_N^{(+)}=-\frac{\nu_B}{2m^2}h_1\mu^2,\quad \nu^{-1}C_N^{(-)}=\frac{h_2\mu^4}{2m}\,,\quad \nu^{-1}D_N^{(+)}=0,\quad D_N^{(-)}=\frac{h_1\mu^2+h_2\mu^4}{2m}\,. \tag{29}$$

Because these corrections are very small, we shall absorb them into the non-Δ background $\bar{C}$ and $\bar{D}$ in Sec. V.

The analogous procedure for the decuplet involves the definition

$$\langle\Delta^+(K)|A_\mu{}^3|p(p)\rangle=-(\tfrac{2}{3})^{1/2}f_\pi\bar{u}^\nu(K)\left[g^*(q^2)\left(g_{\nu\mu}-\frac{q_\nu q_\mu}{q^2-\mu^2}\right)+(\text{three other covariants})\right]u(p)\,, \tag{30}$$

where $q=K-p$ and $g^*(\mu^2)\equiv g^*=$ pion-nucleon-Δ coupling constant.

We choose the three other covariants to be divergenceless, at least on the decuplet mass shell. Their contributions to C and D, though unknown, will be suppressed by $[(M_\Delta-m_N)/m_N]$ (Ref. 25) and we ignore them. The relevant s-channel structure is then just the spin-$\frac{3}{2}$ projection operator ($M_\Delta=M, m_N=m$)

$$g^*(q^2)^2P_{\mu\nu}^{(3/2)}(K)=-g^*(q^2)^2\left[\left(g_{\mu\nu}-\frac{K_\mu K_\nu}{M^2}\right)(\not K+M)+\tfrac{1}{3}\left(\gamma_\mu+\frac{K_\mu}{M}\right)(\not K-M)\left(\gamma_\nu+\frac{K_\nu}{M}\right)\right]$$
$$=g^*(q^2)^2\left[\left(\frac{K_\mu K_\nu}{M^2}-g_{\mu\nu}\right)\times\tfrac{2}{3}(M+m+\not Q)+\tfrac{1}{6}(m+M)[\gamma_\mu,\gamma_\nu]\right.$$
$$\left.-\tfrac{1}{6}[\gamma_\mu\not Q\gamma_\nu-\gamma_\nu\not Q\gamma_\mu]-\tfrac{1}{6}\left\{\Delta_\mu\left(\gamma_\nu+\frac{K_\nu}{M}\right)-\left(\gamma_\mu+\frac{K_\mu}{M}\right)\Delta_\nu\right\}-\frac{1}{6M}\{K_\mu[\not Q,\gamma_\nu]+[\gamma_\mu,\not Q]K_\nu\}\right], \tag{31}$$

leading eventually to the contributions (using Table V in the Appendix)

$$C_\Delta^{(+)}(\nu,q'\cdot q)=\frac{2g^{*2}}{9m^2}\frac{\nu_\Delta}{\nu_\Delta{}^2-\nu^2}\alpha(q'\cdot q)-\frac{2}{9mM^2}\{[h_0^*(\mu^2+\tfrac{1}{2}(M^2-m^2))((M+m)^2-q'\cdot q)+h_0^*(\mu^2-\tfrac{1}{2}q'\cdot q)(M(M+m))$$
$$+\tfrac{1}{2}h_1^*\mu^2(M^2-m^2-q'\cdot q)((M+m)^2-q'\cdot q)\}\,, \tag{32}$$

$$\nu^{-1}C_\Delta^{(-)}=-\frac{g^{*2}}{9m^2}\frac{\alpha(q'\cdot q)}{\nu_\Delta{}^2-\nu^2}+\frac{1}{9M^2}(h_1^*+h_2^*\mu^2)[(M+m)^2-q'\cdot q]\,, \tag{33}$$

$$\nu^{-1}D_\Delta^{(+)}=\tfrac{2}{3}\frac{g^{*2}}{m}\frac{\beta(q'\cdot q)}{\nu_\Delta{}^2-\nu^2}\,, \tag{34}$$

$$D_\Delta^{(-)}=-\tfrac{1}{3}\frac{g^{*2}}{m}\frac{\nu_\Delta\beta(q'\cdot q)}{\nu_\Delta^2-\nu^2}+\tfrac{2}{9}\frac{m}{M}g^{*2}+\frac{2h_0^*m\nu_\Delta}{9M^2}\,, \tag{35}$$

where

$$\alpha(q'\cdot q)=(E_\pi{}^2-q'\cdot q)[(M+m)^2-q'\cdot q]$$
$$+(\mu^2-q'\cdot q)[(M+m)E_\pi-\tfrac{1}{2}(\mu^2+q'\cdot q)]\,, \tag{36}$$

$$\beta(q'\cdot q)=E_\pi{}^2-\tfrac{1}{3}(M+m-E_\pi)^2-q'\cdot q\ , \tag{37}$$

$$\nu_\Delta=\frac{M^2-m^2-q'\cdot q}{2m}\ , \tag{38}$$

and $E_\pi=(M^2-m^2+\mu^2)/2M$ is the c.m. energy of the pion at resonance.

We use the value of g^{*2} obtained by HJS ($g^{*2}/4\pi=0.264\mu^{-2}=13.5\ \mathrm{GeV}^{-2}$) by fitting data away from the peak in the narrow-resonance approximation. This is 40% lower than the value obtained by a narrow-width calculation of the Δ decay rate (used, e.g., in Ref. 1). Since we are not evaluating dispersion relations for F and B, however, we prefer to use $M=1231$ MeV instead of the HJS value $M=1219$ MeV. Our results are insensitive to this choice in any case.

It is characteristic of our method that nonpole contributions depend somewhat upon the behavior of coupling constants off the pion mass shell—i.e., corrections to the generalized Goldberger-Treiman relation. That this result differs from naive perturbation theory should not bar its acceptance, for it is well known that a similar phenomenon occurs in calculating the generalized Born contribution to pion electroproduction[26] and weak pion production.[27]

At any rate, the coefficients h_i and the analogously defined h_i^* should be determined. Since $g(0)=mg_A/f_\pi$ is known, this is easy for h_i. Defining

$$r\equiv\frac{\mu^2g'(0)}{g}\approx\frac{g-g(0)}{g}=0.065\approx3\left(\frac{\mu}{m}\right)^2\ ,$$

we see that g' is large, so that we might neglect g'' to give

$$h_0=g(0)^2\approx(1-r)^2g^2,\quad h_1=2g'g=2rg^2\ ,$$

and

$$h_2=g'^2+g''g\approx r^2g^2\ .$$

In the absence of a determination of $g^*(0)$, a reasonable bound for r^* is $|r^*|<0.065$. On this basis, we find that our results are not sensitive to the value of r^*.

We conclude this section with some general observations which will be useful in assessing the comparison with experiment. We begin by stating five theorems:

(i) $C^{(+)}(\nu=0,q'\cdot q=0)\sim O(\mu^4)$.

(ii) All contributions to $\nu^{-1}C^{(-)}(\nu=0,q'\cdot q=0)$ from $\frac{1}{2}^\pm$ resonances are likewise of $O(\mu^4)$. (Other contributions, notably the $\frac{3}{2}^+$, may be of order μ^2.)

(iii) The $\frac{3}{2}^-$, $\frac{5}{2}^\pm$, and all higher-spin resonances are suppressed by angular momentum.

(iv) The direct-channel resonances contribute only poles to $D^{(+)}$.

(v) All nonpole resonance contributions to $C^{(+)}$, $\nu^{-1}C^{(-)}$, and $D^{(-)}$ are constant in ν.

The first theorem was proved by CD and verified for field-theory models in Ref. 1. The second theorem significantly sharpens the original Adler-Weisberger relation and follows from the absence of the $R_{\mu\nu}{}^3$ covariant of (A22) in the spin-$\frac{1}{2}$ contribution to $M_{\mu\nu}$ (27). Such terms completely vanish in field-theory models.[1] The third theorem has been previously invoked at threshold to help justify decuplet saturation.[3]

The fourth theorem seems to hold only for our method, since it arises directly from our technique of identifying divergenceless covariants. It is in harmony with the absence of any current-algebra contribution to the Ward identity for $B^{(+)}$. The theorem ensures that a test of this identity reduces to a test of the USDR for $B^{(+)}$.

The fifth theorem leads to the expectation that most of the ν variation in F and B will arise from the lowest-lying pole contribution, the Δ. HJS have already found this to be the case, to very good accuracy everywhere except perhaps in $F^{(+)}$. Hence, we do not need to discuss most of the higher-order HJS expansion coefficients.

Furthermore, our form for $q'^\mu M_{\mu\nu}q^\nu$ as given by (A22) provides an explanation for the fact that the decuplet does not account for all the ν and t variation in $F^{(+)}$, whereas it does for the other three amplitudes. The demonstration of this point along with proof of the theorems is reserved for the Appendix.

IV. LOW-ENERGY DATA ANALYSIS

In order to evaluate the background πN amplitudes at various low-energy configurations of ν and t, HJS suggest the expansions[8]

$$\begin{aligned}\overline{A}^{(+)}(\nu,t)&=a_1^++a_2^+t+a_3^+\nu^2+a_4^+\nu^2t+a_5^+\nu^4+\cdots\ ,\\ \nu^{-1}\overline{B}^{(+)}(\nu,t)&=b_1^++b_2^+t+b_3^+\nu^2+b_4^+\nu^2t+b_5^+\nu^4+\cdots\ ,\\ \nu^{-1}\overline{A}^{(-)}(\nu,t)&=a_1^-+a_2^-t+a_3^-\nu^2+a_4^-\nu^2t+a_5^-\nu^4+\cdots\ ,\\ \overline{B}^{(-)}(\nu,t)&=b_1^-+b_2^-t+b_3^-\nu^2+b_4^-\nu^2t+b_5^-\nu^4+\cdots\ ,\end{aligned} \tag{39}$$

where the constant coefficients are determined directly from fixed-t dispersion relations and are listed in Table I.[28] Likewise one can expand the "forward" background amplitudes as

$$\begin{aligned}\overline{F}^{(+)}(\nu,t)&=f_1^++f_2^++f_3^+\nu^2+f_4^+\nu^2t+f_5^+\nu^4+\cdots\ ,\\ \nu^{-1}\overline{F}^{(-)}(\nu,t)&=f_1^-+f_2^-t+f_3^-\nu^2+f_4^-\nu^2t+f_5^-\nu^4+\cdots\ ,\end{aligned} \tag{40}$$

and these coefficients are also given in Table I. Using (10, 11) one can compute the $f_i^\pm$'s directly

TABLE I. Coefficients of low-energy expansions, Eqs. (39) and (40).

$i=$	1	2	3	4	5
a_i^+	26.1 ± 0.3	1.32 ± 0.10	4.4	0	1.12
b_i^+	−3.28	0.19	−0.92	0.09	−0.28
f_i^+	−1.40 ± 0.15	1.27 ± 0.15	1.12	0.19	0.20
a_i^-	−8.4	−0.45	−1.15	0.02	−0.29
b_i^-	7.9	0.29	0.99	−0.06	0.25
f_i^-	−0.52 (−0.44)	−0.16	−0.16	−0.04	−0.04

from the $a_i^\pm$'s and $b_i^\pm$'s. However for the cases where the cancellation between the a_i and b_i is almost complete, the f_i can be found from the HJS coefficients c_i, also found directly from dispersion relations. Table I reflects the HJS numbers, save for three notable exceptions[28]:

$$f_1^+ = a_1^+ - g^2/m = (-1.40 \pm 0.15)\mu^{-1} , \quad (41)$$

$$f_2^+ = a_2^+ = (1.27 \pm 0.15)\mu^{-3} , \quad (42)$$

$$f_1^- = -0.52\,\mu^{-2} , \quad (43)$$

instead of the HJS values $a_1^+ - g^2/m = -1.6 \pm 0.3$, $a_2^+ = 1.13 \pm 0.10$, and $f_1^- = c_1^- = -0.44 \pm 0.02$. The reason for these discrepancies will now be explained.

Given $a_1^+ = (26.1 \pm 0.3)\mu^{-1}$ (Ref. 7) and the usual[19] value of $g^2/4\pi = 14.64 \pm 0.6$, one finds $f_1^+ = -1.3 \pm 1.4$, and although this central value agrees with (41), the errors are much too large for this to be a meaningful determination of f_1^+. Instead one first finds the amplitude $\overline{F}^{(+)}$ (th) at the physical threshold $\nu = \mu$, $t = 0$ from the s-wave scattering length $\frac{1}{3}(a_1 + 2a_3)$ by

$$4\pi\left(1 + \frac{\mu}{m}\right)\frac{1}{3}(a_1 + 2a_3) = -g^2\frac{\mu^2}{4m^3}(1 - \mu^2/4m^2)^{-1} + \overline{F}^{(+)}(\text{th}). \quad (44)$$

The πNN coupling constant error is then insignificant, leading to a pole contribution of $-0.15\mu^{-1}$ to the right-hand side (RHS) of (44). The average of the three most accurate determinations of $a_1 + 2a_3$ (Table II) is $a_1 + 2a_3 = -0.02\mu^{-1}$, giving $-0.10\mu^{-1}$ for the LHS of (44) and therefore

$$\overline{F}^{(+)}(\text{th}) = 0.05\mu^{-1} . \quad (45)$$

Next one writes a once subtracted dispersion relation for $\overline{F}^{(+)}$ (th):

$$\overline{F}^{(+)}(\text{th}) - f_1^+ = \frac{\mu^2}{\pi}\int_\mu^\infty d\nu' \frac{\sigma_{\pi^- p}(\nu') + \sigma_{\pi^+ p}(\nu')}{\nu'(\nu'^2 - \mu^2)^{1/2}} = 1.46,\ 1.45 , \quad (46)$$

as obtained by HJS[8] and by Samaranayake and Woolcock.[29] Combining (46) with (45), one finds $f_1^+ = -1.4\mu^{-1}$.

This estimate of f_1^+ should be contrasted with the HJS number -1.66 used to find $\sigma_N \sim 40$ MeV in Ref. 14 or -1.6 ± 0.3 in Ref. 8, and the CD value of -0.95. The different values are due to the relative importance of the low-energy data in determining the scattering length. While the numbers in Table II give $a_1 + 2a_3 \sim -0.02\mu^{-1}$, Jakob[30] uses the latest CERN 71 phase shifts and follows the HJS philosophy of deemphasizing the low-energy data to find $a_1 + 2a_3 \sim -0.04\mu^{-1}$. On the other hand, the most recent determinations of $a_1 + 2a_3$ are near zero.[30a] We therefore believe a fair estimate at the present time is

$$a_1 + 2a_3 = (-0.02 \pm 0.03)\mu^{-1} , \quad (47)$$

which leads to (41).

At this point we would also like to point out that the "no cusp" solution of (46) is

$$\overline{F}^{(+)}(\text{th}) - f_1^+ = f_3^+ + f_5^+ = -1.27\mu^{-1} , \quad (48)$$

which would imply $f_1^+ = -1.23\mu^{-1}$. While cusp effects do exist, we see that they are rather small. Hence we shall continue to use the no-cusp solution as a guide to the size of the other low-energy πN scattering length determinations. Furthermore, since we shall be approximating cut corrections to the background amplitudes C and D of (21)–(24) by a series of resonance poles which have no cusp effects, it will prove advantageous to use the no-cusp values of the HJS parameters when testing the Ward identities against the resonance satura-

TABLE II. s-wave scattering lengths (in μ^{-1}).

	H [a]	L [b]	HSS [c]	SW [d]
$a_1 - a_3$	0.271 ± 0.007	0.266 ± 0.017	0.288 ± 0.010	0.277 ± 0.009
$a_1 + 2a_3$	−0.002 ± 0.008	0.056 ± 0.022	−0.021 ± 0.010	−0.026 ± 0.008

[a] Experimental values determined by J. Hamilton (H), Phys. Lett. 20, 687 (1966).
[b] C. Lovelace (L), *Pion-Nucleon Scattering* (Wiley, New York, 1967).
[c] HSS (Ref. 31).
[d] SW (Ref. 33).

tion theorems (ii)–(v) of Sec. III.

Now we find $f_2^+ = a_2^+$ from the p-wave $k^2\cos\theta$ scattering-length formula at threshold (a generalization of the result of ACM[13]):

$$\tfrac{2}{3}\pi\left[\frac{1}{4m^2}(a_1+2a_3)+(a_{11}+2a_{31})+2\left(1+\frac{3\mu}{2m}\right)(a_{13}+2a_{33})\right]$$
$$=\frac{g^2}{4m^3}(1-\mu^2/4m^2)^{-1}(1-\mu/2m)^{-1}+\left(\frac{\partial\overline{F}^{(+)}(\text{th})}{\partial t}+\frac{\mu}{2m}\frac{\partial\overline{F}^{(+)}(\text{th})}{\partial\nu^2}\right). \quad (49)$$

The LHS of (49) is $1.69\mu^{-3}$ from the work of Höhler *et al.* (HSS)[31] or $1.88\mu^{-3}$ from the work of Collins *et al.* (CSW)[32] (see Table III). If we assume that cusp effects are small, then (49) becomes

$$f_2^+ + f_4^+ + \frac{\mu}{2m}(f_3^+ + 2f_5^+) = (1.53)_{\text{HSS}}, (1.72)_{\text{CSW}}. \quad (50)$$

Assuming the value for $f_{3,4,5}^+$ in Table I, (50) leads to the no-cusp values

$$f_2^+(\text{no cusp}) = (1.23)_{\text{HSS}}, (1.42)_{\text{CSW}}$$
$$= 1.32 \pm 0.10\ , \quad (51)$$

and the central value in (51) agrees with Cheng and Dashen.[12] The value $f_2^+ = 1.13$ obtained by interpreting mutually inconsistent data in the manner of HJS[14] requires a significant cusp suppression from their threshold value of $f_2^+ = 1.23$.[8] Taking into account the low-energy data points, we estimate the p-wave cusp suppression to be 0.05 ± 0.05 which, when combined with (51), leads to our final value (42).

We can further probe the effect of the cusp by evaluating the p-wave $k^2\sin\theta$ scattering length formula

$$\frac{8\pi}{3}m^2(1+\mu/m)[(a_{11}+2a_{31})-(a_{13}+2a_{33})]$$
$$= -\frac{g^2}{\mu}[(1-\mu/2m)^{-1}+\mu/2m]$$
$$+\mu(m+\mu)\nu^{-1}\overline{B}^{(+)}(\text{th}) - \tfrac{1}{2}\overline{F}^{(+)}(\text{th}) \quad (52)$$

TABLE III. p-wave scattering lengths (in μ^{-3}).

	HW [a]	RWF [b]	HSS [c]	CSW [d]
$a_{11}-a_{31}$	-0.073 ± 0.015	-0.003	-0.051	-0.049
$a_{13}-a_{33}$	-0.255 ± 0.015	-0.243	-0.243	-0.250
$a_{11}+2a_{31}$	-0.177 ± 0.015	-0.117	-0.168	-0.160
$a_{13}+2a_{33}$	0.396 ± 0.015	0.402	0.396	0.431

[a] Experimental values determined by J. Hamilton and W. S. Woolcock (HW), Rev. Mod. Phys. 35, 737 (1963).
[b] L. D. Roper, R. M. Wright, and B. T. Feld (RWF), Phys. Rev. 138, B190 (1965).
[c] HSS (Ref. 31).
[d] CSW (Ref. 32).

at threshold. The experimental value of the LHS of (52) is $(-246\mu^{-1})_{\text{HSS}}$ or $(-257\mu^{-1})_{\text{CSW}}$ compared with the RHS of (52) found by the no-cusp values in Table I, $-248\mu^{-1}$. Cusp effects would be harder to detect here because the g^2 term dominates the RHS of (52); yet a 10% cusp effect would alter it by $4\mu^{-1}$.

At the on-shell point $\nu=0$, $t=\mu^2$, we find

$$\overline{F}^{(+)}(0,\mu^2) = f_1 + f_2 = (-0.13 \pm 0.21)\mu^{-1}\ . \quad (53)$$

Comparing (53) with the Adler consistency condition (ACC) (13), we see that off-shell effects of the q^2 extrapolation from 0 to μ^2 are small. Put another way, if these effects were negligible then we would have

$$f_2^+(\text{ACC}) = -f_1^+(\text{ACC}) \approx -1.3 \text{ to } -1.4\ \mu^{-1}\ . \quad (54)$$

Our choice for f_1^- corresponds to Adler's value of R_1.[10] While HJS may be correct for *physical* πN scattering, the Adler value of $f_1^- = -0.52$ seems to correspond to the no-cusp limit of HJS (Fig. 1 of Ref. 8). As explained earlier, we shall need this number to test the on-shell Adler-Weisberger relation (22) when saturating with resonances. One could in principle find f_1^- directly from scattering lengths. The s-wave isotopic-odd scattering length $\frac{1}{3}(a_1 - a_3)$ obeys

$$4\pi(1+\mu/m)\tfrac{1}{3}(a_1-a_3)\mu^{-1}$$
$$=\frac{g^2}{2m^2}(1-\mu^2/4m^2)^{-1}+\nu^{-1}\overline{F}^{(-)}(\text{th})\ . \quad (55)$$

The mean value of $a_1 - a_3$ given in Table II agrees with the Samaranayake and Woolcock[33] value of $0.277\mu^{-1}$. Thus (55) gives

$$\nu^{-1}\overline{F}^{(-)}(\text{th}) = -0.72\mu^{-2} \quad (56)$$

compared with the no-cusp solution (with $f_1^- = -0.52$) at $\nu=\mu$, $t=0$ (Ref. 1):

$$\nu^{-1}\overline{F}^{(-)}(\text{th}) = f_1^- + f_3^- + f_5^- = -0.72\mu^{-2}. \quad (57)$$

While cusp effects again appear to be small, the error associated with (56) is too large to decide between the Adler and HJS value of f_1^-.

It is interesting that our version of the *off-shell* Adler-Weisberger relation (17) appears to be best satisfied with the HJS value of $f_1^- = -0.44$, while the original version [with f_π replaced by mg_A/g in (17)] is best satisfied with $f_1^- = -0.52$; that is, both lead to the present experimental value of $g_A = 1.24$.

To complete the picture, we evaluate the p-wave threshold scattering lengths for the isotopic-odd amplitudes. The $k^2\cos\theta$ formula is

$$\tfrac{2}{3}\pi\left[\frac{1}{4m^2}(a_1-a_3)+(a_{11}-a_{31})+2\left(1+\frac{3\mu}{2m}\right)(a_{13}-a_{33})\right]$$
$$=-\frac{g^2}{4m^2\mu}\left(1-\frac{\mu^2}{4m^2}\right)^{-1}\left(1+\frac{\mu^2}{4m^2}\right)\left(1-\frac{\mu}{2m}\right)^{-1}$$
$$+\mu\left(\frac{\partial}{\partial t}\,\nu^{-1}\overline{F}^{(-)}(\text{th})+\frac{\mu}{2m}\,\frac{\partial}{\partial\nu^2}\,\nu^{-1}\overline{F}^{(-)}(\text{th})+\frac{1}{4m\mu}\,\nu^{-1}\overline{F}^{(-)}(\text{th})\right), \tag{58}$$

and the $k^2\sin\theta$ formula is

$$\tfrac{8}{3}\pi m^2(1+\mu/m)[(a_{11}-a_{31})-(a_{13}-a_{33})]$$
$$=\frac{g^2}{2m}(1-\mu/2m)^{-1}+(m+\mu)\overline{B}^{(-)}(\text{th})-\tfrac{1}{2}\mu\nu^{-1}\overline{F}^{(-)}(\text{th}). \tag{59}$$

The LHS of (58) gives $(-1.36)_{\text{HSS}}$, $(-1.38)_{\text{CSW}}$, and the no-cusp assumption on the RHS of (58) yields $-1.36\mu^{-3}$. The LHS of (59) is $(83.8)_{\text{HSS}}$, $(87.7)_{\text{CSW}}$ and the no-cusp assumption for the RHS of (59) gives $85.9\mu^{-1}$. Cusp effects of 10% would change the RHS of (58) and (59) by $0.03\mu^{-3}$ and $7\mu^{-1}$, respectively. Certainly this would not be acceptable in the latter case.

Thus we see that all six s- and p-wave scattering lengths are consistent with reasonably small cusp effects (of the order of 10% or smaller).

V. ON-SHELL TESTS OF THE WARD IDENTITIES

Now we are able to compare the current-algebra constraints (21)–(24) with on-shell data. Since these identities are functions of ν and t, we shall evaluate them at four points which will enable us to probe their ν and t dependence: the Cheng-Dashen point CDP, $\nu=0$, $t=2\mu^2\,(q'\cdot q=0)$; the symmetry point SP, $\nu=t=0$; the threshold point TP, $\nu=\mu$, $t=0$; and the subthreshold point STP, $\nu=\mu$, $t=2\mu^2\,(q'\cdot q=0)$.

First we investigate the $\overline{F}^{(-)}$ Adler-Weisberger Ward identity (22). Writing $\nu^{-1}C^{(-)}=\nu^{-1}C^{(-)}_{\Delta}+\nu^{-1}\overline{C}^{(-)}$, our theory (33) implies that at the CDP

$$\nu^{-1}C^{(-)}_{\Delta}(0,2\mu^2)=(-1.24)_{\text{P}}+(1.13)_{\text{NP}}=-0.11\,, \tag{60}$$

where P (NP) indicates the pole (nonpole) part of the Δ contribution. Taking into account the measured slope $F_1^{V\prime}(0)=0.046\mu^{-2}$ (Ref. 34) or $f_\pi^{-2}F_1^{V\prime}(0)=0.11$, we find from (22) that

$$f_1^-+2f_2^-=-0.84$$
$$=(1.16+0.11)-2.04-0.11+\nu^{-1}\overline{C}^{(-)}(0,2\mu^2)\,, \tag{61}$$

giving for the non-Δ background,

$$\nu^{-1}\overline{C}^{(-)}(0,2\mu^2)=0.04\,. \tag{62}$$

The smallness of (62) is a reflection of the $O(\mu^4)$ theorem for $\nu^{-1}C^{(-)}(0,2\mu^2)$ (theorem ii) and therefore (61) represents a *definitive* test of the Adler-Weisberger Ward identity. At the STP we find $\nu^{-1}C^{(-)}_{\Delta}(\mu,2\mu^2)=(-1.50)_{\text{P}}+(1.13)_{\text{NP}}=-0.37$ and (22) then implies $\nu^{-1}\overline{C}^{(-)}(\mu,2\mu^2)=0.02$. This small variation in ν between the CDP and STP is due to the suppression of the C_{11} term in (A22) for higher resonant ν_R values $(\mu/\nu_R)^2$ (see Appendix). This is therefore a second definitive test of (22). At the SP we have $\nu^{-1}C^{(-)}_{\Delta}(0,0)=(-0.93)_{\text{P}}+(1.13)_{\text{NP}}=0.20$, and (22) leads to $\nu^{-1}\overline{C}^{(-)}(0,0)=0.26$, and at the TP $\nu^{-1}C^{(-)}_{\Delta}(\mu,0)=(-1.17)_{\text{P}}+(1.13)_{\text{NP}}=-0.04$ or $\nu^{-1}\overline{C}^{(-)}(\mu,0)=0.20$ from (22). Both non-Δ backgrounds are significantly larger than (62) (but still small) because of the C_9 term in (A22) which is not suppressed away from $q'\cdot q=0$. Thus the entire ν and t dependence of the Adler-Weisberger Ward identity is completely understandable in our approach.

Next we probe the on-shell $\overline{B}^{(-)}$ Ward identity (24).[35] Our theory (35) gives at the CDP,

$$D^{(-)}_{\Delta}(0,2\mu^2)=(3.89)_{\text{P}}+(0.54)_{\text{NP}}=4.43, \tag{63}$$

and using[34] $F_1^{V\prime}(0)+F_2^{V\prime}(0)=0.266\,\mu^{-2}$, (24) becomes with $D^{(-)}=D^{(-)}_{\Delta}+\overline{D}^{(-)}$,

$$b_1^-+2b_2^-=8.50$$
$$=6.07-2.04+4.43+\overline{D}^{(-)}(0,2\mu^2)\,, \tag{64}$$

giving

$$\overline{D}^{(-)}(0,2\mu^2)=0.04\,. \tag{65}$$

This non-Δ background is small compared with the Δ contribution (63), and therefore (64) represents a reasonable test of the $\overline{B}^{(-)}$ Ward identity. Because C_{10} in (A22) is not suppressed at any special kinematic point, we must also check to see that there is no wild variation of $\overline{D}^{(-)}$ at other values of ν and t compared with (65). At the STP we have $D^{(-)}_{\Delta}(\mu,2\mu^2)=(4.71)_{\text{P}}+(0.54)_{\text{NP}}=5.25$ giving $\overline{D}^{(-)}(\mu,2\mu^2)=0.34$; at the SP $D^{(-)}_{\Delta}(0,0)=(4.07)_{\text{P}}+(0.65)_{\text{NP}}=4.72$, implying $\overline{D}(0,0)=-0.23$; and at the TP $D^{(-)}_{\Delta}(\mu,0)=(4.97)_{\text{P}}+(0.65)_{\text{NP}}=5.62$, leading to $\overline{D}(\mu,0)=0.11$ by (24). All of these non-Δ backgrounds are small compared with the corresponding Δ contributions $D_{\Delta}^{(-)}$. Since this should be the case because of the absence of the covariant $\nu^2[\not{q}', \not{q}]$ in (A22) (see Appendix), we conclude that these four points collectively represent a *definitive* test of the $B^{(-)}$ Ward identity (24). According to HJS, field-theory values of $D_{\Delta}^{(-)}$ can be up to twice the size of (63), thus violating the $\overline{B}^{(-)}$ Ward identity. It is important that $\overline{D}_{\Delta}^{(-)}$ be approximately one-half the LHS of (64) if the $\overline{B}^{(-)}$ Ward identity is to be valid. Put another way, it is one of the deep constraints of current algebra that the LHS of (64) is not dominated phenomenologically by the Δ resonance.

As was noted before, the $\overline{B}^{(+)}$ Ward identity (23) is not constrained by the current commutator or nucleon background terms, and it is therefore proper that all nonpole background contributions $\nu^{-1}D^{(-)}$ vanish [theorem (iv)]. Thus (23) is simply a test of how well Δ saturates $\overline{B}^{(+)}$, but not of current algebra per se. At the CDP we find from (34)

$$\nu^{-1} D_{\Delta}^{(+)}(0, 2\mu^2) = (-3.25)_P , \qquad (66)$$

which differs from HJS because we evaluate this pole at $M = 1231$ MeV rather than at 1219 MeV. Then (23) becomes

$$\begin{aligned} b_1^+ + 2b_2^+ &= -2.90 \\ &= -3.25 + \nu^{-1}\overline{D}^{(+)}(0, 2\mu^2) , \end{aligned} \qquad (67)$$

giving $\nu^{-1}\overline{D}^{(+)}(0, 2\mu^2) = 0.35$ or a 10% correction to the Δ contribution. The other three points just map out b_2^+, b_3^+, b_4^+, and b_5^+ as given in HJS and therefore $\nu^{-1}\overline{D}^{(+)} \approx 0.35$ at these points as well.

Lastly we investigate the $\overline{F}^{(+)}$ Ward identity (21) and determine the value of σ_N. Because of the $O(\mu^4)$ theorem of Sec. III for $\overline{F}^{(+)}$ at $\nu = q'\cdot q = 0$, we neglect $C^{(+)}(0, 2\mu^2)$ and write (21) at the CDP as

$$f_1^+ + 2f_2^+ = (1.14 \pm 0.33)\,\mu^{-1} = \frac{\sigma_N}{f_\pi^{\,2}} \qquad (68)$$

or (with $f_\pi \approx 94$ MeV, as used in other determinations of σ_N)

$$\sigma_N = 73 \pm 21 \text{ MeV}. \qquad (69)$$

While A_{19} of (A22) causes $C^{(+)}$ to be given by a once subtracted dispersion, a USDR for this term causes a small error $O(\mu^4)$. From (32) we find

$$\begin{aligned} C^{(+)}(0, 2\mu^2) = \frac{2\mu^4}{9M^2m^2\nu_\Delta} \Big[& 2h_1^*\,\nu_\Delta m(M+m)(2M+m) \\ & + \tfrac{1}{4} g^{*2}(M^2 + m^2 + 4Mm) \\ & + h_2^*\,\nu_\Delta^{\,2} m^2 (M+m)^2 \Big] \\ + O(\mu^4) . \end{aligned} \qquad (70)$$

We expect the first term in (70) to be dominant if g^* differs appreciably from g_0^*. The nucleon Goldberger-Treiman difference of 6% is most likely an upper bound (see Sec. III), giving a 5-MeV reduction of the σ term. However, the finite-width suppression of g^* by 20% makes this difference significantly smaller, i.e., $r^* \sim \pm 0.01$, indicating that the first term in (70) might be $\sim \pm 1$ MeV, which is the same size as the second term. The third term in (70) as well as the $O(\mu^4)$ subtraction constant presumably give much less than a 1 MeV correction. We conclude that indeed the entire background amplitude $C^{(+)}(0, 2\mu^2)$ is negligible (which agrees with field-theory models[36]) and that σ_N is given by (69).

It should be pointed out that the non-Δ backgrounds $\overline{C}^{(+)}$ vary more rapidly in ν and in t than do $\nu^{-1}\overline{C}^{(-)}$ or $\overline{B}^{(-)}$. We find that in order for (69) to remain valid with σ_N independent of t,

$$\begin{aligned} &\overline{C}^{(+)}(0, 2\mu^2) = 0 , \quad &&\overline{C}^{(+)}(\mu, 2\mu^2) = 0.34 , \\ &\overline{C}^{(+)}(0, 0) = -0.88 , \quad &&\overline{C}^{(+)}(\mu, 0) = -0.38 . \end{aligned} \qquad (71)$$

The large variation of (71) in t is a reflection of the C_9 term of (A22) and the somewhat smaller variation in ν is due to an enhancement of the higher resonances by ν_R in $C_{11}^{(+)}$ relative to $C_{11}^{(-)}$ in (A22).

At this point we relate the HJS phenomenological parameterization of the LHS of (21) to the work of ACM. The latter begin with the Weinberg "smoothness" assumption[13,37]:

$$\overline{F}^{(+)}(\nu, t; q^2, q'^2) = A + Bt + C(q^2 + q'^2) + D\nu^2 . \qquad (72)$$

They then demand that $\overline{F}^{(+)}(0, \mu^2; 0, \mu^2) = 0$ and $\overline{F}^{(+)}(0, 2\mu^2; \mu^2, \mu^2) = -F(0,0;0,0) = \sigma_N/f_\pi^{\,2} = -A$. However $F^{(+)}(0,0;\mu^2,\mu^2) = A + 2C = f_1^+ = -1.40 \pm 0.15$. The combination $B + (\mu/2m)D$ is determined from the $k^2\cos\theta\, p$-wave scattering length. In their second paper,[13] ACM include the $t\nu^2$ and ν^4 dependence of (72), which effectively leads to (51) or $B = f_2^+ = 1.32 \pm 0.10$. Ignoring the cusp suppression, the ACM analysis gives $-A = f_1^+ + 2f_2^+ = 1.24 \pm 0.25$, which predicts $\sigma_N = 79 \pm 16$ MeV. The fact that $C = f_1^+ + f_2^+ = -0.08 \pm 0.21$ again reflects that off-shell effects are small.

In Sec. IV we have related scattering lengths to the low-energy amplitude and in this section we have demonstrated that the latter are consistent with the current-algebra Ward identities. There are no further constraints. However, in order to compare our results with the work of Weinberg,[37] Schnitzer,[3] Raman,[3] and Peccei,[6] we now combine the two analyses and compute threshold scattering lengths directly in terms of the Ward identities. Following Ref. 1, we write the isotopic-odd s-wave scattering length from (55) with (22) at the TP:

$$\tfrac{1}{3}(a_1 - a_3) = \mu[4\pi(1+\mu/m)]^{-1} \times\left[\frac{g^2\mu^2}{8m^4}(1-\mu^2/4m^2)^{-1} + \frac{1}{2f_\pi^2} + \nu^{-1}C^{(-)}(\mu,0)\right]. \qquad (73)$$

Weinberg neglects the small nucleon-pole term and the Δ contribution by PCAC (partial conservation of axial-vector current). This leads to $\frac{1}{3}(a_1 - a_3) = 0.080\mu^{-1}$. He also replaces f_π by f_π^{GT} which gives $\frac{1}{3}(a_1 - a_3) = 0.091\mu^{-1}$, which is in excellent agreement with experiment, $0.092\mu^{-1}$. From our viewpoint, the threshold amplitude $\nu^{-1}C^{(-)}(\mu, 0)$ converts $0.080\mu^{-1}$ to $0.091\mu^{-1}$ (the nucleon contribution is $0.001\mu^{-1}$). Likewise the isotopic-even s-wave scattering length can be written as

$$\tfrac{1}{3}(a_1 + 2a_3) = [4\pi(1+\mu/m)]^{-1} \times\left[-\frac{g^2\mu^2}{4m^3}(1-\mu^2/4m^2)^{-1} + \frac{\sigma_N}{f_\pi^2} + C^{(+)}(\mu,0)\right]. \qquad (74)$$

Weinberg assumes that all three terms on the RHS of (74) are small, so that $\frac{1}{3}(a_1 + 2a_3) \approx 0$. We have found that the σ term is not small but almost completely cancels against $C^{(+)}(\mu, 0)$ as given by (71). Also, three of the four p-wave scattering lengths are dominated by the nucleon pole and agree reasonably well with experiment,[3,6] but the fourth, $a_{P1/2}^{(-)}$, has a suppressed nucleon pole similar to (73) and (74) and therefore is a sensitive test of models for $D^{(-)}(\mu, 0)$. Combining (59) and (24) we write at the TP

$$\tfrac{1}{3}(a_{11} - a_{31}) - \tfrac{1}{3}(a_{13} - a_{33}) \approx \frac{1}{8\pi m}\left[-\frac{g^2\mu}{4m^3} + \frac{1+\kappa_V}{2f_\pi^2} + D^{(-)}(\mu,0)\right]. \qquad (75)$$

Following Schnitzer[3] and Peccei,[6] we approximate $D^{(-)}(\mu, 0) \approx D_\Delta^{(-)}(\mu, 0)$ (which we find is true in any case), and using our value of $D_\Delta^{(-)}(\mu, 0) = 5.62$, the RHS of (75) becomes

$$\tfrac{1}{3}(a_{11} - a_{31}) - \tfrac{1}{3}(a_{13} - a_{33}) = -0.001 + 0.032 + 0.033 = 0.064\mu^{-3}, \qquad (76)$$

which is in perfect agreement with experiment (Table III). We also obtain the experimental value of $\frac{1}{3}(a_{13} - a_{33}) = -0.081\mu^{-3}$ from (58) and therefore (76) predicts $\frac{1}{3}(a_{11} - a_{31}) = -0.017\mu^{-3}$, again in agreement with experiment. Contrast this with the Schnitzer "dispersion theory model" value of $\frac{1}{3}(a_{11} - a_{31}) = -0.005$ and the Peccei field-theory prediction of $\frac{1}{3}(a_{11} - a_{31}) = +0.003$. Once more it is clear that our dispersion-theory approach succeeds where other models fail.

VI. CONCLUSION

We have presented a dispersion-theory analysis of the current-algebra Ward identities for πN scattering. Our results include two definitive tests of the on-shell forward (Adler-Weisberger) Ward identity for $F^{(-)}$ and two definitive tests of the on-shell spin-flip Ward identity for $B^{(-)}$. We have also extracted the $\pi N \sigma$ term from the on-shell Ward identity for $F^{(+)}$ and find $\sigma_N = 73 \pm 21$ MeV. The two s-wave and four p-wave threshold scattering lengths are completely incorporated into our analysis of the low-energy data in terms of the HJS-type coefficients f_i^+ and b_i^+. Conversely, we can also predict all six scattering lengths from the Ward identities and obtain perfect agreement with experiment.

Our value for the σ term is approximately midway between the CD and HJS values of 110 MeV and 40 MeV, respectively. We have followed the method of HJS and have expanded the amplitude about the symmetry point $\nu = t = 0$, but do not agree with their extrapolation of the data. In particular we reject their value of $f_1^+ = -1.66\mu^{-1}$,[14] which is based upon the estimate $a_1 + 2a_3 = -0.075\mu^{-1}$. Instead we take $a_1 + 2a_3 = (-0.02 \pm 0.03)\mu^{-1}$ which gives $f_1^+ = (-1.40 \pm 0.15)\mu^{-1}$. We also find that taking into account the various determinations of the p-wave isotopic-even scattering lengths increases the HJS value of f_2^+ to 1.27 ± 0.15, closer to the CD estimate. Thus we conclude that the low-energy methods of HJS are in substantially better agreement with the broad-area subtraction technique of CD than had been previously realized. We have also related the "smoothness method" of ACM to the on-shell technique of CD and HJS. The ACM method ignores f_2^+ cusp effects and leads to $\sigma_N = 79 \pm 16$ MeV.

Owing to the $O(\mu^4)$ theorem, the dispersion-theory $C_\Delta^{(+)}$ amplitude does not play a role in obtaining σ_N. Nevertheless, our expression for $C_\Delta^{(+)}$, (32), is intimately linked with a large σ term. At the Adler consistency point $\nu = 0$, $t = \mu^2$, (54) demands that $C^{(+)}$ cancel σ_N. Since $C^{(+)}(0, \mu^2) = C_\Delta^{(+)}(0, \mu^2) + \overline{C}^{(+)}(0, \mu^2) = -0.8 - 0.4$ from (32) and (71), we see that this is indeed the case.

The implication of ACM that $\sigma_N(t=0) = \sigma_N(t = 2\mu^2)$ may not be correct, as it is possible that σ_N may have a slight dependence upon the variable t. Our analysis is on-shell and avoids such a statement. Pagels and Pardee[38] have shown that the 2π contributions to the σ term could be nonanalytic in μ^2 and decrease the σ term by 14 MeV from $t = 2\mu^2$ to $t = 0$, leading to a reduction of our value to $\sigma_N(t=0)$

~60 MeV. This is consistent with Hakim[39] who finds $\sigma_N(t=0)=51\pm 9$ MeV by applying the lab-frame analysis of Fubini and Furlan.[40]

We now discuss some fundamental implications of such a large nucleon σ term. In terms of the σ model,[41,42] $\sigma_N = 65-75$ MeV is not difficult to understand. The prediction $\sigma_N = (\mu^2/m_\sigma^2)m_N \sim 70$ MeV implies $m_\sigma \sim 500$ MeV which in turn predicts the width $\Gamma_\sigma \sim 300$ MeV in the σ model. Thus the $I=0$, σ particle would be very hard to detect.

The "experimental" value for σ_N is a factor of four larger than the favored value[12] of $\sigma_N \sim 10-20$ MeV as predicted by the $(3,\bar{3})+(\bar{3},3)$-breaking scheme of Gell-Mann, Oakes, and Renner,[15]

$$\sigma_N = \tfrac{1}{3}(\sqrt{2}+c)[\sqrt{2}\,(u_0)_{NN} + (u_8)_{NN}] \ , \tag{77}$$

where $H = H_0 + H' = u_0 + cu_8$, with $c = -1.25$ as determined by the *quadratic* pseudoscalar mass formula. In a recent paper[43] we have shown that a model-independent estimate of the isotopic zero $KN\sigma$ term $\sigma(KN_0)$ implies

$$(u_0/u_8)_{NN} \approx 1 \ , \tag{78}$$

independent of c and f_K. Combining this with (77) and $\sigma_N \sim 70$ MeV leads to $c \sim -1.0$. This is consistent with the von-Hippel–Kim[44] estimate[43] of $\sigma(KN_0)\approx 0$, $\sigma(KN_1)\approx 170$ MeV, which in turn implies $c \sim -0.95$ or $\sigma(\pi N)\sim 90$ MeV.

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APPENDIX: DISPERSION THEORY OF AXIAL-VECTOR NUCLEON AMPLITUDES

It has been conjectured[45] and demonstrated for massive particles[21] that hadronic reactions of the type $A+B\to C+D$ can be expressed in terms of invariant amplitudes which are free of kinematic singularities (KSF) in ν and t. Even processes involving on-shell $q^2=0$ photons can be specified in terms of gauge-invariant amplitudes which are free of kinematic singularities (and zeros) in ν and t.[24,46,47] However *virtual* photon (or axial-vector) processes *do* involve intrinsic singularities because of the constraint of current conservation.

To illustrate the way in which such kinematic singularities can be treated, we consider two simple examples. In spinless electroproduction $\gamma_\mu(q) + B(p) \to C(p') + D(q')$, there are two independent current-conserving covariants K_μ obeying $q^\mu K_\mu = 0$:

$$K_\mu^1 = q\cdot q' P_\mu - q\cdot P q'_\mu \ , \tag{A1}$$

$$K_\mu^2 = q^2 q'_\mu - q\cdot q' q_\mu \ . \tag{A2}$$

However there is one further covariant,

$$K_\mu^3 = q^2 P_\mu - q\cdot P q_\mu \ , \tag{A3}$$

which can be eliminated in terms of the other two by the relation

$$q\cdot q' K_\mu^3 = q^2 K_\mu^1 + q\cdot P K_\mu^2 \ . \tag{A4}$$

Hence, given a perturbative calculation which formally leads to the KSF amplitudes ($\nu = q\cdot P$),

$$M_\mu = A_1(\nu, q'\cdot q, q^2)K_\mu^1 + A_2(\nu, q'\cdot q, q^2)K_\mu^2 + A_3(\nu, q'\cdot q, q^2)K_\mu^3 \ , \tag{A5}$$

one can eliminate K^3 in favor of K^1 and K^2 by (A4) and write

$$M_\mu = A_1'(\nu, q'\cdot q, q^2)K_\mu^1 + A_2'(\nu, q'\cdot q, q^2)K_\mu^2 \ . \tag{A6}$$

Then A_1' and A_2' contain an intrinsic kinematic singularity in $q\cdot q'$ which can be removed by the constraint condition[27,48]

$$\nu A_1' - q^2 A_2' = \text{finite as } q\cdot q' \to 0 \ , \tag{A7}$$

as can be seen by expressing $A_{1,2}'$ in terms of $A_{1,2,3}$. In practice, finding the exact KSF expansion (A5) is very difficult and alternatively being saddled by the constraint condition (A7) is at best cumbersome. Instead we shall respect this kinematic singularity by expanding $A_3(\nu, q\cdot q, q^2)$ in a power series in $q'\cdot q$ and converting all but the $(q'\cdot q)^0$ term into A_1 and A_2 by (A4). The "residual" term must be included in the resulting expansion:

$$M_\mu = A_1''(\nu, q'\cdot q, q^2)K_\mu^1 + A_2''(\nu, q'\cdot q, q^2)K_\mu^2 + B(\nu, q^2)K_\mu^3 \ , \tag{A8}$$

where *all* three amplitudes $A_{1,2}''$ and B are then KSF in ν, $q'\cdot q$ and q^2. (Strictly speaking, only $\mathrm{Im}B$ is independent of $q'\cdot q$.) Eliminating K_μ^1 in (A4) alternatively leads to an equivalent KSF expansion in $K_\mu^{2,3}$ and the residual term $C(\nu, q'\cdot q)K_\mu^1$.

The forward Compton scattering of virtual-charged photons off a spinless target $\gamma_\nu(q)+N(p) \to \gamma_\mu(q)+N(p)$ gives further insight into such intrinsic kinematic singularities. Bjorken[9] has observed that the four "natural" covariants in the expansion

$$M_{\mu\nu} = A(\nu, q^2)p_\mu p_\nu + B(\nu, q^2)(p_\mu q_\nu + q_\mu p_\nu) + C(\nu, q^2)q_\mu q_\nu + D(\nu, q^2)g_{\mu\nu} \tag{A9}$$

cannot be expressed in terms of the same number of current-conserving (CC) and non-current-conserving (NCC) amplitudes. Instead one must write

$$M_{\mu\nu} = M_{\mu\nu}^{CC} + M_{\mu\nu}^{NCC} + M_{\mu\nu}^{RES} , \tag{A10}$$

where (setting $\nu = p\cdot q$, $m=1$)

$$\begin{aligned} M_{\mu\nu}^{CC} &= A_1(\nu, q^2)[q_\mu q_\nu - q^2 g_{\mu\nu}] \\ &\quad + A_2(\nu, q^2)[q^2 p_\mu p_\nu - \nu(p_\mu q_\nu + q_\mu p_\nu) + \nu^2 g_{\mu\nu}] , \\ M_{\mu\nu}^{NCC} &= B_1(\nu, q^2)(p_\mu q_\nu + q_\mu p_\nu) + B_2(\nu, q^2) g_{\mu\nu} , \\ M_{\mu\nu}^{RES} &= C(\nu) p_\mu p_\nu . \end{aligned} \tag{A11}$$

The additional "residual" amplitude $C(\nu)$ must be included; without it B_1 and B_2 would contain a kinematic singularity in q^2. To see the effect of this term we compute $q'^\mu M_{\mu\nu} q^\nu$ [which can be interpreted as the background πN forward amplitude (20)]:

$$q'^\mu M_{\mu\nu} q^\nu = q^2 \nu B_1(\nu, q^2) + q^2 B_2(\nu, q^2) + \nu^2 C(\nu) . \tag{A12}$$

At $q^2=0$ it is clear that the entire (nonpole) amplitude is given by $C(\nu)$. It is important to realize that the current-conserving amplitudes A_1 and A_2 remain KSF in ν and q^2.

Now we are in position to treat the problem of interest: virtual-charged Compton (axial-vector) scattering off a spin-$\frac{1}{2}$ nucleon target as a function of the three invariants t, ν (again set $m=1$), and q^2 (we take $q'^2 = q^2$ to simplify matters slightly). In analogy to (A9) there are 20 "natural" covariants (although there are 22 obvious covariants, there are two "equivalence theorems"[21-24] which reduce the number to 20). We then write

$$M_{\mu\nu}(20) = M_{\mu\nu}^{CC1}(12) + M_{\mu\nu}^{CC2}(6) + M_{\mu\nu}^{NCC}(2) + M_{\mu\nu}^{RES} , \tag{A13}$$

where there are 12 spin-1–spin-1 transitions, 6 spin-1–spin-0 transitions (current-conserving only in both momenta $q'^\mu M_{\mu\nu}^{CC2} q^\nu = 0$), 2 spin-0–spin-0 transitions plus a number of additional residual amplitudes of both the electroproduction and forward Compton type as yet to be determined.

First we define eight auxiliary amplitudes ($m=1$, $\not{Q} = \gamma\cdot Q$):

$$\begin{aligned} R_{\mu\nu}^1 &= q'_\mu q'_\nu + q_\mu q_\nu , \\ R_{\mu\nu}^2 &= q_\mu P_\nu + P_\mu q'_\nu , \\ R_{\mu\nu}^3 &= q'_\mu P_\nu + P_\mu q_\nu , \\ R_{\mu\nu}^4 &= (q_\mu P_\nu + P_\mu q'_\nu) - (q_\mu \gamma_\nu + \gamma_\mu q'_\nu) , \\ R_{\mu\nu}^5 &= (q'_\mu P_\nu + P_\mu q_\nu) - (q'_\mu \gamma_\nu + \gamma_\mu q_\nu) , \\ R_{\mu\nu}^6 &= 2P_\mu P_\nu - (\gamma_\mu P_\nu + P_\mu \gamma_\nu) , \\ R_{\mu\nu}^7 &= q_\mu [\not{Q}, \gamma_\nu] + [\gamma_\mu, \not{Q}] q'_\nu , \\ R_{\mu\nu}^8 &= q'_\mu [\not{Q}, \gamma_\nu] + [\gamma_\mu, \not{Q}] q_\nu . \end{aligned} \tag{A14}$$

Next we display the six doubly current-conserving covariants of $M_{\mu\nu}^{CC2}$ corresponding to spin-1–spin-0 transitions which we label by $L_{\mu\nu}$:

$$\begin{aligned} L_{\mu\nu}^1 &= 2q'\cdot q q'_\mu q_\nu - q^2 R_{\mu\nu}^1 , \\ L_{\mu\nu}^2 &= q'\cdot q R_{\mu\nu}^3 - \nu R_{\mu\nu}^1 , \\ L_{\mu\nu}^3 &= [\not{q}', \not{q}] q'_\mu q_\nu - q^2 R_{\mu\nu}^8 , \\ L_{\mu\nu}^4 &= q'_\mu [\not{q}, \gamma_\nu] + [\gamma_\mu, \not{q}'] q_\nu , \\ L_{\mu\nu}^5 &= [\not{q}', \not{q}] R_{\mu\nu}^3 - 4\nu R_{\mu\nu}^5 , \\ L_{\mu\nu}^6 &= [\not{q}', \not{q}] R_{\mu\nu}^1 - 2q'\cdot q R_{\mu\nu}^8 , \end{aligned} \tag{A15}$$

where $q'^\mu L_{\mu\nu} q^\nu = \epsilon'^\mu L_{\mu\nu} \epsilon^\nu = 0$, but $q'^\mu L_{\mu\nu} \neq 0$, $L_{\mu\nu} q^\nu \neq 0$. Then we express the 12 current-conserving KSF covariants of $M_{\mu\nu}^{CC1}$ in terms of the above covariants ($[\gamma\not{Q}\gamma]_{\mu\nu} \equiv \gamma_\mu \not{Q} \gamma_\nu - \gamma_\nu \not{Q} \gamma_\mu$):

$$\begin{aligned} K_{\mu\nu}^1 &= q_\mu q'_\nu - q'\cdot q g_{\mu\nu} , \\ K_{\mu\nu}^2 &= q'\cdot q P_\mu P_\nu - \nu(R_{\mu\nu}^2 - \nu g_{\mu\nu}) , \\ K_{\mu\nu}^3 &= L_{\mu\nu}^2 - q^2(R_{\mu\nu}^2 - \nu g_{\mu\nu}) + q^2 \nu g_{\mu\nu} , \\ K_{\mu\nu}^4 &= L_{\mu\nu}^1 - q'\cdot q q'_\mu q_\nu + q^4 g_{\mu\nu} , \\ K_{\mu\nu}^5 &= q'\cdot q L_{\mu\nu}^4 - q^2(R_{\mu\nu}^7 - 2R_{\mu\nu}^4) , \\ K_{\mu\nu}^6 &= q'\cdot q(R_{\mu\nu}^6 - \tfrac{1}{2}[\gamma\not{Q}\gamma]_{\mu\nu}) + \tfrac{1}{2}\nu([\not{q}', \not{q}] g_{\mu\nu} - 4R_{\mu\nu}^4) , \\ K_{\mu\nu}^7 &= [\not{q}', \not{q}] P_\mu P_\nu - \nu(2R_{\mu\nu}^6 + [\gamma\not{Q}\gamma]_{\mu\nu}) , \\ K_{\mu\nu}^8 &= L_{\mu\nu}^5 - 2q^2[\gamma\not{Q}\gamma]_{\mu\nu} , \\ K_{\mu\nu}^9 &= 2L_{\mu\nu}^3 + q^2 L_{\mu\nu}^4 + q^4[\gamma_\mu, \gamma_\nu] - [\not{q}', \not{q}] q'_\mu q_\nu , \\ K_{\mu\nu}^{10} &= 2R_{\mu\nu}^4 + R_{\mu\nu}^7 - [\not{q}', \not{q}] g_{\mu\nu} - q'\cdot q[\gamma_\mu, \gamma_\nu] , \\ K_{\mu\nu}^{11} &= [\not{q}', \not{q}] R_{\mu\nu}^2 - 4\nu R_{\mu\nu}^4 - 2q'\cdot q[\gamma\not{Q}\gamma]_{\mu\nu} , \\ K_{\mu\nu}^{12} &= L_{\mu\nu}^6 + 2q^2(R_{\mu\nu}^7 - [\not{q}', \not{q}] g_{\mu\nu}) , \end{aligned} \tag{A16}$$

where $q'^\mu K_{\mu\nu}^i = K_{\mu\nu}^i q^\nu = 0$. The detailed explanation and prescription for obtaining $K_{\mu\nu}^i$ as a gauge-invariant set of covariants is given in Refs. 24, 46, and 47. The covariants K^1, K^2, K^5, K^7, K^{10}, and K^{11} correspond to the KSF set for real photon Compton scattering,[24,46,47] and K^1 and K^2 become the two KSF covariants for forward virtual Compton scattering in (A11). There are actually two more covariants, $[\not{q}', \not{q}] K_{\mu\nu}^1$ and

$$q'\cdot q(P_\mu[\not{Q}, \gamma_\nu] + [\gamma_\mu \not{Q}] P_\nu) - \nu R_{\mu\nu}^7 + [\not{q}', \not{q}](\nu g_{\mu\nu} - \tfrac{1}{2} R_{\mu\nu}^2),$$

which can be expressed in terms of the above 12 *gauge-invariant* $K_{\mu\nu}^i$ by the "equivalence theorems" of Refs. 21, 24, and 46. This is one of the crucial points in our approach. Once having identified the two additional gauge-invariant covariants to be eliminated by the equivalence theorems, we need *not* concern ourselves with the equivalence theorems, per se. The remaining two spin-0–spin-0 amplitudes $M_{\mu\nu}^{NCC}$ can be specified by the covariants $N_{\mu\nu}$ which obey $\epsilon'^\mu N_{\mu\nu} = N_{\mu\nu} \epsilon^\nu = 0$ and are not gauge-invariant. They are obviously

$$N^1_{\mu\nu}=q'_\mu q_\nu \ ,$$
$$N^2_{\mu\nu}=[\not{q}',\not{q}]q'_\mu q_\nu \ . \tag{A17}$$

Finally we determine the additional residual amplitudes $M^{\rm RES}_{\mu\nu}$ needed to avoid introducing the intrinsic kinematic singularities into the above set of 20 covariants. In analogy with electroproduction, there are the additional covariants

$$L^7_{\mu\nu}=q^2R^3_{\mu\nu}-2\nu q'_\mu q_\nu \ ,$$
$$K^{13}_{\mu\nu}=q^4P_\mu P_\nu-\nu^2q'_\mu q_\nu-\nu L^7_{\mu\nu} \ , \tag{A18}$$
$$K^{14}_{\mu\nu}=q^2(2R^6_{\mu\nu}-[\gamma\not{Q}\gamma]_{\mu\nu}-\nu[\gamma_\mu,\gamma_\nu])+\nu L^4_{\mu\nu} \ ,$$

which are related to our above set by

$$q'\cdot q L^7_{\mu\nu}=q^2L^2_{\mu\nu}-\nu L^1_{\mu\nu} \ ,$$
$$q'\cdot qK^{13}_{\mu\nu}=q^4K^2_{\mu\nu}-q^2\nu K^3_{\mu\nu}+\nu^2K^4_{\mu\nu} \ , \tag{A19}$$
$$q'\cdot qK^{14}_{\mu\nu}=2q^2K^6_{\mu\nu}+q^2\nu K^{10}_{\mu\nu}+\nu K^5_{\mu\nu} \ .$$

Following (A8) we therefore include the electroproduction-type residual terms (EP)

$$M^{\rm RES}_{\mu\nu}({\rm EP})=B_1(\nu,q^2)L^7_{\mu\nu}+B_2(\nu,q^2)K^{13}_{\mu\nu}+B_3(\nu,q^2)K^{14}_{\mu\nu} \ . \tag{A20}$$

By inspection of our sets $L_{\mu\nu}(6)$ and $K_{\mu\nu}(12)$, one can also infer the forward Compton-type residual terms (FC) analogous to $C(\nu)p_\mu p_\nu$ which one must add to the amplitude in order to ensure KSF-invariant amplitudes:

$$\begin{aligned} M^{\rm RES}_{\mu\nu}({\rm FC}) &= C_1(\nu,q'\cdot q)R^1_{\mu\nu}+C_2(\nu,q'\cdot q)(R^2_{\mu\nu}-\nu g_{\mu\nu}) \\ &+C_3(\nu,q^2)R^3_{\mu\nu}+C_4(\nu,q'\cdot q)(2R^4_{\mu\nu}-R^7_{\mu\nu}) \\ &+C_5(\nu,q'\cdot q)[\gamma\not{Q}\gamma]_{\mu\nu} \\ &+C_6(\nu,q^2)(R^6_{\mu\nu}-\tfrac{1}{2}[\gamma\not{Q}\gamma]_{\mu\nu}) \\ &+C_7(\nu,q'\cdot q)(R^7_{\mu\nu}-[\not{q}',\not{q}]g_{\mu\nu}) \\ &+C_8(\nu,q'\cdot q)R^8_{\mu\nu} \\ &+[C_9(\nu,q'\cdot q)+C'_9(\nu,q'\cdot q)q^2]g_{\mu\nu} \\ &+[C_{10}(\nu,q'\cdot q)+C'_{10}(\nu,q'\cdot q)q^2][\gamma_\mu,\gamma_\nu] \\ &+[C_{11}(\nu)+C'_{11}(\nu)q^2]P_\mu P_\nu \ . \end{aligned} \tag{A21}$$

Table IV shows the result of expanding each of a set of 22 natural covariants in terms of ours, calculating the invariants at a resonance position $\nu=\nu_R$ and multiplying with $q'^\mu(\ )q^\nu$. To prove that our choice of (A15), (A16), (A17), (A20), and (A21) leads to a KSF expansion (A13), one simply solves for the natural covariants in terms of our set and verifies that no kinematic singularities in ν, $q'\cdot q$, or q^2 are thereby introduced.

Finally we compute $q'_\mu M^{\mu\nu}q_\nu$ from (A13). All of the contributions come directly from (A17) and

TABLE IV. Contributions from natural covariants of current-nucleon elastic scattering.

Covariants contributing to spinor 1	Contribution
$g_{\mu\nu}$	$q'\cdot q$
$q_\mu q'_\nu$	$(q'\cdot q)^2$
$q'_\mu q_\nu$	q^4
$q'_\mu q'_\nu+q_\mu q_\nu$	$2q^2(q'\cdot q)$
$g^2(q_\mu P_\nu+P_\mu q'_\nu)$	$[2g^2\nu_R+h_0(\nu-\nu_R)](q'\cdot q)$
$g^2(q'_\mu P_\nu+P_\mu q_\nu)$	$[2g^2\nu_R+2h_0(\nu-\nu_R)]q^2$
$g^2P_\mu P_\nu$	$g^2\nu_R^2+(h_0+h_1q^2)(\nu+\nu_R)(\nu-\nu_R)$
Covariants contributing to spinor $[\not{q}',\not{q}]$	**Contribution**
$g_{\mu\nu}[\not{q}',\not{q}]$	$q'\cdot q[\not{q}',\not{q}]$
$q_\mu q'_\nu[\not{q}',\not{q}]$	$(q'\cdot q)^2[\not{q}',\not{q}]$
$q'_\mu q_\nu[\not{q}',\not{q}]$	$q^4[\not{q}',\not{q}]$
$(q'_\mu q'_\nu+q_\mu q_\nu)[\not{q}',\not{q}]$	$2q^2(q'\cdot q)[\not{q}',\not{q}]$
$\gamma_\mu\not{Q}\gamma_\nu-\gamma_\nu\not{Q}\gamma_\mu$	0
$(q_\mu P_\nu+P_\mu q'_\nu)[\not{q}',\not{q}]$	$2\nu_R(q'\cdot q)[\not{q}',\not{q}]$
$(q'_\mu P_\nu+P_\mu q_\nu)[\not{q}',\not{q}]$	$2\nu_Rq^2[\not{q}',\not{q}]$
$[\gamma_\mu,\gamma_\nu]$	$[\not{q}',\not{q}]$
$q_\mu[\not{Q},\gamma_\nu]+[\gamma_\mu,\not{Q}]q'_\nu$	$q'\cdot q[\not{q}',\not{q}]$
$q'_\mu[\not{Q},\gamma_\nu]+[\gamma_\mu,\not{Q}]q_\nu$	$q^2[\not{q}',\not{q}]$
$q_\mu P_\nu+P_\mu q'_\nu-(q_\mu\gamma_\nu+\gamma_\mu q'_\nu)$	$\tfrac{1}{2}q'\cdot q[\not{q}',\not{q}]$
$q'_\mu P_\nu+P_\mu q_\nu-(q'_\mu\gamma_\nu+\gamma_\mu q_\nu)$	$\tfrac{1}{2}q^2[\not{q}',\not{q}]$
$P_\mu[\not{Q},\gamma_\nu]+[\gamma_\mu,\not{Q}]P_\nu$	$\nu[\not{q}',\not{q}]$
$g^2P_\mu P_\nu[\not{q}',\not{q}]$	$[g^2\nu_R^2+h_0\nu_R(\nu-\nu_R)][\not{q}',\not{q}]$
$g^2\{2P_\mu P_\nu-(\gamma_\mu P_\nu+P_\mu\gamma_\nu)\}$	$[\tfrac{1}{2}g^2\nu_R+\tfrac{1}{2}h_0(\nu-\nu_R)][\not{q}',\not{q}]$

(A21) with $M^{\rm NCC}_{\mu\nu}=A_{19}N^1_{\mu\nu}+A_{20}N^2_{\mu\nu}$:

$$\begin{aligned} q'_\mu M^{\mu\nu}q_\nu &= C_1(\nu,q'\cdot q)2q^2(q'\cdot q) \\ &+C_2(\nu,q'\cdot q)\nu q'\cdot q+C_3(\nu,q^2)2q^2 \\ &+C_6(\nu,q^2)\tfrac{1}{2}\nu[\not{q}',\not{q}]+C_8(\nu,q'\cdot q)\tfrac{1}{2}q^2[\not{q}',\not{q}] \\ &+[C_9(\nu,q'\cdot q)+q^2C'_9(\nu,q'\cdot q)]q'\cdot q \\ &+[C_{10}(\nu,q'\cdot q)+q^2C'_{10}(\nu,q'\cdot q)][\not{q}',\not{q}] \\ &+[C_{11}(\nu)+C'_{11}(\nu)q^2]\nu^2+A_{19}(\nu,q'\cdot q,q^2)q^4 \\ &+A_{20}(\nu,q'\cdot q,q^2)q^4[\not{q}',\not{q}] \ . \end{aligned} \tag{A22}$$

On the assumption that $F^{(+)}$ is the only *pion amplitude* requiring a subtraction in ν, (A22) leads us to expect that all amplitudes except the crossing-even parts of C_1, C_9, and A_{19} will obey USDR. The contribution of C_1 is suppressed relative to C_9 by an extra factor μ^2 on shell. Furthermore, only A_{19} contributes at $q'\cdot q=0$.

The subtraction in C_9 is just what is needed to explain the fact that the decuplet accounts for only $\frac{2}{3}$ of the experimental t variation in $C^{(+)}$. The subtraction in A_{19} will share the general μ^4 suppression, and may even have a small effect when compared to a large decuplet pole contribution. Hence the working assumption that all amplitudes are unsubtracted is not an essential input to obtain our results.

Since the C_i are KSF by instruction, from (A22) we see that only A_{19} will contribute to $C^{(+)}$ in the limit $q'\cdot q=\nu=0$, which establishes theorem (i).

To begin the proof of theorem (ii), note that only C_3, C_{11}, and A_{19} contribute to $C^{(-)}$ when $q'\cdot q=0$, and that the factor ν^2 prevents C_{11} from contributing to $\nu^{-1}C^{(-)}$ at $\nu=0$. The proof of the theorem thus consists of showing that R_3 is absent in the expansion of $\gamma_\mu(\not K\pm M)\gamma_\nu$, where $K_\mu=P_\mu+Q_\mu$ in the s channel. Since it is easy to see this for $\gamma_\mu\gamma_\nu$, we need only establish it for the expression (27), a straightforward task from Table IV.

Theorem (iv) arises from the absence in (A22) of any term of the form $\nu^2[\not q', \not q]$. The reason is easily seen to be the fact that $P_\mu P_\nu[\not q', \not q]$ can be written directly in terms of $K^7_{\mu\nu}$ without inducing a residual term. Theorem (v) results from the absence of any terms of order ν^3 or higher in (A22).

Note that the nucleon nonpole corrections $C_N^{(\pm)}$, $D_N^{(\pm)}$ satisfy the theorems separately, so that they can be treated as part of the backgrounds $\overline{C}^{(\pm)}$ and $\overline{D}^{(\pm)}$ of Sec. (V).

We now show that high-mass resonances are more likely to cause significant ν^2 variation for $\overline{C}^{(+)}$ than for $\nu^{-1}\overline{C}^{(-)}$, $\nu^{-1}\overline{D}^{(+)}$, or $\overline{D}^{(-)}$. First note that the dominant contribution to $C^{(\pm)}$ away from $\nu=0$ comes from C_{11}, which also is responsible for most of the ν variation. Writing the contribution of a resonance at ν_R as

$$C_{11}^R(\nu)=\frac{\alpha^{(+)}\nu_R}{\nu_R{}^2-\nu^2}+\frac{\alpha^{(-)}\nu}{\nu_R{}^2-\nu^2}\ ,$$

so that for $\nu_R\gg\nu$

$$C_{11}^R(\nu)\nu^2\approx\frac{\alpha^{(+)}}{\nu_R}\nu^2+\left(\frac{\alpha^{(-)}}{\nu_R{}^2}\nu^2\right)\ \nu\ ,$$

we see that an extra factor ν_R enhances the contribution to $C^{(+)}$ over that to $\nu^{-1}C^{(-)}$. Put another way, the dispersion relation for $C_{11}^{(+)}(\nu)$ converges more slowly than that for $C_{11}^{(-)}(\nu)$.

No similar situation exists for $D^{(\pm)}$ because of the absence of a term in (A13) proportional to $\nu^2[\not q', \not q]$. The greatest ν variation in D will come from C_6:

$$C_6^R(\nu)=\frac{\beta^{(+)}\nu_R}{\nu_R{}^2-\nu^2}+\frac{\beta^{(-)}\nu}{\nu_R{}^2-\nu^2}\ ,$$

contributing $\beta^{(+)}/\nu_R$ to $\nu^{-1}D^{(+)}$ and $\nu^2\beta^{(-)}/\nu_R{}^2$ to $D^{(-)}$. The resulting variation in $D^{(-)}$ is suppressed by an extra factor $1/\nu_R$. It should be emphasized that this result for $D^{(\pm)}$ is valid only with our approach.

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