necessitate that $m(\mathcal{P}'_1) - m(\mathcal{P}'_2)$ be small on a quark-mass scale, however.]

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Singularities in three-body final-state amplitudes*

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Three-body amplitudes have two-body threshold singularities over and above those coming from the usual treatments of final-state interactions. These are analyzed both formally and numerically. In the numerical model studies, they account for an important variation of the amplitude. The singularity comes from the "next-to-last" rescattering, and hence, may be represented correctly by any approximate amplitude that has that rescattering, even if the approximation scheme diverges. This may account for the correct shape (not magnitude) of multibody spectra determined with a divergent multiple-scattering series.

I. INTRODUCTION

How to extract two-body information from a multibody final state is an old problem. Empirically, we have learned much from such situations in nearly every branch of physics, but when there are more than two strongly interacting particles in the final state, there exists little in the way of a firm theoretical basis for the analysis. In this paper we investigate the nature and consequences of an important singularity in the final-state amplitude that has been ignored in most previous analyses. This is a threshold singularity in the pair subenergy of each final-state pair. It is therefore on the boundary of the physical region, and produces considerable variation of the amplitudes over that region. Its neglect can lead to incorrect final-state parameters. We carry out our analysis in the context of the three-body problem, but the existence and nature of the singularity is by no means restricted to that case.

The existence of this singularity was already implicit in 1967[1,2] in a different guise, but we were not then aware of its importance for phenomenology. Much more recently we have demonstrated its presence in a general way through unitarity.[3] In this paper we explore further its origin, nature, and numerical importance.

It is customary to decompose a three-body final-state amplitude into a sum of terms, depending on which pair interacts last,

$$M=\sum_{i=1}^{3} f_i\,\tau_i(\sigma_i)\,, \qquad \text{(a)}$$

where τ_i is the j-kth pair's two-body t matrix ($i\neq j\neq k$) and σ_i is that pair's center-of-mass energy. f_i is the coefficient of τ_i in the decomposition. This form is closely related to the Faddeev or multiple-scattering expansion of nuclear physics and to the isobar expansion of particle physics.[4] Most empirical analyses proceed by assuming that f_i is slowly varying and that f_i and f_j ($i\neq j$) are totally independent. But what we have already shown is that for small σ_i (Ref. 1–3)

$$f_j=A_j+i(\sigma_j)^{1/2}B_j$$

and

$$B_j=\sum_{k\neq j} f_k\,\tau_k\ . \qquad \text{(b)}$$

This means that f_i has a square-root singularity in the pair subenergy of the j-kth pair and the coefficient of the singularity is the "non-i" term in the decomposition. Near $\sigma_i=0$ we can write

$$M\cong(A+iBq)\,e^{i\delta}\,\frac{\sin\delta}{q}+B\,,$$

where $q=(\sigma_i)^{1/2}$, $\tau_i=(e^{i\delta}\sin\delta)/q$ and we have dropped the i label to keep the expression simple. Elementary algebra then gives

$$M=A\,e^{i\delta}\,\frac{\sin\delta}{q}+B\,e^{i\delta}\cos\delta\ .$$

This result on the coherence of the amplitude shows that in some sense it is the entire amplitude that carries the phase δ and not just a part. This is a kind of Watson theorem which has been known for some time in some circles,[1,2,5] but its significance for empirical analyses has not been exploited.

In this paper we shall show how the $(\sigma_i)^{1/2}$ emerges from a standard singularity analysis of the multiple-scattering series. This will generalize the result from the previous Faddeev-equation analysis[1] and flesh out the unitarity proof.[3] We shall see that the singularity comes from the very last rescattering in f. We will then investigate the singularity in a simple numerical three-body model and see that it is numerically quite important. In fact, for a wide range of dynamical parameters, the variation of f is largely controlled by the singularity.

Since the singularity of f comes from the last rescattering, the structure of the singularity is well represented in any theory that has that last rescattering, for example, the first term in the multiple-scattering series. We show numerically that even if that series diverges badly, the singularity structure is well determined by the first few terms. Since the major shape of f is determined by that singularity, the first few multiple-scattering terms can give a good shape for f, and hence the spectrum, even when the series diverges. This may account for the success of truncated multiple-scattering approximations for spectral shapes in a wide class of three-body problems for which the series is known to diverge or converge slowly.

In Sec. II we discuss the singularity analysis necessary to obtain the $\sigma^{1/2}$ term in f. In Sec. III we present a numerical model that shows the importance of the singular term and the numerical validity of a simple parametrization of f. Section IV shows that the first few multiple-scattering terms can faithfully reproduce the singularity shape even though the series diverges. Section V presents a discussion of the results and some problems for the future.

II. SINGULARITY ANALYSIS

In this section we shall discuss the dependence of a three-body amplitude on the pair subenergies from the point of view of the singularity structure of the perturbation series. This will yield complementary information to that already obtained in a Schrödinger equation type of analysis[1] and from unitarity.[3] The singularity structure of three-body amplitudes has been previously analyzed, but largely from the point of view of the total energy.[6] In our analysis here we shall fix the total energy and look at the dependence on the invariant pair subenergies, that is, on the energy of a pair in its own center-of-mass system. In particular we shall look for singularities in the subenergies that are in (or on the boundaries of) the physical regions, since these will be most likely to produce rapid dependence of amplitudes.

In order to have an explicit case to study, we follow AFR and consider the weak decay of a particle into three equal-mass particles that are then allowed to interact strongly. We will follow the notation of AFR except that we take $\hbar = m = 1$, where m is the mass of any of the three particles. We take nonrelativistic kinematics for the present. How all these restrictions may be relaxed will be discussed later.

Consider an nth-order graph for the decay process as shown in Fig. 1(a). Only $\vec{q}_0$ and $\vec{q}_{-1}$ are external momenta. They enter the amplitude through potential terms; for example, for Yukawa potentials we have terms like

$$[(\vec{q}_{-1} - \vec{q}_1)^2 + \mu^2]^{-1}$$

and (1a)

$$[(\vec{q}_0 - \vec{q}_2)^2 + \mu^2]^{-1} ,$$

where μ is the Yukawa range. They also enter

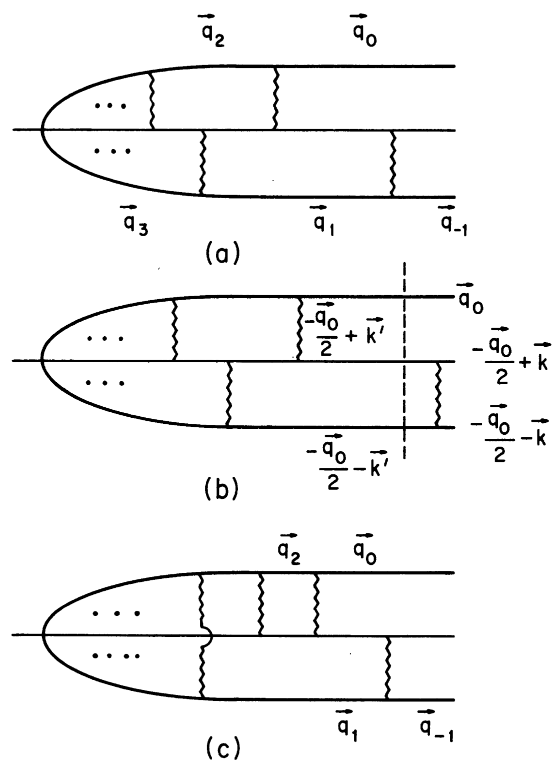

FIG. 1. Some typical nth-order diagrams for the three-body decay process. (a) A simple skeleton graph. (b) The same as (a) with a variable change. The dashed line gives the location of the singular propagator. (c) A more complex graph with the same subenergy singularity as (a) or (b).

through a propagator

$$(E - q_1^2 - q_0^2 - \vec{q}_1 \cdot \vec{q}_0)^{-1} . \qquad (1b)$$

Since (1a) cannot vanish for physical momenta, there are no singularities associated with the potential terms in or on the boundaries of the physical region. For that reason the precise form of the potential terms does not matter. If, for example, we used Yamaguchi separable potentials of range β rather than the Yukawa potentials, we would replace (1a) by

$$[(\vec{q}_{-1} + \tfrac{1}{2}\vec{q}_0)^2 + \beta^2]^{-1}[(\vec{q}_1 + \tfrac{1}{2}\vec{q}_0)^2 + \beta^2]^{-1}$$

and (1a′)

$$[(\vec{q}_0 + \tfrac{1}{2}\vec{q}_1)^2 + \beta^2]^{-1}[(\vec{q}_2 + \tfrac{1}{2}\vec{q}_1)^2 + \beta^2]^{-1} ,$$

but again these cannot vanish for physical momenta. The only remaining dependence of the amplitude associated with Fig. 1(a) on external momenta is from (1b), the propagator, and in AFR it is shown that this gives a singularity of the type[7]

$$(E - \tfrac{3}{4} q_0^2)^{1/2} . \qquad (2)$$

To see this more clearly we relabel the momenta as in Fig. 1(b). This relabeling does not affect the singularity. The location of the propagator [Eq. (1b)] is indicated by a dashed line. It becomes

$$[E - \tfrac{3}{4} q_0^2 - k'^2]^{-1} . \qquad (1b')$$

Clearly the vanishing of this denominator in the d^3k' integral will give the singularity of Eq. (2). Furthermore, energy conservation is

$$E = \tfrac{3}{4} q_0^2 + k^2 , \qquad (3)$$

where k^2 is the c.m. energy of the lower pair. Hence Eq. (2) represents a two-body threshold singularity in that pair's subenergy. This singularity is then on the boundary of the physical region. Since the momenta labels $\vec{q}_0$ and $\vec{q}_{-1}$ do not penetrate into the diagram of Fig. 1(a), there are no further subenergy singularities associated with it. This is shown explicitly in AFR, where it is further shown that most of the problem is involved in getting the nature of the total energy singularity structure.

Since the potential between $\vec{q}_2$ and $\vec{q}_0$ in Fig. 1(a) cuts $\vec{q}_0$ and $\vec{q}_{-1}$ off from the rest of the graph, our discussion is independent of what happens to the left of that point. Hence, it applies equally well to graphs of the type shown in Fig. 1(c) and to more complex graphs. However, Fig. 2(a) is a new type that we must study. It is clear that it can acquire a $(E - \frac{3}{4} q_0^2)^{1/2}$ type singularity from each of the propagators indicated by the dotted lines. In fact the sum of all rescatterings between the lower pair can be summed into the two-body t matrix. The graph then can be represented as

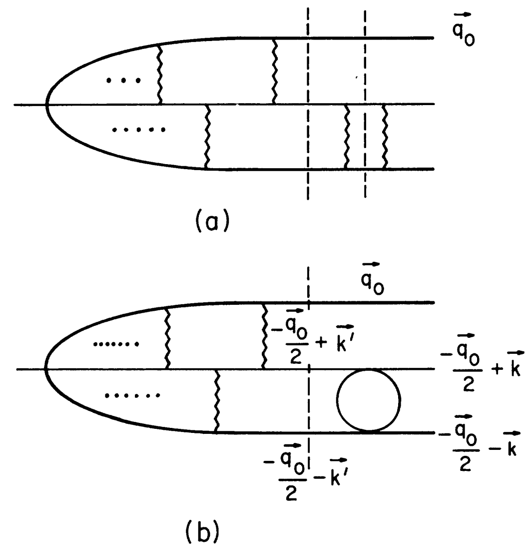

FIG. 2. (a) A typical graph for the three-body decay with more than one rescattering subenergy singularity (located at dashed lines). (b) The sum of all such graphs. The large circle represents the two-body t matrix.

in Fig. 2(b). The amplitude associated with this graph can be written

$$\int \frac{h(\vec{q}_0, \vec{k}')(\vec{k}'|t(k^2)|\vec{k})}{E - \frac{3}{4} q_0^2 - k'^2} \frac{d^3k'}{(2\pi)^3} , \qquad (4)$$

where $(\vec{k}'|t(k^2)|\vec{k})$ is the half-on-shell two-body t matrix of the lower pair, and we have used energy conservation, Eq. (3), to express the energy argument of the t matrix as k^2. $h(\vec{q}_0, \vec{k}')$ is everything to the left of the propagator indicated by the dotted line in Fig. 2(b). The arguments given above show that it has no physical region $\vec{q}_0$ singularity. Since the $\vec{k}'$ dependence of $(\vec{k}'|t(k^2)|\vec{k})$ involves potential terms like (1a) or (1a′), the $\vec{k}'$ integral in (4) can only give physical subenergy singularities by virtue of the propagator. This will, of course, give the same pair-energy threshold singularity as before in Eq. (2).

The sum of *all* graphs which end with a scattering between the lower pair can be written in the form (4). In that case $h(\vec{q}_0, \vec{k}')$ is the sum of all graphs that do not end in such a scattering. Hence, at the singular point $(E = \frac{3}{4} q_0^2)$ in the full amplitude, the coefficient of the singularity is $h(\vec{q}_0, 0)(0|t(0)|0)$, where h is the sum of all graphs not ending in a scattering of the bottom pair and $(0|t(0)|0)$ is the on-shell lower-pair t matrix at zero energy. This important connection between the singular part of these graphs and the sum of all graphs that do not end with a scattering of the lower pair was developed in the context of the Faddeev equation several years ago.[1]

If we specialize to a case where the two-body

t matrix is dominated by a single partial wave (we take s waves to simplify the kinematics), we can write

$$(\vec{k}'|t(k^2)|\vec{k}) = A(k,k')\,\tau(k^2)\,, \tag{5}$$

where $\tau(k^2)$ is the on-shell t matrix and $A(k,k')$ is the half-shell function. Clearly $A(k,k)=1$. Further A has only potential or unphysical singularities in k and k'. Using (5) we can write (4) as

$$\int \frac{h(\vec{q}_0,\vec{k}')A(k,k')}{E-\frac{3}{4}q_0^{\,2}-k'^2}\frac{d^3k'}{(2\pi)^3}\tau(k^2)\,, \tag{6}$$

which is the isobar form (a). The analytic properties of A guarantee that the coefficient of $\tau(k^2)$ in (6) has only the $(E-\frac{3}{4}q_0^{\,2})^{1/2}$ singularity as a function of the subenergy. This form (6) connects our treatment here with the unitarity discussion of Aaron and Amado.[3]

We have seen that the amplitude for the decay of one particle to three has a threshold square-root singularity in the subenergies of the pairs over and above those of the two-body t matrix. This singularity comes from the propagator just before the last pair scatterings. There are no further physical pair subenergy singularities from farther "into" the multiple scattering diagram essentially because the external labels do not penetrate in. It is clear that very similar arguments would apply to n-body amplitudes ($n \geq 3$), since again the external labels do not get into the graphs. Superficially it might appear that the same is true for two bodies. Figure 3 shows a typical two-body amplitude, and we see that $\vec{q}_0$, the external-momenta label, does not penetrate in. But of course energy conservation is $E=q_0^{\,2}$, and E appears everywhere. The extra freedom to define the subenergy independently of the total energy that occurs for $n \geq 3$ is essential to our discussion.

We now turn to the question of the rather restrictive case we have discussed. Clearly, the equality of the masses, which has kept the algebra simple, is of no essential importance. There will always be a pair subenergy threshold square-root singularity associated with the propagator of an equation like (4). Furthermore, since it is a threshold singularity, it will be the same in relativistic as in nonrelativistic theory as long as the relativistic theory has a proper nonrelativistic limit. Furthermore, nowhere have we used the structure of the decay vertex, and hence, precisely the same arguments would apply in a production or breakup amplitude ($2\to 3$). In general, then, in an amplitude for producing more than two particles, the coefficient of a particular final-pair t matrix can be expected to have a square-root singularity in that pair subenergy, and that singularity comes from the very last propagator before the t matrix. Furthermore, unitarity relates the coefficient of the singularity to the other parts of the amplitude.

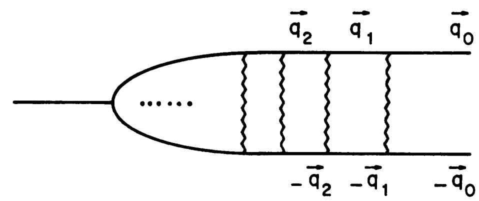

FIG. 3. A typical two-body decay amplitude graph.

III. A NUMERICAL MODEL

In this section we examine a simple numerical model of a three-body system to demonstrate explicitly the importance of the threshold singularity and to study its features. The model we take is that of the weak decay of a particle to three identical bosons ($\hbar = m = 1$). The final particles interact strongly via Yamaguchi separable potentials like those in (1a′). Hence, our model, although similar to that of Amado and Noble,[8] is simpler, since it does not have pairwise final-state resonances. For the primary decay vertex we take the simple symmetric form

$$\Gamma(\vec{p}, -\tfrac{1}{2}\vec{p}+\vec{q}, -\tfrac{1}{2}\vec{p}-\vec{q}) = \gamma_0(\alpha^2+\tfrac{3}{4}p^2+q^2)^{-1}\,, \tag{7}$$

where α is the weak decay range parameter and γ_0 its strength. The full decay amplitude M to first order in γ_0, but to all orders in the final-state interaction, can be written

$$M = \Gamma + \sum_i f(p_i)\,\Pi(E-\tfrac{3}{4}p_i^{\,2})\,v(q_i^{\,2})\,, \tag{8}$$

where v is the Yamaguchi vertex $v(q^2)=(q^2+\beta^2)^{-1}$ and Π is the Yamaguchi propagator

$$\Pi(\sigma) = \left[\frac{1}{\lambda} + \frac{1}{(2\pi)^3}\frac{1}{2}\int \frac{d^3q\, v(q^2)}{\sigma - q^2}\right]^{-1}\,, \tag{9}$$

with λ the coupling strength. f is defined by Eq. (8). This equation is represented diagrammatically in Fig. 4(a). The quasi-two-body amplitude f satisfies the well-known integral equation

$$f(\vec{p}) = \frac{\gamma_0}{(2\pi)^3}\int \frac{v(q^2)\,d^3q}{(\alpha^2+\frac{3}{4}p^2+q^2)(E-\frac{3}{4}p^2-q^2)} + \frac{1}{(2\pi)^3}\int \frac{d^3p'\,v((\vec{p}+\frac{1}{2}\vec{p}')^2)\,v((\vec{p}'+\frac{1}{2}\vec{p})^2)\,\Pi(E-\frac{3}{4}p'^2)}{E-p^2-p'^2-\vec{p}\cdot\vec{p}'}f(\vec{p}')\,, \tag{10}$$

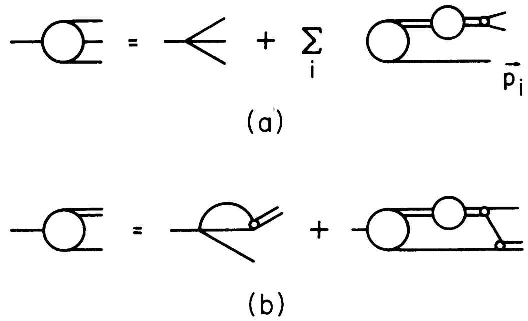

FIG. 4. (a) Diagrammatic representation of Eq. (8). (b) Diagrammatic representation of Eq. (10).

which is represented diagrammatically in Fig. 4(b).[9] $f(\vec{p})$ is essentially the coefficient of $\tau(k^2)$ in Eq. (6), since for Yamaguchi potentials,

$$A(p,p')=\frac{p^2+\beta^2}{p'^2+\beta^2}\ . \tag{11}$$

Hence, we expect f to have a square-root singularity at $E=\frac{3}{4}p^2$. To examine this singularity we solve Eq. (10) numerically. We first project the s-wave part of the kernel of (10) and then invert the equation numerically by matrix inversion. The physical range of p is $0\leq p\leq(\frac{4}{3}E)^{1/2}$. It is well-known that the numerical solution of an equation of this form encounters singularities of the kernel particularly in this range of p. We use the contour-deformation technique originally proposed by Hetherington and Schick,[10] adapted to this problem by Aaron and Amado[11] and refined by Cahill and Sloan.[12] The cautions introduced by Cahill and Sloan are particularly important, since we do the partial wave integration numerically. If this is not done, there can be problems with the standard computer definition of the branches of the complex logarithmic function. Considerable care is required to make sure all these points are correctly treated.

In our numerical solution we have put the Yamaguchi range $\beta=1$ and redefined the coupling parameter $\lambda=16\pi\nu$ so that $\nu=1$ corresponds to a zero-energy bound state of the two-body system. We have then studied the numerical solution of (10) for $E=1$, 0.1, 0.01, for $\nu=0.98$ and 0.5 and for $\alpha=0.1$, 1, 10. This choice is meant to give some indication of the range of possible variation not to cover fully the parameter space.

The real and imaginary parts of the amplitudes f are plotted in Fig. 5 for some of these choices. $f(p)$ is not plotted against p but against $k=(E-\frac{3}{4}p^2)^{1/2}$. The physical range of k is $(E)^{1/2}$ to 0. The strong linear dependence for small k in Fig. 5 shows the importance of the threshold singularity. One also sees that in no sense is f slowly varying. If we had been studying l waves rather than s waves, we should expect $f(p)$ to go like p^l for small p as well as having the $k=0$ singularity. This is clearly seen in Fig. 5 of Aaron and Amado,[11] where an $l=1$ wave is plotted against p. Both the p behavior for small p and the $(E-\frac{3}{4}p^2)^{1/2}$ sin-

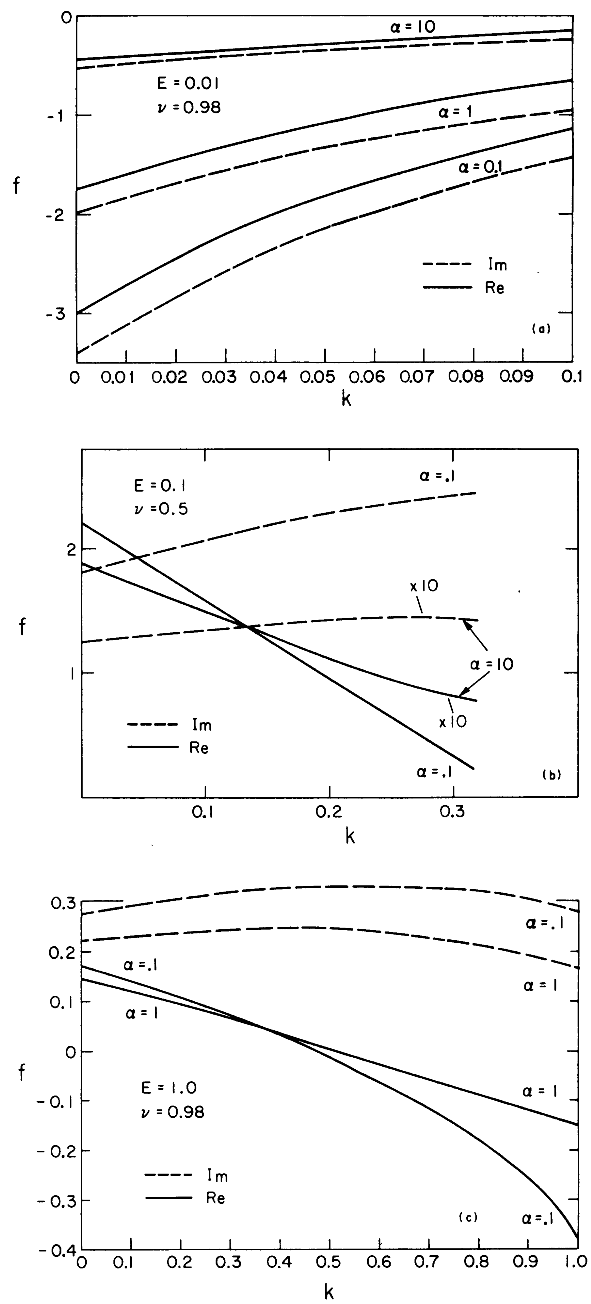

FIG. 5. The real and imaginary parts of the particle amplitude f for various energies E and coupling strengths ν as a function of the two-body submomentum k. α is the weak decay range. In some cases the amplitude has been multiplied by 10 to keep it on scale. (a) $E=0.01$, $\nu=0.98$; (b) $E=0.1$, $\nu=0.5$; (c) $E=1$, $\nu=0.98$.

gularity are easily seen.

It is, of course, not reasonable to expect $f(p)$ to be linearly proportional to $k=(E-\frac{3}{4}p^2)^{1/2}$ for all k. For large k, for example, f must eventually go to zero. Hence a possible parametrization for f might be

$$f(p)=a(b+k)^{-1} \ . \tag{12}$$

To test this form out we have studied f^{-1}. The real and imaginary parts of f^{-1} are plotted as a function of k in Fig. 6 for the same choice of parameters. We see that the curves are linear for some cases over the entire range of k, but departures from linearity also indicate that (12) is by no means exact. Much more work remains to be done to understand and improve the parametrization of f.

Since for identical particles the sum of all terms not ending in the scattering of a given pair is expressed in terms of the same function f, it is easy to check numerically that the coefficient of the singular part of f is indeed correctly related numerically to f at the appropriate shifted momentum in this numerical model. Since our equation is essentially Schrödinger's equation, we would be very surprised if that were not so.

We have seen in a numerical model that the singular part of the isobar or quasi-two-body amplitude in a three-body decay is large and important. In fact, the easiest way to parametrize that amplitude seems to be in terms of that singular term.

IV. THE MULTIPLE SCATTERING SERIES

We have seen formally in previous papers and in Sec. II that the isobar or quasi-two-body amplitude $f(p)$ defined in Eq. (a) has a strong threshold singularity in the two-body subenergy $E-\frac{3}{4}p^2$ and has no other physical-region singularities. We have also seen in Sec. III that this singularity dominates the numerical behavior of f. Furthermore, this singularity comes from the very last rescattering in f. This suggests that the nature of this singularity might be correctly given by any simple approximation to f that contain that last scattering, even if that approximation does a poor job on the over-all magnitude of f. Such an approximation is easily generated by the first terms of the multiple-scattering series. Let us rewrite Eq. (10) in schematic form:

$$f=\Gamma+B\pi f \, . \tag{13}$$

The singularity comes from the propagator in Γ and in B, but since only the $B\pi$ term contains the scattering information, we would expect the singularity to be better represented in terms containing $B\pi$. The Newman or multiple-scattering series for (13) is

$$f=\Gamma+B\pi\Gamma+B\pi B\pi\Gamma+\cdots \ . \tag{14}$$

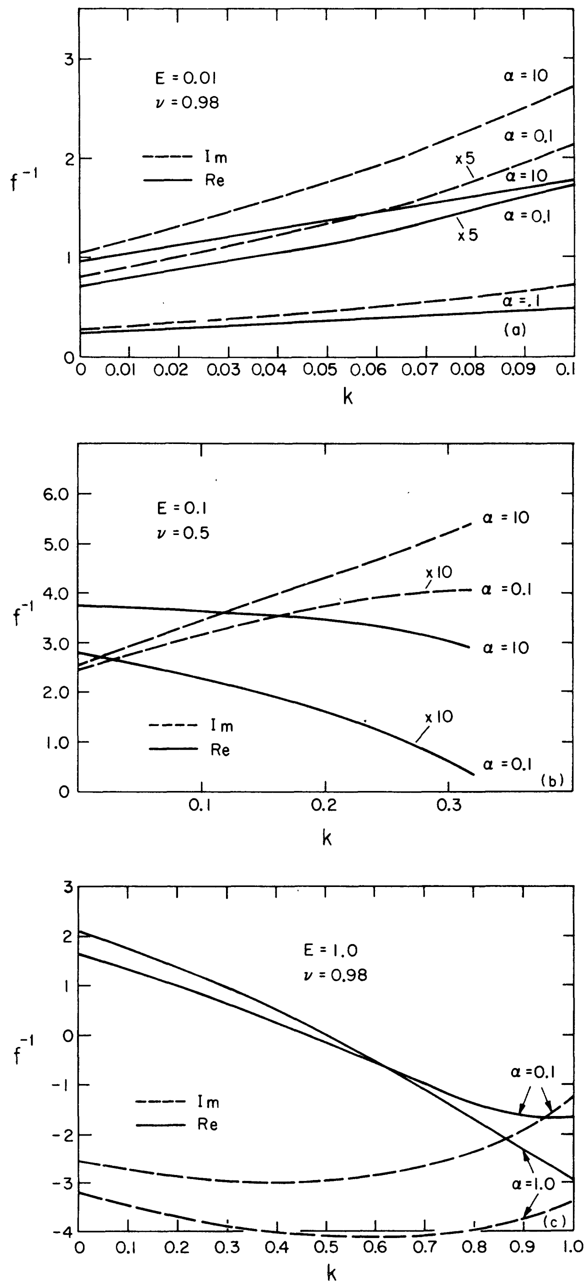

FIG. 6. The real and imaginary parts of the inverse of the partial amplitudes f^{-1} for various energies E and coupling strengths ν as a function of the two-body submomenta k. α is the weak decay range. In some cases the amplitude has been multiplied by 5 or 10 to keep it on scale. (a) $E=0.01$, $\nu=0.98$; (b) $E=0.1$, $\nu=0.5$; (c) $E=1$, $\nu=0.98$.

This is represented schematically in Fig. 7. The dashed lines represent the propagators that give the singularity. This series diverges badly for most of the numerical cases we have considered. The question is whether, in spite of that divergence, the singularity is well given by the series; that is, whether the shape of $f(p)$ for small $k=(E-\frac{3}{4}p^2)^{1/2}$ is correctly given even if the magnitude is not.

We have examined this question numerically in two ways. First we define the logarithmic slope of f,

$$S=\frac{1}{f}\left.\frac{df}{dk}\right|_{k=0}. \tag{15}$$

This quantity gives the "shape" of f at small k with the magnitude divided out. We have calculated S numerically by differencing (our mesh is coarse and the answers are not very accurate) both for the exact S (S_E) and for the S from the first term of Eq. (12) (S_1), the first plus second (S_2), and the first plus second plus third (S_3). These are given in Table I for a range of parameters defined in Sec. III. As we expect, S_1 is a poor approximation, since the first term contains little of the dynamics, but S_2 and S_3 are remarkably good approximations to S_E. It should be borne in mind that the multiple-scattering series is diverging badly for many of these cases, and hence agreement as to magnitude and sign of the S's is really quite good.

An alternative way to investigate the "correct" shape of the approximate term is to study their predictions for $|f|^2=R$. This is a quantity closely related to the rate. We define an exact, R, R_E, and R_1,R_2,R_3, in terms of the first, first plus second, and first plus second plus third terms in (14). In Fig. 8 we plot various R's on a logarithmic scale vs k. If the R's are proportional, they should all be parallel on the logarithmic scale. We see that except for R_1, which has no

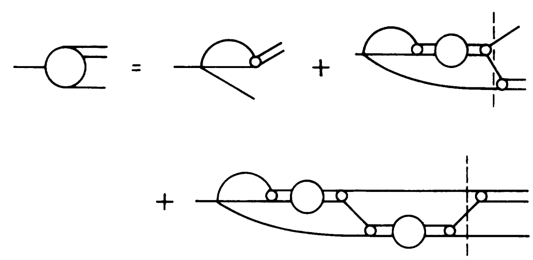

FIG. 7. Diagrammatic representation of the multiple-scattering series [Eq. (14)]. The dashed lines represent the singular propagator.

dynamics, they are parallel to a remarkable degree. We also see that the normalizations that would be required to make R_2 or R_3 agree with R_E are quite large, indicating the bad absolute value (in fact often divergence) of the multiple-scattering series. Still the very close tracking of the R's over the whole range of k indicates substantially correct variation with k, which translates into correct shape of the final-state spectrum.

Three-body experiments involving threshold enhancements are often analyzed using the first few terms of the multiple-scattering series even when that series is known to diverge or converge at best very slowly. Shapes are often correct even if over-all magnitudes are not and large factors are needed to normalize data and theory. We believe the dominance of the singularity discussed here and its relatively faithful reproduction by the multiple-scattering series may go a long way to explain these successes. There are, of course, many questions left to investigate, since we have not dealt with the coherence of the various final-state terms, or with more than one partial wave, but we still believe that we have the germ of an explanation of the remarkable durability of the shape of the final-state spectrum under seemingly invalid approximations.

TABLE I. The real and imaginary parts of the slope parameter S defined in Eq. (15) for the first (S_1), the first plus second (S_2), and first plus second plus third (S_3) multiple scattering terms of Eq. (14) and the exact S (S_E) for various energies E, coupling strength ν, and weak ranges α. The first entry in each case is the real part. Units are such that $\hbar=m=1$.

ν/E	α	S_1		S_2		S_3		S_E	
0.98	10	−0.030	1.3	−9.5	1.2	−9.4	0.94	−8.9	0.58
0.01	1	−0.045	2.0	−9.6	1.1	−9.4	0.91	−9.0	0.52
	0.1	−0.14	4.0	−9.3	1.1	−9.2	0.89	−9.4	0.11
0.98	10	−0.28	1.2	−0.79	0.70	−0.59	0.58	−0.48	0.75
1	1	−0.36	1.6	−0.64	0.74	−0.26	0.56	−0.25	0.91
	0.1	−0.42	1.9	−0.50	0.71	−0.028	0.51	−0.089	1.1
0.5	10	−0.094	1.3	−1.4	1.8	−1.4	1.6	−1.3	1.5
0.1	1	−0.14	1.9	−1.4	2.0	−1.4	1.8	−1.2	1.7
	0.1	−0.29	3.2	−1.4	2.5	−1.2	2.3	−1.0	2.2

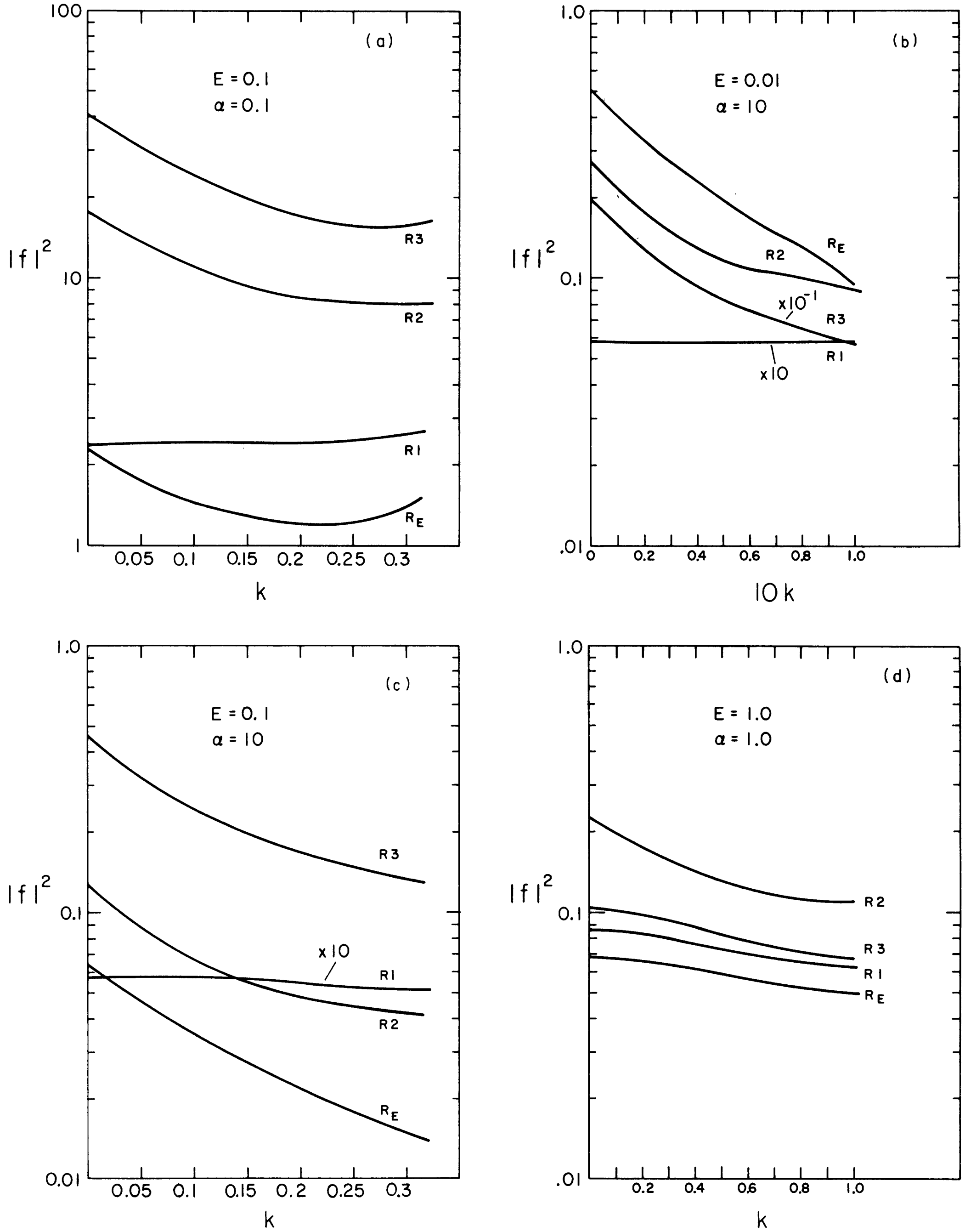

FIG. 8. The absolute squares of f for the first (R_1), first plus second (R_2), and first plus second plus third (R_3) multiple-scattering terms and for the exact f (R_E) as a function of the submomentum k for various energies E, and weak decay ranges α, all for $\nu = 0.98$. (a) $E = 0.1$, $\alpha = 0.1$; (b) $E = 0.01$, $\alpha = 10$; (c) $E = 0.1$, $\alpha = 10$; (d) $E = 1$, $\alpha = 1$.

V. DISCUSSION

We have shown both analytically and numerically that the "isobar" amplitude f of (a) has an important pair subenergy threshold singularity. We have also seen that it may be possible to parametrize f so as approximately to include the rapid dependence implied by the singularity. In executing this parametrization it is important to maintain the coherence implied by (b). Further parametrizations have been discussed elsewhere.[3,2] None of these forms completely encompass the full content of the discontinuity relations implied by the singularity analysis of Sec. II, by Schrödinger's equation or by unitarity. It would certainly be useful to develop a formalism that did include it, and we are presently working to that end.

The numerical importance of the singularity we found in Sec. III is related to the identical-particle model, since for identical particles, the "non-i" part of the total amplitude at $q_i = 0$ is given in terms of f_i at some other momentum, and for small energies, f_i has not changed significantly at these other momenta. Empirically, the presence of the singular term in f will clearly be most important in the analysis of threshold enchancements like the n-n scattering length or in cases of final-state two-body resonances with width comparable with the phase space, that is, of broad overlapping resonance bands. For narrow resonances or high energies the effects of the singular part of f should be less. It should also be noted that in many isobar analyses with $l \neq 0$ two-body states, q^l factors are usually removed, and it is then necessary to redo some of the kinematics of the paper. This is discussed in Aaron and Amado.[3]

Clearly much more work is needed in order to assess the full analytic and numerical impact of the singular variation of f_i both for empirical analysis and for a deeper understanding of the theoretical structure of multibody amplitudes, but we hope that the work presented here and related work has shown that that singular part and its coherent relation to the other parts of the amplitude cannot be ignored.

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