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[8]In the quark model

$$\int_0^1 d\xi\, \xi[\lambda(\xi)+\bar{\lambda}(\xi)] = 9\int_0^1 d\xi\, F_2^{\gamma N}(\xi) - \tfrac{5}{2}\int_0^1 d\xi\, F_2^{\nu N}(\xi)\Big|_{\Delta S=0},$$

where $\lambda(\xi)$ and $\bar{\lambda}(\xi)$ are the distributions of λ and $\bar{\lambda}$ quarks in the nucleon, and $F_2^{\gamma N}(\xi)$ and $F_2^{\nu N}(\xi)$ are structure functions averaged over neutrons and protons. D. Perkins [*Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 4, p. 189] has estimated

$$\int_0^1 d\xi\, F_2^{\gamma N}(\xi) = 0.14\pm 0.02,\quad \int_0^1 d\xi\, F_2^{\nu N}(\xi)\Big|_{\Delta S=0} = 0.49\pm 0.07$$

implying that

$$\int_0^1 d\xi\, \xi[\lambda(\xi)+\bar{\lambda}(\xi)] = 0.02\pm 0.25.$$

[9]See for example, Kuti and Weisskopf (Ref. 3) and references therein.

[10]M. Kugler informs us that this relation can also be obtained using the Melosh transformation from current to constituent quarks.

[11]These sum rules are expected to converge in conventional Regge theory. Only singularities at $\alpha \geq 1$ would cause a divergence, and as noted earlier, the Pomeranchukon at $\alpha = 1$ decouples from $g_1(\xi)$.

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Universal slope and the nonlinear Regge trajectories

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We have investigated constraints imposed by duality on the slopes of nonlinear trajectories. In Argand-loop analysis for square-root-type trajectories we show analytically that the leading and daughter trajectories in the s channel have the same form, except for an additive constant, as the input trajectory in the t channel. This result is also true for all leading nonlinear trajectories in a model proposal by Cohen-Tannoudji, Henyey, Kane, and Zakrzewski.

I. INTRODUCTION

In a model world of zero-width resonances, duality requires a universal slope for all *linear* trajectories. Thus, in Argand-loop analysis the slope of the output trajectory has been shown equal to the slope of the input trajectory.[1] The dual resonance amplitudes (Veneziano or Virosoro representation) also require a universal slope for all linear trajectories.[2,3] One important consequence of a universality of slope is that a unique scale parameter emerges for high-energy processes. However, in the real world, resonances have finite widths and consequently trajectories are nonlinear. This prompts the question: Does duality also require a unique scale parameter for nonlinear trajectories?

The purpose of the present paper is to examine the constraints imposed by duality on the slopes of nonlinear trajectories. Firstly, we investigate this problem without using a definite dual resonance representation. We show analytically that the partial-wave projection of a crossed-channel nonlinear Regge-pole contribution can result in partial-wave amplitudes which produce spiral traces in the Argand diagram. The resulting Regge trajectory in the direct channel is identical to the input trajectory apart from a constant, i.e., the input and output trajectories are parallel. Evaluation of the partial-wave projections of Regge amplitudes requires a definite form for the Regge trajectory. We have taken square-root type of trajectories and our proof can be extended to nth-root type of trajectories.

We then study a model recently proposed by CHKZ.[4] Crossing symmetry is exactly satisfied by this model and Regge asymptotic behavior is ensured by the requirement[5]

$$|\alpha(t)| \lesssim |C|t|^{1/2}\ln(t)|, \text{ as } |t|\to\infty\,. \tag{1}$$

We derive a condition on the asymptotic behavior of nonlinear Regge trajectories, employing a model proposed by Mandelstam.[6] We show that this asymptotic condition, when applied to the CHKZ amplitude, leads to the equation

$$\mathrm{Re}\,\alpha_\rho(x) = \mathrm{Re}\,\alpha_\theta(x) + C_\theta \quad (\theta = N_\alpha,\ \Delta,\ N_\gamma)\ , \qquad (2)$$

where the C_θ's are constants.

In Sec. II we investigate the structure of maxima in different Argand diagrams for square-root trajectories. In Sec. III we derive the asymptotic behavior of trajectories. We then use this condition in the CHKZ representation. Finally, these results are discussed in Sec. IV.

II. PARTIAL-WAVE AMPLITUDES

We study the s-channel partial-wave projection of the Regge amplitude

$$A(s,t) = \frac{\beta(t)}{\Gamma(\alpha)\sin\pi\alpha}\,(\pm 1 - e^{-i\pi\alpha})\left(\frac{s}{s_0}\right)^{\alpha}, \qquad (3)$$

where $\alpha = \alpha(t)$ and the other symbols carry their usual meanings. We choose a simple form for the trajectory

$$\alpha(t) = \eta_0 + i\eta(t-\Sigma)^{1/2}\ . \qquad (4)$$

The partial-wave projection is defined by

$$a_l(s) = \tfrac{1}{2}\int_{-1}^{+1} P_l(Z_s)A(s,t)\,dZ_s\ . \qquad (5)$$

In order to calculate this integral analytically, we approximate the smooth factor (for negative t)

$$\frac{\beta(t)}{\Gamma(\alpha(t))\sin\pi\alpha(t)} \approx \text{constant}\ . \qquad (6)$$

Following the method of Ref. 7 we obtain the partial-wave amplitude $a_l(s)$ for large s ($s \gg l$)

$$A_l^{\pm}(s) = \int_{-1}^{+1} dz_s P_l(Z_s)\left(\pm\frac{s}{s_0}\right)^{\alpha(t)},$$
$$a_l(s) = \pm A_l^{+}(s) - A_l^{-}(s)\ , \qquad (7a)$$

where

$$A_l^{-}(s) = 2p\left(\frac{-s}{s_0}\right)^{\eta-\eta' q} \times \int_{-1}^{+1} dx \left(\frac{-s}{s_0}\right)^{-\eta' px} (-1)^l P_l(a-(px+q)^2)(px+q) \qquad (7b)$$

and

$$q+p = (a+1)^{1/2},\quad q-p = (a-1)^{1/2}\ , \qquad (8)$$

$$a = 1 + \frac{2\Sigma}{s-\Sigma}\ ,\quad \eta' = \eta\left(\frac{s-\Sigma}{2}\right)^{1/2}. \qquad (9)$$

Using the expansion

$$(px+q)P_l(a-(px+q)^2) = \sum_{m=0}^{2l+1} k_{l,m}P_m(x)\ , \qquad (10)$$

where $k_{l,m}$ are constants,[8] in Eq. (7), we obtain

$$A_l^{-}(s) = 2p\left(\frac{-s}{s_0}\right)^{\eta_0-\eta' q}(-1)^l \times \sum_{m=0}^{2l+1} k_{l,m}2(i)^m j_m(i\eta' p\ln(-s/s_0))\ . \qquad (11)$$

Now, using

$$j_m(z) \sim \frac{1}{z}\cos[z-(m+1)\tfrac{1}{2}\pi]\ ,\quad |\arg(z)| < \pi,\quad |z| \to \infty\ , \qquad (12a)$$

the s-asymptotic behavior of $A_l^{-}(s)$ for $s \gg l$ is given by

$$A_l^{-}(s) \sim \frac{-2\sqrt{2}}{\eta'\ln(-s/s_0)}\left(\frac{-s}{s_0}\right)^{\eta_0} \times \sum_{m=0}^{2l+1} k_{l,m}\left[(-1)^{l+m-1} + \left(\frac{s}{s_0}\right)^{-\eta'\sqrt{2}} e^{i\pi(\eta'\sqrt{2}-l)}\right]. \qquad (12b)$$

In Eq. (12b) the first term inside the square bracket gives a large background, while the second term gives resonance maxima. The imaginary part of $A_l^{-}(s)$ associated with these resonance maxima is given by

$$\mathrm{Im}A_l^{-}(s)|_{\text{resonance}} \simeq \sin\pi[\eta_0 + \eta(s-\Sigma)^{1/2} - l] \times \left(\frac{s}{s_0}\right)^{\eta(s-\Sigma)^{1/2}+\eta_0}, \qquad (13a)$$

which exhibits maxima at

$$\eta_0 + \eta(s-\Sigma)^{1/2} - l = 2n + \tfrac{1}{2},\quad n = 0, 1, 2, \ldots\ . \qquad (13b)$$

We thus find that the output trajectory has the same form as the input trajectory except for a constant. Using similar arguments we can also show that if the input trajectory has the form $\alpha(t) = \eta_0 + \eta(\Sigma - t)^{1/n}$, then the output trajectory is given by $\alpha(s) = \text{const} + \eta(\Sigma - s)^{1/n}$. Notice also that in the direct channel, there appears an infinite set of daughter trajectories parallel to the leading trajectory.

III. THE DUAL AMPLITUDE

It was shown by Mandelstam[6] that linear trajectories in a dual amplitude have a universal slope. We shall show that when Mandelstam's model is extended to rising nonlinear trajectories, asymptotically, all trajectories become parallel. Using the CHKZ model we shall then prove that if trajectories are asymptotically parallel, they are parallel everywhere.

Consider the situation in which an external particle of mass μ is combined with a series of external particles with spins σ and masses m_σ. The intermediate states comprise a Regge sequence of spins $\sigma+n$ ($n=0, 1, 2, \ldots$) and masses $M_{\sigma+n}$, where

$$\mathrm{Re}\,\alpha_1(m_\sigma)=\sigma\ , \tag{14}$$

$$\mathrm{Re}\,\alpha_2(M_{\sigma+n})=\sigma+n\ . \tag{15}$$

Suppose now that $\mathrm{Re}\,\alpha_1(Z)\sim K_1Z^{m_1}$ and $\mathrm{Re}\,\alpha_2(Z)\sim K_2Z^{m_2}$ as $Z\to\infty$; then the following expression is obtained for the binding energy $(M_{\sigma+n}-m-\mu)$ in the limit $\sigma\to\infty$:

$$B(\sigma)=\left(\frac{\sigma}{K_2}\right)^{1/m_2}-\left(\frac{\sigma}{K_1}\right)^{1/m_1}-\mu\ . \tag{16}$$

Now, following Mandelstam, we assume that $B(\sigma)$ remains finite as $\sigma\to\infty$. Then we have

$$m_1=m_2 \text{ and } K_1=K_2\ , \tag{17}$$

i.e., the two nonlinear trajectories $\mathrm{Re}\,\alpha_1(x)$ and $\mathrm{Re}\,\alpha_2(x)$ are parallel at infinity.

We consider the CHKZ amplitude for πN kinematics involving different trajectories in the t and the s and u channels. For our purpose here the $A^-(s,t,u)$ amplitude suffices. It is given by[9]

$$\frac{A^-(s,t,u)}{4\pi}=\sum_{x=N_\alpha,\Delta,N_\gamma}\gamma_{x\rho}K^-_{x\rho}(s,t)+\sum_{x,y=N_\alpha,\Delta,N_\gamma}\gamma_{x,y}(1-\delta_{xy})M^-_{xy}(s,u)\ , \tag{18}$$

where

$$K^-_{\sigma\rho}(s,t)=\int_0^1 dx\, f_\rho(sx)f_\sigma(tx')x^{-\alpha_\rho(tx')}\times(1-x)^{-\alpha_\sigma(sx)-1/2}-s\to u \tag{19}$$

and

$$M^-_{\theta\phi}(s,u)=\int_0^1 dx\,\gamma_\phi(sx)f_\theta(ux')x^{-\alpha_\theta(ux')}\times(1-x)^{-\alpha_\theta(sx')+1/2}-s\to u\ . \tag{20}$$

In the above equations $x'=1-x$.

To ensure correct pole structure for $A^-(s,t,u)$, the residue functions occurring in (19) and (20) satisfy the following conditions:

$$f_\rho(y)=\alpha_\rho(y)f(y)\ , \tag{21a}$$

$$f_\theta(y)=[\tfrac{1}{2}-\alpha_\theta(y)]f(y)\ , \tag{21b}$$

$$\gamma_\theta(y)=\beta_\phi(y)f(y)\ , \tag{22}$$

where $f(y)$ is the usual f function which vanishes faster than any inverse power of y as $|y|\to\infty$. The factor $\beta_\phi(y)$ in Eq. (22) is a ghost-eliminating factor associated with either the s- or the u-channel trajectories. We take the most general form for $\beta(y)$, i.e.,

$$\beta_\phi(y)=\sum_{n=0}^{N}\rho_n^\phi[\alpha_\phi(y)]^n,\quad \phi=N_\alpha,\Delta,N_\gamma \tag{23}$$

where the ρ_n^ϕ's are constants.

Demanding that the N_α trajectory have even signature and the N_γ and Δ trajectories have odd signature leads to the following restrictions for small negative values of u:

$$\int_0^\infty d\mu\,\mu^{-\alpha_\sigma(u)-1/2}f(-\mu)\,[\alpha_\rho(-\mu)-C_\sigma\beta_\sigma(-\mu)]=0,\qquad \sigma=N_\alpha,\Delta,N_\gamma \tag{24}$$

and

$$C_{N_\alpha}=-\gamma_{N_\alpha\rho}-\gamma_{N_\alpha N_\gamma}-\gamma_{N_\alpha\Delta}\ , \tag{25a}$$

$$C_\Delta=-\gamma_{\Delta\rho}+\gamma_{\Delta N_\alpha}+\gamma_{\Delta N_\gamma}\ , \tag{25b}$$

$$C_{N_\gamma}=-\gamma_{N_\gamma\rho}+\gamma_{N_\gamma\Delta}+\gamma_{N_\gamma N_\alpha}\ , \tag{25c}$$

where the γ's are defined in Eq. (18).

As Eq. (24) is satisfied for continuous values of u, using an inverse Mellin transform we obtain

$$\alpha_\rho(-\mu)=\sum_{n=0}^{N}a_n^\theta[\alpha_\theta(-\mu)]^n,\ \text{ for all } \mu>0\ , \tag{26}$$

where $a_n^\theta=\rho_n^\theta C_\theta$. From Eq. (17) we know that

$$\mathrm{Re}\,\alpha_\rho(\mu)=\mathrm{Re}\,\alpha_\theta(\mu)=K_1\mu^m$$

as $\mu\to\infty$, hence, in Eq. (26)

$$a_n^\theta=0 \text{ for } n>1,\quad a_n^\theta=1 \text{ for } n=1\ .$$

Thus,

$$\mathrm{Re}\,\alpha_\rho(\mu)=\mathrm{Re}\,\alpha_\theta(\mu)+a_0^\theta\ , \tag{27}$$

where $\theta=N_\alpha,\Delta,N_\gamma$.

IV. DISCUSSION

We have shown that duality requires a unique scale parameter for nonlinear trajectories. Our investigations in Sec. II depend on a definite form of nonlinear trajectory. However, our analysis of Sec. II is independent of any dual representation. In Argand-loop analysis we find that not only the input and output trajectories have the same slope, but all the daughter trajectories generated in the direct channel are also parallel.

In Sec. III our investigations were dependent on a given dual representation (CHKZ). In this case, our results are true only for the leading trajectories. This is due to the multipole structure of the daughter trajectories in the CHKZ representation. In this case, the condition Eq. (27) is not satisfied.

However, inclusion of cuts in the CHKZ representation does not change our results. This is true for the rather general case where cuts are constructed from moving poles.

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[8]The constant $k_{l,m}$ is defined by the following equation:

$$k_{l,m}=\sum_{r=0}^{l} C_{l,r}\sum_{\mu=0}^{r}\binom{r}{\mu} a^{\mu} \sum_{\lambda=0}^{2r-2\mu+1}\binom{2r-2\mu+1}{\lambda} q^{\lambda} \times p^{2r-2\mu+1-\lambda} d_{2r-2\mu-\lambda+1,m},$$

where $C_{l,r}$ and $d_{l,m}$ are given by

$$P_l(y)=\sum_{r=0}^{l} C_{l,r}y^r \quad \text{and}$$

$$y^l=\sum_{m=0}^{l} d_{l,m}P_m(y).$$

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Particle ratios in energetic hadron collisions*

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We construct a simple statistical model in order to estimate, for very energetic collisions, the ratios of particles with various quantum numbers produced with center-of-mass momenta less than a few GeV. Two conclusions are that (1) isobar decay can account for a large fraction of the SU(3) violation observed experimentally, and (2) at high transverse momentum the dominance of pions diminishes.

In this paper we construct an extremely simple statistical picture of hadron production in the central region of rapidity. It illustrates how several features of the produced-particle spectrum in hadron collisions can be understood without resorting to sophisticated dynamical models. While we are not sure whether this is a model which can be used for serious quantitative study, we do believe it contains qualitative features of some generality.

Subsequent to carrying out this work, we realized that Anisovich and Shekhter[1] have developed a very similar picture. We recommend that the interested reader compare that work with this. Not surprisingly our versions differ in various ways: The initial assumptions differ somewhat, and comparison of our results with theirs provides an indication of the sensitivity of the basic idea to details. Also, Anisovich and Shekhter considered not only the central rapidity region (for low-$p_\perp$ secondaries) but the target and projectile fragmentation regions as well. On the other hand, we have applied the idea to the production of particles of high and low $p_\perp$ in the central rapidity region.

The model is very simple. Imagine that, as a re-